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MHD CASSON OR CARREAU FLUIDS FLOW WITH MICROORGANISMS OVER A PERMEABLE SHRINKING SURFACE

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Abstract:

The characteristics of dual solutions of the flow of Casson or Carreau fluids mixed with microorganisms over a permeable shrinking surface are investigated considering the effects of the magnetic field, heat dissipation, Dufour, and thermophoretic diffusivity. By using similarity transformations, the governing equations are converted into a set of ordinary differential equations. These reduced equations are then solved by the well-known shooting technique along with the fourth-order Runge-Kutta method. A comparison is provided and the present solutions hold a good agreement with available published results. The local skin friction coefficient, local Nusselt number, local Sherwood number, and local density number of the microorganisms are found to increase with the increase of non-Newtonian Casson parameter, Dufour, and thermophoretic diffusivity number, however, it decreases with Weissenberg number. The remarkable finding which is not revealed yet is that dual solutions exist in the flow of Casson or Carreau fluids with microorganisms over a permeable shrinking sheet subject to a certain combination of parameters. Moreover, the domain of the occurrence of dual solutions broadens on account of higher values of all physical parameters of the problem except the Eckert number.

Keywords: Microorganisms; Casson fluid; Carreau fluid; Magnetohydrodynamic; Permeable surface.

NOMENCLATURE

B_0	Magnetic field strength	T	Fluid's temperature
C	Fluid's concentration	T_w	Wall temperature
C_s	Concentration susceptibility	C_w	Wall Concentration
C_p	Heat capacities of the fluid	M_w	Wall density of microorganisms
D_B	Brownian diffusion	T_∞	Temperature outside the boundary layer
D_n	Microorganisms diffusion	C_∞	Concentration outside the boundary layer
D_T	Thermal diffusion	M_∞	Density outside the boundary layer
D_f	Dufour number	u, v	Velocities in x and y directions
Ec	Eckert number	We	Weissenberg number
g	Acceleration due to the gravity	***Greek symbols***	
Gr_T	Momentum buoyancy parameter	σ	Magnetic field strength
Gr_C	Concentration buoyancy parameter	ν & ν_c	Viscosity of the fluid and microorganisms
Ha	Hartman Number	δ	Boundary layer thickness
$\overline{K}$	Thermal diffusion ratio	ρ	Density of the fluid
$\bar{K}$	Permeability constant	κ'	Permeability constant
M	Microorganisms mass density	b	Constant
N_b	Brownian diffusivity number	β	Shrinking parameter
N_t	Thermophoretic diffusivity number	Γ	Variable time

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Nn	Density number of the microorganisms	l	Characteristic length
Nu	Nusselt number	Φ	Heat dissipation
Pe	Peclet number	κ	Thermal conductivity
Pr	Prandtl number	β_T	Volumetric coefficients due to thermal
Sb	Schmidt number for microorganisms	β_C	Volumetric coefficients due to C
Sc	Schmidt number	β_M	Microorganisms coefficient due to M
Sh	Sherwood number	σ_1	Mass concentration parameter

1. Introduction

Flow over a horizontal or vertical plate, stretching or shrinking sheet, porous surface, with or without the presence of magnetic/electric field are of interest of many researchers over the years. Also, many researchers carried out a theoretical or numerical study to find the dual solution of the above mentioned problem. Ishak et al. (2008) studied numerically the mixed convection flow on a permeable vertical plate and observed the dual solution of the problem. Later, Ishak et al. (2009) carried out a similar study for micropolar fluids. In these study, they first transformed the set of nonlinear PDE into nonlinear ODE and then used the finite difference method to formulate and solve the problem. Their results reveal that dual solutions can be observed for the assisting flow. Also, Xu and Liao (2008) studied the flow over a flat plate and obtained the dual solution using the Homotopy analysis method. Their results are compared with numerical solutions and found a good agreement. It suggests that this method can be very useful for solving nonlinear system. Moreover, Subhashini et al. (2012) numerically investigated the dual solution behavior of flow over a porous flat plate with suction or injection. They used the nonlinear shooting method with the Runge-Kutta method of 4th order to solve the converted non-linear ODE. It is observed that the dual solutions of velocity, temperature and concentration profiles can be found for some particular parameters. Later, Subhashini and Sumathi (2014) used numerical approach to find the dual solution of nanofluids flow over moving vertical plate. They used a numerical approach discussed in Subhashini et al. (2012). They considered three different types of nanofluids and found that the existence of dual solution is possible. Some of the similar research is mentioned in Khashiie et al. (2021).

Turkyilmazoglu (2012) studied theoretically the dual and triple solutions behavior of electrically conducting non-Newtonian MHD fluid flow past a shrinking sheet. It is seen that multiple solution exists for some small group of parameters. Moreover, Zheng et al. (2012) carried out a theoretical investigation to observe the dual solution for micropolar fluid flow over stretching/shrinking sheet. They also used Homotopy analysis method for theoretical framework of the problem. Bhattacharyya (2011) numerically studied the mass transfer behavior in the presence of chemical reaction of flow over stretching or shirking sheet. He also transformed the nonlinear system of PDE into a nonlinear system of ODE using similarity transformations and then solved the system by the well-known nonlinear shooting method. It is found that the dual solutions of velocity and concentration can be observed for some parameters. Singh and Chamkha (2013) investigated numerically the dual solution of flow over vertical porous shrinking sheet. Subhashini et al. (2013) carried out a numerical study of the dual solutions of non-Newtonian fluid flow over stretching sheet with variable thickness. Akbar et al. (2014) carried out numerical procedure to find the dual solution of MHD Carreau fluid flow over the porous shrinking sheet. Singh and Chamkha (2013), Subhashini et al. (2013) and Akbar et al. (2014) used the same numerical approach proposed by Subhashini et al. (2012). Sandeep and Sulochana (2015) studied unsteady micropolar fluid flow over stretching/shrinking sheet with non-uniform heat source/sink. They used the nonlinear shooting method with the available packages in Matlab. All the above mentioned authors observed that dual solution is feasible. El-Aziz (2016) carried out a research on MHD incompressible fluid flow over stretching/shrinking sheet with non-uniform heat source/sink. He used the nonlinear shooting method with Runge-Kutta-Fehlberg fifth order scheme. He showed the dual solutions for different parameters and presented them graphically in terms of velocity and temperature. Recently, Lund et al. (2019) studied the Williamson fluid flow over an exponentially vertical shrinking sheet with variable thickness and thermal conductivity. They used Runge-Kutta shooting method and used maples for simulations. They also found dual solutions for different parameters. In addition, some of the relevant research works have been cited in (Ghosh et al., 2016; Freidoonimehr & Rahimi, 2017; Khan et al., 2017; Hamid et al., 2019; Haq et al., 2020; Roy, 2020; Mousavi et al., 2021; Asogwa & Ibe, 2020; Nagendra et al., 2022; Roy & Saha, 2022).

