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Short Communication

Fejér and Dirichlet Kernels: Their Associated Polynomials

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Abstract

We show that the Fejér kernel generates the fifth-kind Chebyshev