From the above literature reviews, it is seen that very little attention has been given by the researcher on the investigation of the dual solutions of Casson or Carreau fluid flow over a porous surface. However, in recent years, presence of microorganism in the fluids creates a huge interest in the CFD research community (Atif et al., 2019; Mondal & Pal, 2020; Yusuf et al., 2021; Biswas et al., 2021). With authors' knowledge, the use of both Casson and Carreau fluids in a same model were presented by Raju and Sandeep (2016). Later, Mahanthesh et al. (2019) used this idea for dusty Casson and Carreau fluids. These works influenced us to use both non-Newtonian fluids (Casson or Carreau fluids) in a same model. Open literature suggests that no research has taken an attempt to investigate the dual solutions for the flow of Casson and Carreau fluids with microorganisms over a shrinking sheet. So, the objective of this study is to examine the dual solutions of the flow of MHD Casson and Carreau fluids combined with microorganisms over permeable stretching sheet. The governing equations are reduced to a set of nonlinear differential equations using similarity transformations. We solve the resulting equation using the nonlinear shooting method. Results are discussed in terms of dual solutions for varying relevant parameters.

2. Mathematical Model

We consider two dimensional steady MHD Casson or Carreau fluids flow with microorganisms over a permeable shrinking surface. The flow configuration with boundary conditions are elucidated in Fig. 1. Here x and y measure the distances along and normal to the shrinking sheet. Also, u and v are the velocity components in the x- and y-direction, $u_w(x)$ is the shrinking velocity of the plate, υ_w is the mass flux through the plate, T_w, C_w, M_w are the wall temperature, concentration, density of microorganisms and T_∞, C_∞, M_∞ are the corresponding quantities outside the boundary layer. In the figure, δ_u, δ_T, δ_C, δ_M are respectively the boundary layer thicknesses for momentum, thermal, concentration and microorganisms. Definition of tensor for Casson and Carreau fluids is presented in Raju and Sandeep (2016) and used in this study.

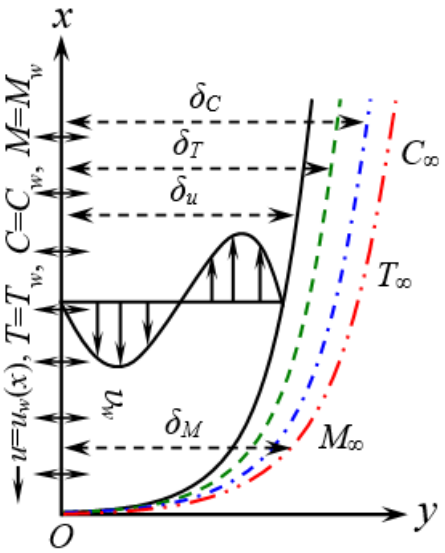

Fig. 1: Physical model with boundary conditions

Moreover, the thermophysical properties of the fluid are taken to be constant. Under the usual Boussinesq approximation, the governing equations for Casson or Carreau fluids flow with microorganisms are, following (Raju & Sandeep, 2016; Mahanthesh et al., 2019; Qasim & Noreen, 2014; Saravana et al., 2019; Waqas et al., 2021):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(1+\frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2}+\frac{3}{2}\nu(n-1)\Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}-\left(\frac{\sigma B_0^2}{\rho}+\frac{\nu}{\kappa'}\right)u +g\beta_T(T-T_\infty)+g\beta_C(C-C_\infty), \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\nu}{C_p}\left(1+\frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2+\frac{\kappa}{\rho C_p}\frac{\partial^2 T}{\partial y^2}+\frac{D_m\bar{\kappa}}{C_s C_p}\frac{\partial^2 C}{\partial y^2}, \tag{3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2}, \tag{4}$$

$$u\frac{\partial M}{\partial x}+v\frac{\partial M}{\partial y}+\frac{b\nu_c}{C_w-C_\infty}\frac{\partial}{\partial y}\left(M\frac{\partial C}{\partial y}\right)=D_n\frac{\partial^2 M}{\partial y^2}, \tag{5}$$

with boundary conditions

$$u=u_w(x)=-bx, v=v_w \; at \; y=0, \tag{6}$$

$$u\to 0 \;\; as \; y\to\infty, \tag{7}$$

$$T=T_w, C=C_w, M=M_w \; at \; y=0, \tag{8}$$

$$T\to T_\infty, C\to C_\infty, M\to M_\infty \; as \; y\to\infty. \tag{9}$$

In the above equations, β is the non-Newtonian Casson parameter, n is the power-law index, σ is the magnetic field strength, ν and ν_c are the viscosity of the fluid and microorganisms, D_m is the chemical molecular diffusivity, D_B, D_n and D_T are material, Brownian, microorganisms and thermal diffusions, C_s is the concentration susceptibility, and C_p are the heat capacities of the material and fluid, ρ is the density of the fluid, κ' is the permeability constant, B_0 is the strength of the magnetic field considered normal to the sheet, C is the concentration of the fluid, M is the mass density of the microorganisms, g is the acceleration due to the gravity, Γ is the time constant, κ is the thermal conductivity, $\bar{\kappa}$ is the thermal diffusion ratio, β_T and β_C are the volumetric coefficients due to thermal and concentration expansions. In this study, we also assume that $\beta\to\infty$ for Carreau fluid and $\Gamma=0$ for Casson fluid.

Let us introduce the following transformations:

$$\begin{gathered} u=bxf'(\eta), v=-\sqrt{b\nu}f(\eta), \eta=\sqrt{\frac{b}{\nu}}y \\ \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}, \varphi(\eta)=\frac{C-C_\infty}{C_w-C_\infty}, \chi(\eta)=\frac{M-M_\infty}{M_w-M_\infty}, \end{gathered} \tag{10}$$

Applying the transformations given in Eq. (10) in the Eqs. (1)-(9), the following non-dimensional non-linear ordinary differential equations and corresponding boundary conditions are obtained:

Non-dimensional governing equations:

$$\left(1+\frac{1}{\beta}\right)f'''+\frac{3}{2}(n-1)We^2f'''(f'')^2+ff''-(f')^2-(Ha^2+\bar{K})f' +Gr_T\theta+Gr_C\varphi=0, \tag{11}$$

$$\frac{1}{Pr}\theta''+f\theta'+Ec\left(1+\frac{1}{\beta}\right)(f'')^2+D_f\,\varphi''=0, \tag{12}$$

$$\frac{1}{S_c}\varphi''+f\varphi'+\frac{1}{S_c}\frac{N_t}{N_b}\theta''=0, \tag{13}$$

$$\chi''+S_b\chi' f-Pe[(\sigma_1+\chi)\varphi''+\chi'\varphi']=0, \tag{14}$$

where,

$$Ha^2=\frac{\sigma B_0^2}{b\rho}, \bar{K}=\frac{\nu}{b\kappa'}, Gr_T=\frac{g\beta_T(T_w-T_\infty)}{bu_w}, Gr_C=\frac{g\beta_C(C_w-C_\infty)}{bu_w}, Pr=\frac{\nu}{\alpha}, \alpha=\frac{\kappa}{\rho C_p},$$

$$Ec=\frac{u_w^2}{C_p(T_w-T_\infty)}, D_f=\frac{D_m\bar{\kappa}}{C_sC_p\nu}\frac{(C_w-C_\infty)}{(T_w-T_\infty)}, N_b=\frac{\tau D_B(C_w-C_\infty)}{\nu}, N_t=\frac{\tau D_T(T_w-T_\infty)}{\nu T_\infty},$$

$$S_c=\frac{\nu}{D_B}, We^2=\frac{bu_w^2}{\nu}\Gamma^2, Pe=\frac{b\nu_c}{D_n}, S_b=\frac{\nu}{D_n}, \sigma_1=\frac{M_\infty}{M_w-M_\infty}.$$

Here Ha is the Hartman number, $\bar{K}$ is the permeability constant, Gr_T is the Grashof number for thermal diffusion, Gr_C is the Grashof number for mass diffusion, Pr is the Prandtl number, Ec is the Eckert number, D_f is the Dufour number, N_b is the Brownian diffusivity number, N_t is the thermophoretic diffusivity parameter, We is the Weissenberg number, Pe is the Peclet number, S_c is Schmidt number, S_b is Schmidt number for microorganisms diffusivity and σ_1 is the mass concentration parameter.

Non-dimensional boundary conditions are

$$f(\eta) = S, f'(\eta) = -1 \; at \; \eta = 0, \tag{15}$$

$$f'(\eta) \to 0 \; as \; \eta \to \infty, \tag{16}$$

$$\theta(\eta) = 1, \varphi(\eta) = 1, \chi(\eta) = 1 \;\; at \; \eta = 0, \tag{17}$$

$$\theta(\eta) \to 0, \;\; \varphi(\eta) \to 0, \chi(\eta) \to 0 \; as \; \eta \to \infty. \tag{18}$$

where, $S = -\frac{v_w}{\sqrt{vb}}$, is the suction or injection parameter.

In this study, the local skin friction coefficient (Hayat et al., 2012), the local Nusselt number (Hayat et al., 2012; Waqas et al., 2021), the local Sherwood number (Hayat et al., 2012; Waqas et al., 2021), and the local density number of the microorganisms (Waqas et al., 2021) are defined by

$$\left.\begin{aligned} Re_x^{1/2} C_f &= \left(1 + \frac{1}{\beta}\right) f''(0), \\ Re_x^{-1/2} Nu_x &= -\theta'(0), \\ Re_x^{-1/2} Sh_x &= -\varphi'(0), \\ Re_x^{-1/2} Nn_x &= -\chi'(0), \end{aligned}\right\} \tag{19}$$

where, Re_x is the local Reynolds number.

3. Numerical Methods and Validations

The system of equations (11)-(14) along with the boundary conditions (15)-(18) are solved by the non-linear shooting method combined with Runge-Kutta technique of order four. The flow chart of the method is shown in Fig. 2. To convert the Eqs. (11)-(14) into first order ODEs, let us introduce the following transformations:

$$f = y_1, f' = y_2, f'' = y_3, \theta = y_4, \theta' = y_5, \varphi = y_6, \varphi' = y_7, \chi = y_8, \chi' = y_9. \tag{20}$$

Using the above transformations, we get the following system of ODEs:

$$\left.\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ y_3' &= \frac{y_2^2 - y_1 y_3 - (Ha^2 + \bar{K}) y_2 - Gr_T y_4 - Gr_c y_6}{\left(1 + \frac{1}{\beta}\right) + \frac{3}{2}(n-1) We^2 y_3^2}, \\ y_4' &= y_5, \\ y_5' &= -Pr\left\{y_1 y_5 + Ec\left(1 + \frac{1}{\beta}\right) y_3^2 + D_f y_7'\right\}, \\ y_6' &= y_7, \\ y_7' &= -Sc\left(y_1 y_7 + \frac{1}{Sc}\frac{N_t}{N_b} y_5'\right), \\ y_8' &= y_9, \\ y_9' &= Pe\{(\sigma_1 + y_8) y_7' + y_7 y_9\} - Sb \; y_1 y_9, \end{aligned}\right\} \tag{21}$$

And the corresponding boundary conditions are

$$\left.\begin{aligned} y_1 = S, y_2 = -1, y_4 = 1, y_6 = 1, y_8 = 1 \quad at \; \eta = 0 \\ y_2 \to 0, y_4 \to 0, y_6 \to 0, y_8 \to 0 \quad as \; \eta \to \infty \end{aligned}\right\} \tag{22}$$

In order to solve the above transformed system using the nonlinear shooting method, we need to guess four initial conditions, such as $y_3(0) = p_1, y_5(0) = p_2, y_7(0) = p_3, y_9(0) = p_4$. The four unknowns p_1, p_2, p_3, p_4 are chosen in such a way that these four boundary conditions are satisfied for $\eta \to \infty$. The numerical simulations are performed for suitable domain $[0, \eta_{max}]$ in place of the domain [0, ∞) where we choose $\eta_{max} \in \mathbb{R}^+$ very carefully so that higher values of η_{max} showed insignificant variation for all the outcomes. Moreover, convergence criteria are set as the following:

$$max\{y_i - 0\} < 10^{-6}, i = 1,2,4,6,8 \tag{23}$$

Validation is carried out by comparing the first and second solutions of $-\theta'(0)$ with the results of Qasim and Noreen (2014) as shown in Table 1. An excellent agreement is observed which motivated us to move to the next step.

Table 1: Comparison of $-\theta'(0)$

S	β	Qasim and Noreen (2014)		Present Method	
		First solution	**Second solution**	**First solution**	**Second solution**
2.4	3.0	0.80814	0.88288	0.80814	0.88377
2.7	3.0	0.91197	1.13797	0.91201	1.13905
3.0	3.0	1.01955	1.39641	1.01957	1.40754
2.5	2.0	0.86406	0.93142	0.86401	0.93123
2.5	10.0	0.82813	0.99868	0.82815	0.99756

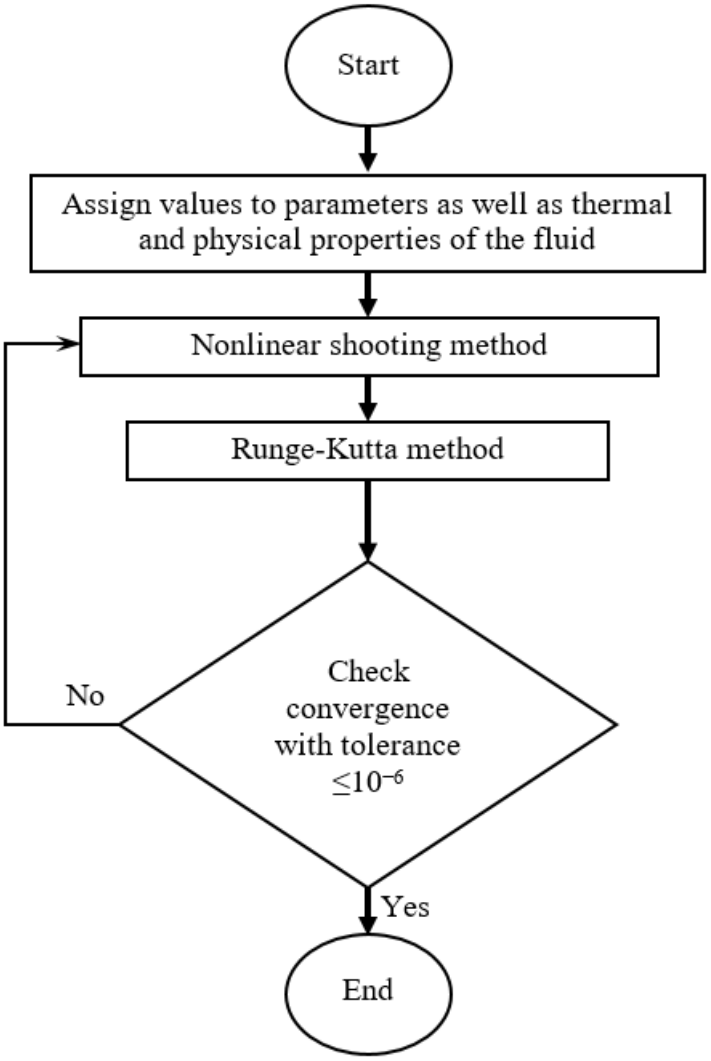

Fig. 2: Flow chart of the solution scheme

4. Results and Discussion

In this investigation, the values of the parameters are collected from the open literatures. Here, we consider Pr = 0.71, *Ha* = 0 to 0.6, *S* = 1 to 6 (Mousavi et al., 2021), β = 0.2 to 0.8, *We* = 1 to 5 (Raju & Sandeep, 2016) and *Ec* = 0 to 0.5 (Ullah et al., 2019).

Influences of N_t on$Re_x^{1/2}C_f$, $Re_x^{-1/2}Nu_x$, $Re_x^{-1/2}Sh_x$, and $Re_x^{-1/2}Nn_x$ are shown in Fig. 3 and Table 2. All of these important quantities decrease owing to the reduction in the suction parameter, *S*. With the increase of N_t

the values of $Re_x^{1/2}C_f$ and $Re_x^{-1/2}Nu_x$ are found to increase whereas the values of $Re_x^{-1/2}Sh_x$ and $Re_x^{-1/2}Nn_x$ decrease. On the other hand, for a fixed set of parameter there exist dual solutions consisting of stable and unstable solutions. The upper branch represents stable solutions and the lower branch shows unstable solutions. It is noted that stable solutions are of physical interest and can be validated by experiment. As N_t increases, dual solutions exist in a wider domain of S. This indicates that when thermophoretic diffusivity is higher, boundary layer separation takes place for a smaller suction parameter. The parameter N_t is defined as the ratio of thermal diffusivity to viscous force. So, the value of N_t becomes higher for larger thermal diffusivity and lower viscous force. As a consequence, when N_t is higher the boundary layer separates from the sheet for smaller suction parameter.

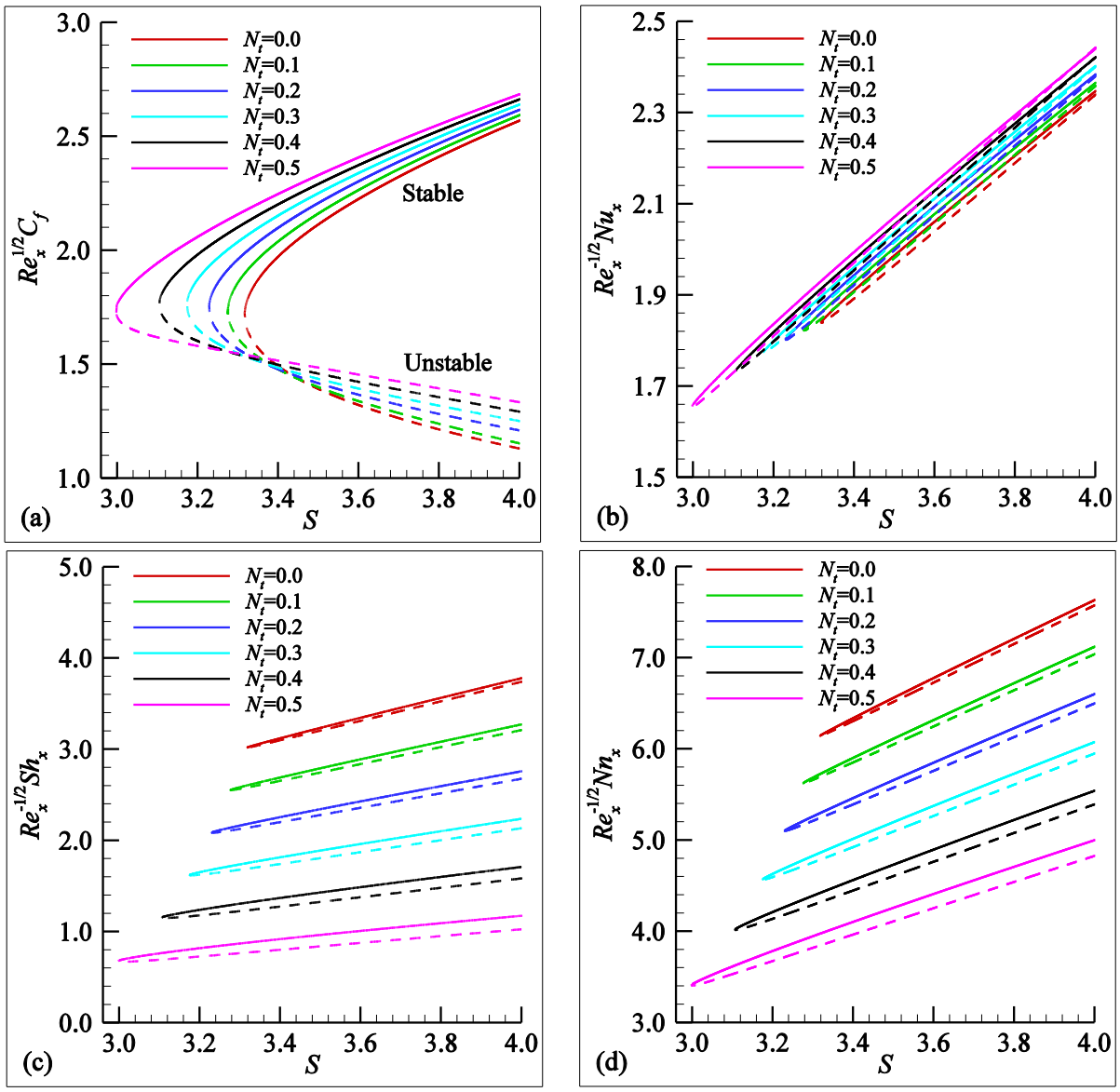

Fig. 3: Values of $Re_x^{1/2}C_f, Re_x^{-1/2}Nu_x, Re_x^{-1/2}Sh_x, and\ Re_x^{-1/2}Nn_x$ when Pr=0.7, Pe=1.0, β=0.5, Ha=0.1, D_f=0.05, Ec=0.1, We=1.0.

Figure 4 and Table 2 depict the changes of $Re_x^{1/2}C_f, Re_x^{-1/2}Nu_x, Re_x^{-1/2}Sh_x,$ and $Re_x^{-1/2}Nn_x$ with non-Newtonian Casson parameter, β. It is clear from the figures that an increase in β enhances the values of $Re_x^{-1/2}Nu_x, Re_x^{-1/2}Sh_x,$ and $Re_x^{-1/2}Nn_x,$ whereas they decrease with lower value of S. Contrary to this, when β is increased the skin friction coefficient decreases in a region for $S\geq$ 3.5 and it increases in a region for S

≤ 3.5. Moreover, the domain of dual solutions increases for higher β. As the increasing β reduces the viscous force and heat dissipation, hence it delays the boundary layer separation in terms of S.

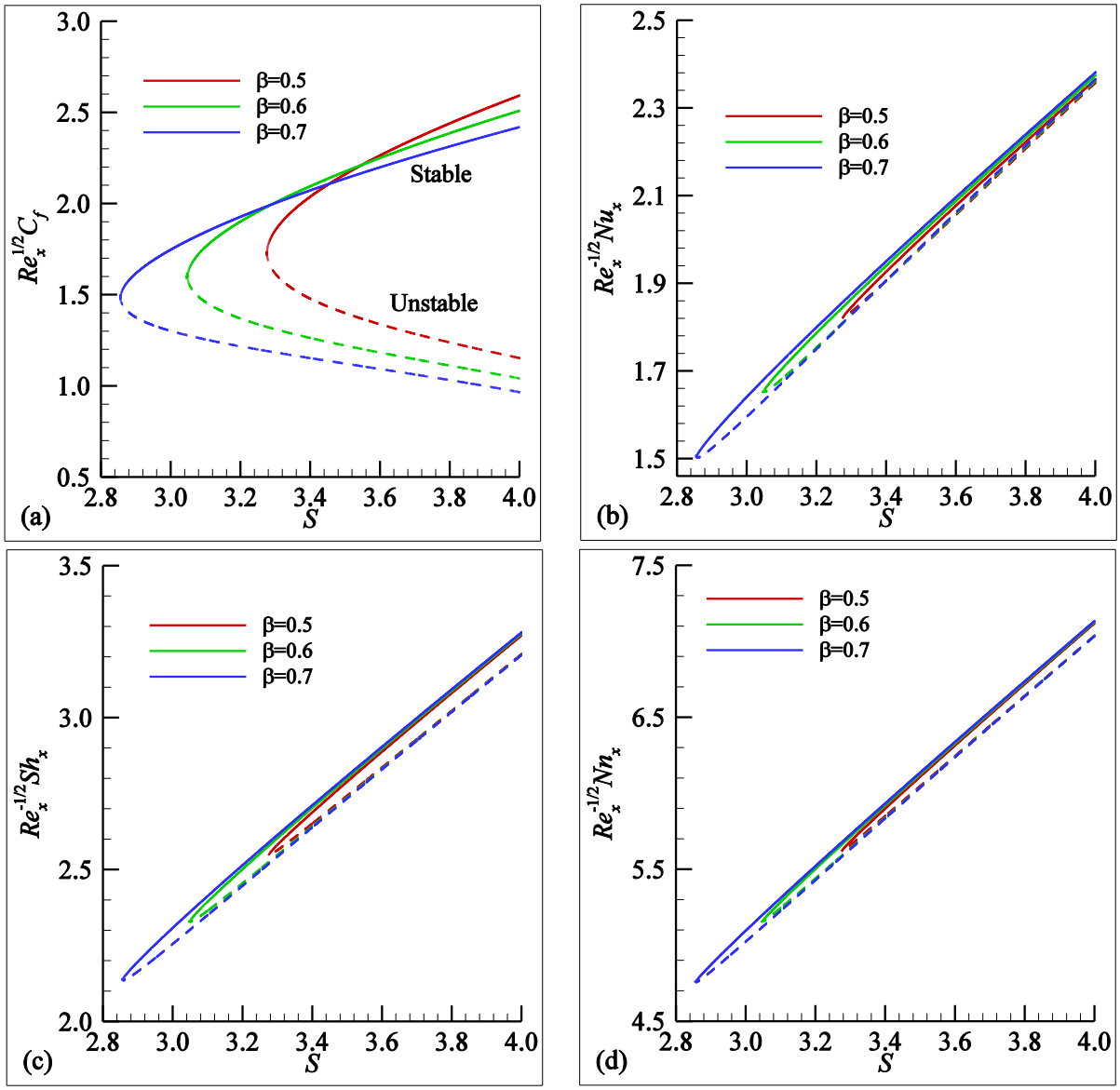

Fig. 4: Values of $Re_x^{1/2}C_f$, $Re_x^{-1/2}Nu_x$, $Re_x^{-1/2}Sh_x$, and $Re_x^{-1/2}Nn_x$ when Pr=0.7, Pe=1.0, Ha=0.1, N_t=0.1, D_f=0.05, Ec=0.1, We=1.0.

The effects of the Dufour number, D_f, on $Re_x^{1/2}C_f$, $Re_x^{-1/2}Nu_x$, $Re_x^{-1/2}Sh_x$, and $Re_x^{-1/2}Nn_x$ are elucidated in Fig. 5 and Table 2. As D_f increases, values of $Re_x^{1/2}C_f$, $Re_x^{-1/2}Sh_x$, and $Re_x^{-1/2}Nn_x$ significantly increase, however, $Re_x^{-1/2}Nu_x$ drastically decreases. Additionally, increasing D_f augments the domain of the existence of dual solutions. For any value of D_f all of the important physical quantities diminish by increasing S. From the definition, D_f measures the relative mass diffusion with the change of enthalpy. As a result, Dufour number, D_f, increases for higher mass diffusion with smaller enthalpy change. Such a phenomena augments $Re_x^{1/2}C_f$, $Re_x^{-1/2}Sh_x$, and $Re_x^{-1/2}Nn_x$ but it diminishes $Re_x^{-1/2}Nu_x$. In addition, for higher D_f the boundary layer separation takes place with smaller S.

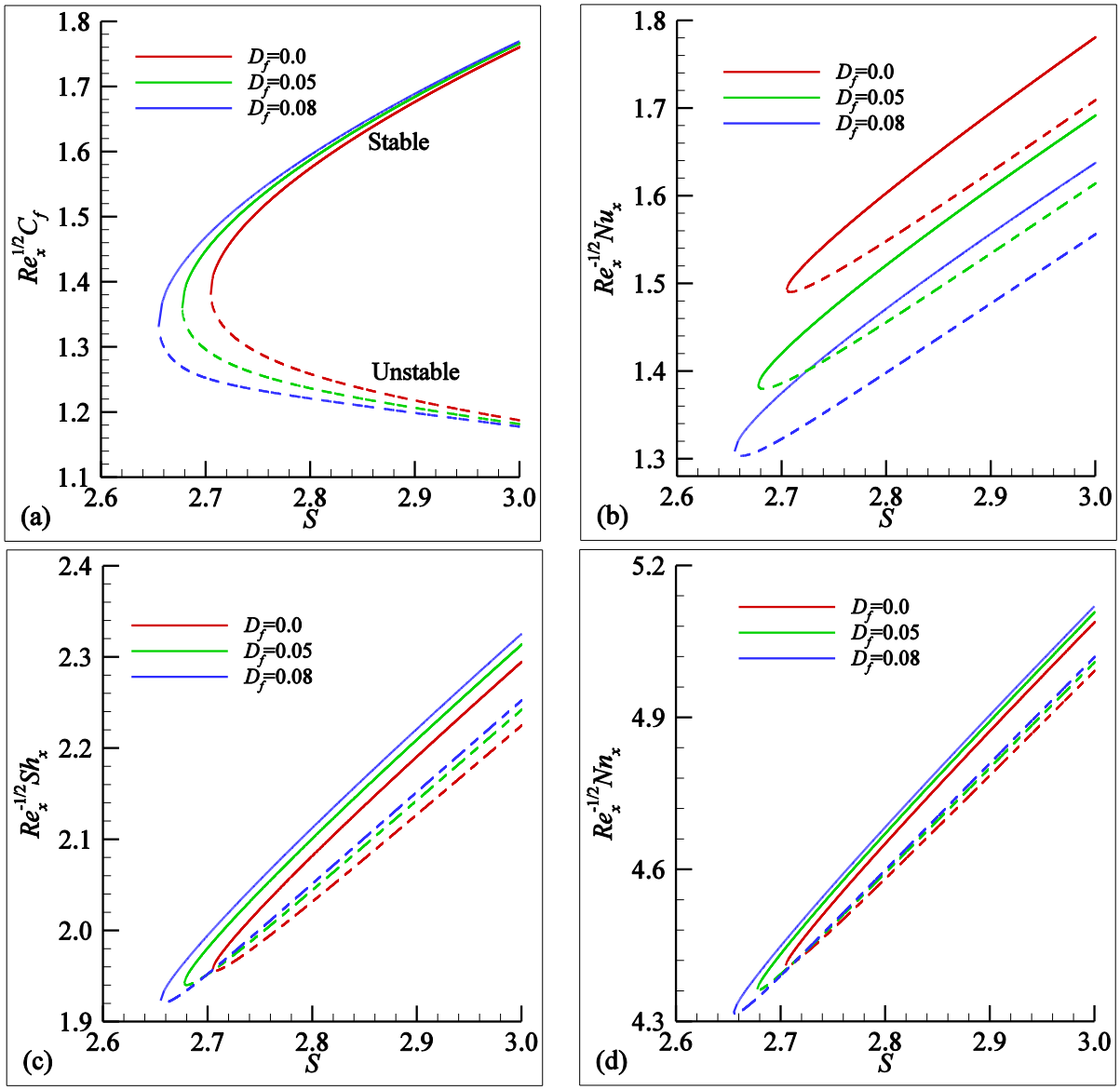

Fig. 5: Values of $Re_x^{1/2}C_f$, $Re_x^{-1/2}Nu_x$, $Re_x^{-1/2}Sh_x$, *and* $Re_x^{-1/2}Nn_x$ when Pr=0.7, *Pe*=1.0, β=0.8, *Ha*=0.1, N_t=0.1, *Ec*=0.1, *We*=1.0.

In Fig. 6 and Table 2, the influences of the Eckert number on $Re_x^{1/2}C_f$, $Re_x^{-1/2}Nu_x$, $Re_x^{-1/2}Sh_x$, and $Re_x^{-1/2}Nn_x$ are presented. Results revealed that increases, higher *Ec* leads to an increase in $Re_x^{1/2}C_f$, $Re_x^{-1/2}Sh_x$, and $Re_x^{-1/2}Nn_x$, but it diminishes the value of $Re_x^{-1/2}Nu_x$. When *S* is increased, the above quantities are found to decrease. Also, dual solutions exist over a larger domain with the increase of *Ec*. It is evident from the definition of *Ec* that when *Ec* is higher the kinetic energy is higher with respect to smaller enthalpy difference. That is why, larger *Ec* gives rise to a significant increase in $Re_x^{1/2}C_f$, $Re_x^{-1/2}Sh_x$, and $Re_x^{-1/2}Nn_x$, whereas a decrease in $Re_x^{-1/2}Nu_x$.

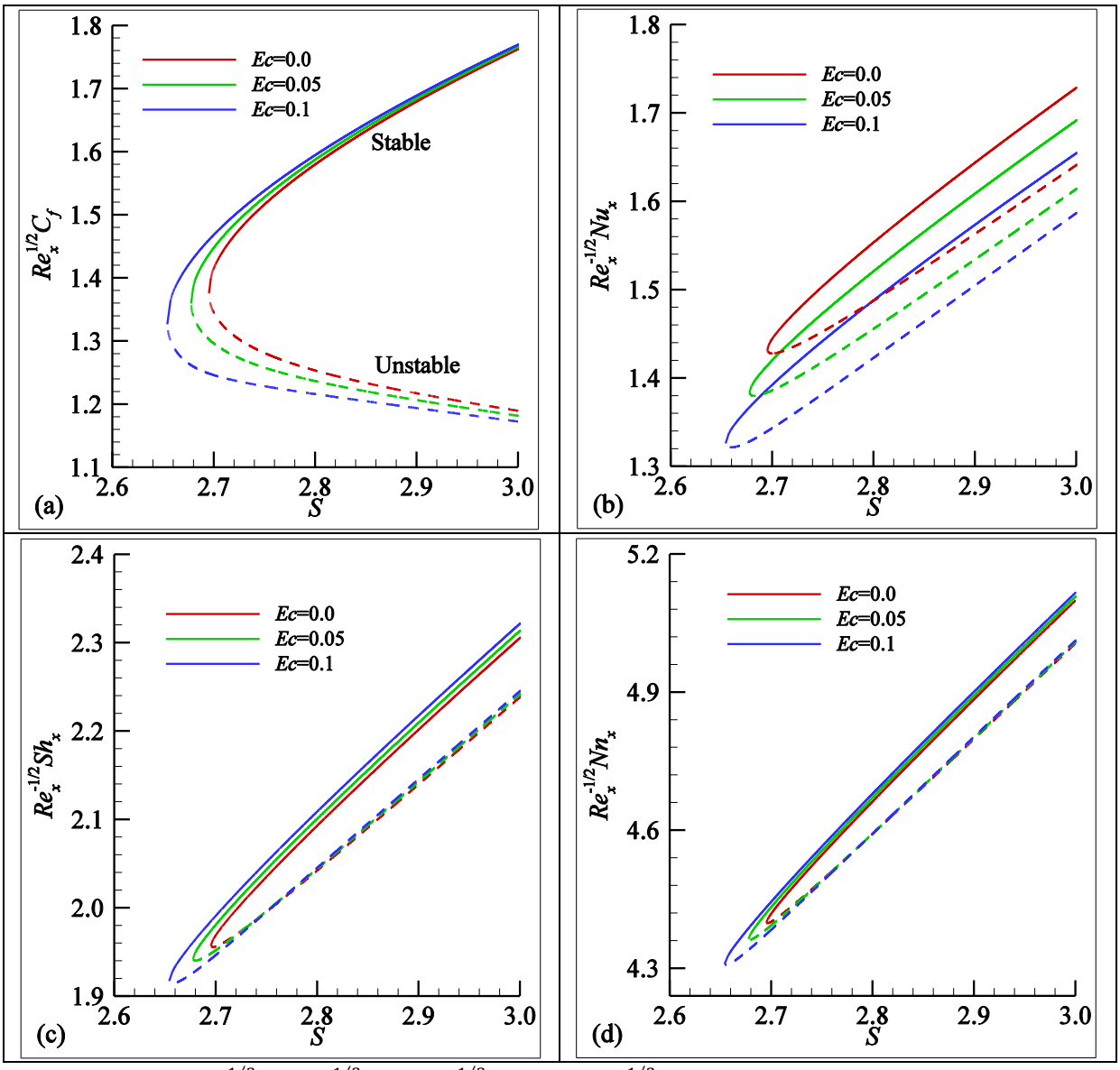

Fig. 6: Values of $Re_x^{1/2}C_f, Re_x^{-1/2}Nu_x, Re_x^{-1/2}Sh_x, and\ Re_x^{-1/2}Nn_x$ when Pr=0.7, *Pe*=1.0, β=0.8, *Ha*=0.1, N_t=0.1, D_f=0.05, *We*=1.0.

Fig. 7 and Table 2 demonstrate the effects of *We* on $Re_x^{1/2}C_f, Re_x^{-1/2}Nu_x, Re_x^{-1/2}Sh_x$, and $Re_x^{-1/2}Nn_x$. It is evident from the results that all of the quantities are found to decrease for higher *We* and decreasing of *S*. Moreover, the domain of dual solutions reduces with increasing *We*. This is due to the fact that the higher shearing rate retards the flow and energy transfer.

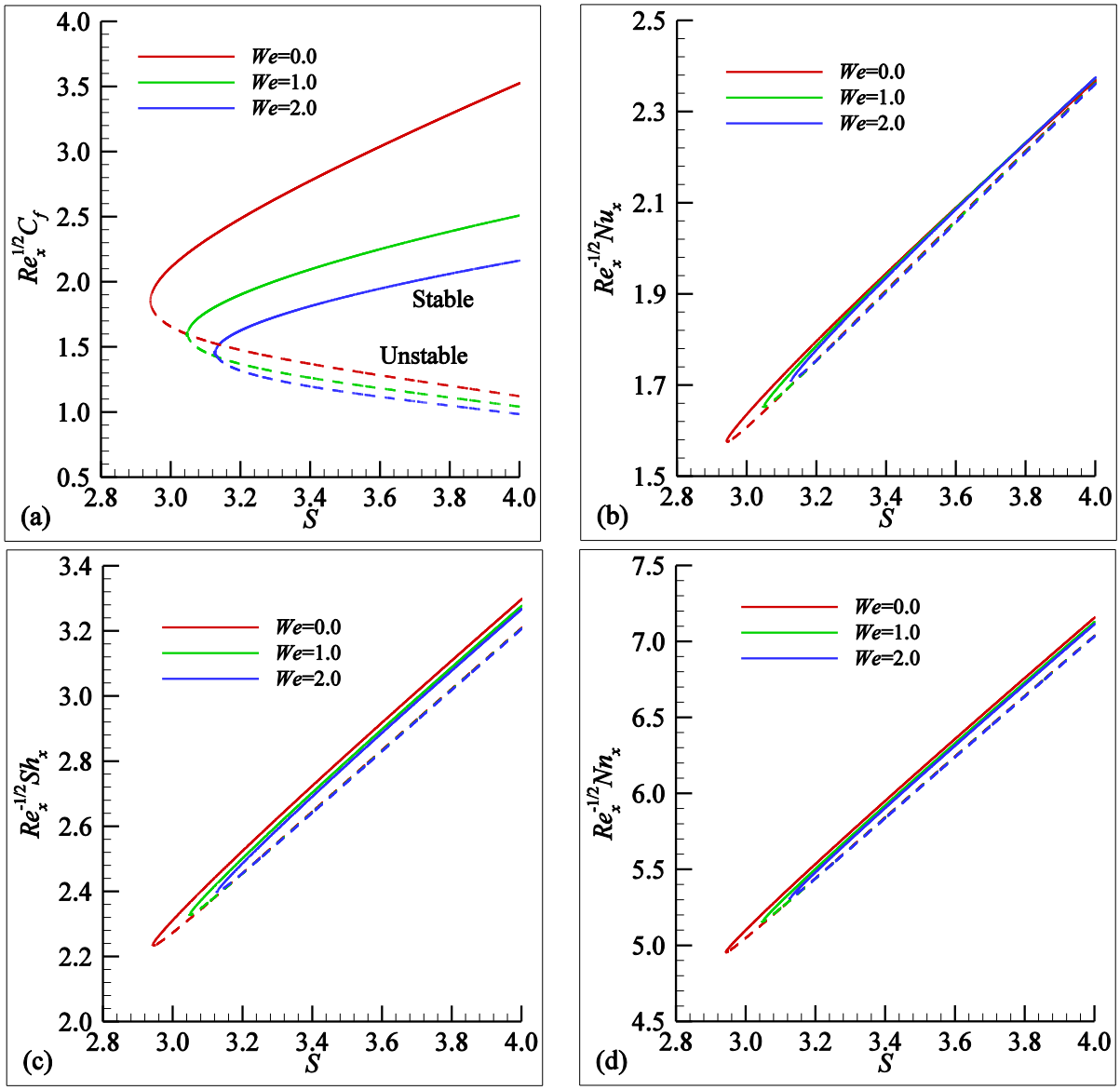

Fig. 7: Values of $Re_x^{1/2}C_f, Re_x^{-1/2}Nu_x, Re_x^{-1/2}Sh_x, and\ Re_x^{-1/2}Nn_x$ when Pr=0.7, *Pe*=1.0, β=0.6, *Ha*=0.1, N_t=0.1, D_f=0.05, *Ec*=0.1.

The changes of $Re_x^{1/2}C_f, Re_x^{-1/2}Nu_x, Re_x^{-1/2}Sh_x,$ and $Re_x^{-1/2}Nn_x$ with variations of *Ha* are shown in Fig. 8 and Table 2 which indicates that higher *Ha* augments all of these quantities but they decrease for lowering *S* irrespective of *We*. The domain of the occurrence of dual solutions increases with *Ha*. It is noted that $Ha^2 = \sigma B_0^2/(b\rho)$ becomes large for higher magnetic strength and lower fluid density which accelerate the flow velocity. As a consequence, when stronger magnetic field is applied, it causes the velocity gradients at the surface and so the local skin friction coefficient is found to be higher.

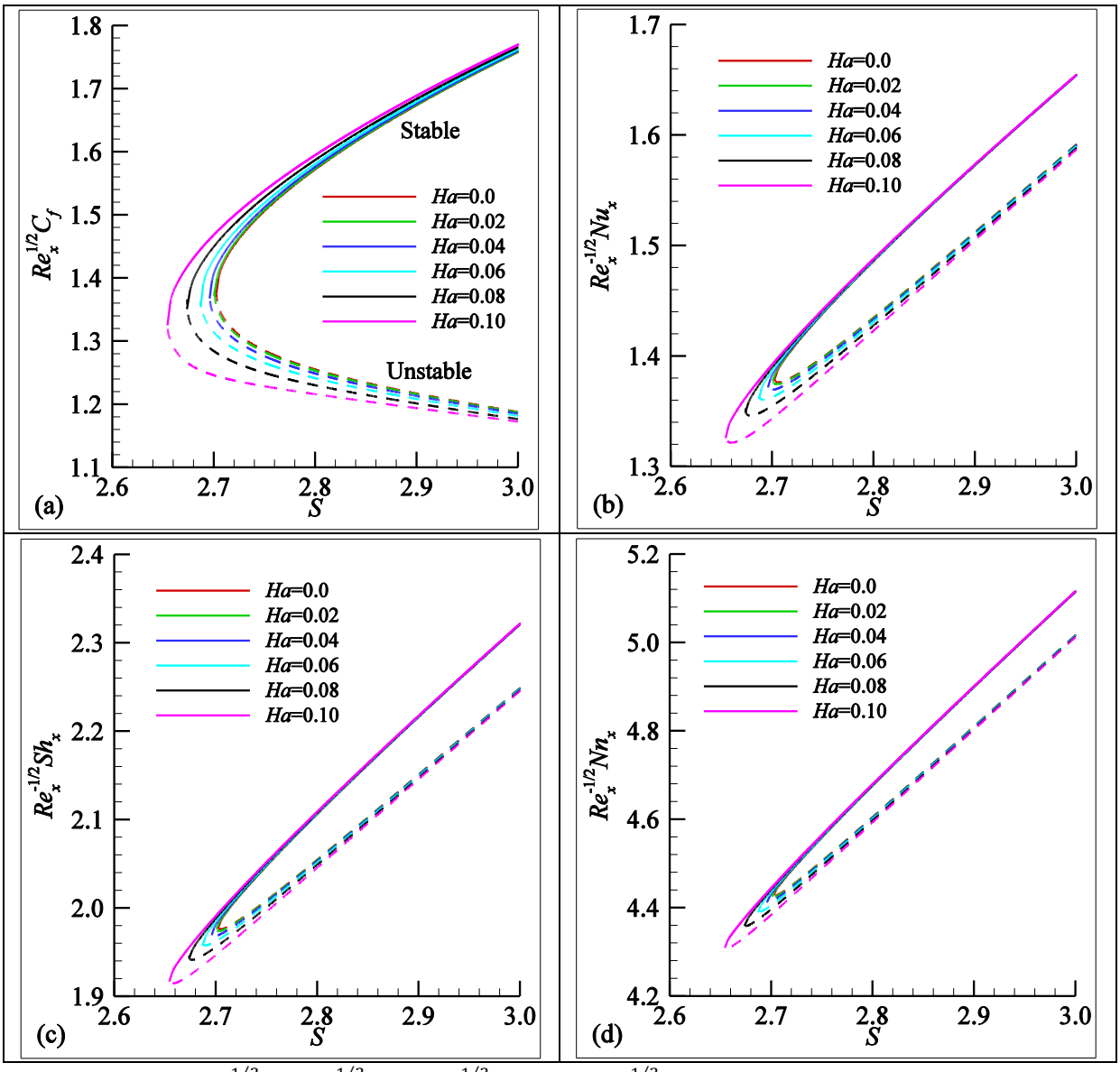

Fig. 8: Values of $Re_x^{1/2}C_f, Re_x^{-1/2}Nu_x, Re_x^{-1/2}Sh_x, and\ Re_x^{-1/2}Nn_x$ when Pr=0.7, Pe=1.0, β=0.8, N_t=0.1, D_f=0.05, Ec=0.1, We=1.0.

Table 2: Variation of $Re_x^{1/2}C_f, Re_x^{-1/2}Nu_x, Re_x^{-1/2}Sh_x, and\ Re_x^{-1/2}Nn_x$ when Pr=0.7, Pe=1.0.

β	Ha	N_t	D_f	Ec	We	S	$Re_x^{1/2}C_f$	$Re_x^{-1/2}Nu_x$	$Re_x^{-1/2}Sh_x$	$Re_x^{-1/2}Nn_x$
0.5	0.10	0.1	0.05	0.10	1.0	3.5	2.16136	2.00313	2.78993	6.11067
0.6							2.17587	2.01507	2.80165	6.12740
0.7							2.13739	2.02403	2.80880	6.13781
0.8	0.0					2.8	1.57223	1.48588	2.10620	4.67591
	0.05						1.57819	1.48645	2.10693	4.67694
	0.10						1.59482	1.48798	2.10895	4.67977
		0.0				3.5	2.11357	1.98648	3.23283	6.55916
		0.2					2.20741	2.01983	2.34156	5.65640
		0.5					2.32940	2.07078	0.96198	4.25690
		0.1	0.0			2.8	1.57479	1.60315	2.08204	4.65143
			0.05				1.58762	1.52097	2.10096	4.67127
			0.08				1.59462	1.47102	2.11233	4.68316
			0.05	0.0		2.8	1.58011	1.55355	2.09299	4.66255

0.05			1.58762	1.52097	2.10096	4.67127
0.10			1.59482	1.48798	2.10895	4.67977
	0.0	3.4	1.36932	1.90841	2.64538	5.84447
	1.0		2.09528	1.94075	2.70372	5.92268
	2.0		1.81304	1.93612	2.69175	5.90579

5. Conclusion

In this study, the flow of MHD Casson or Carreau fluids with microorganisms over a permeable shrinking surface is considered. By solving the dimensionless governing equations by shooting method, we investigate the effects of non-Newtonian Casson parameter, Eckert number, Dufour number, thermophoretic diffusivity parameter, Weissenberg number and Hartmann number on the local skin friction coefficient, local Nusselt number, local Sherwood number, and local density number of the microorganisms. Results are concluded as follows:

(i) An increase in non-Newtonian Casson parameter, Dufour number, thermophoretic diffusivity parameter and Hartmann number causes a significant increase in local skin friction coefficient, local Nusselt number, local Sherwood number, and local density number of the microorganisms.

(ii) Higher Weissenberg number diminishes all of the above quantities.

(iii) However, local skin friction coefficient, local Nusselt number, local Sherwood number, and local density number of the microorganisms are substantially decreased by increasing suction parameter whatever the values of other parameters.

(iv) Dual solutions are observed over a wider region for higher values of non-Newtonian Casson parameter, Dufour number, thermophoretic diffusivity parameter, Eckert number and Hartmann number. On the contrary, the existence of dual solutions shrinks by larger Weissenberg number.

The present study can be extended for unsteady flow and stagnation point flow over a stretching/shrinking sheet.

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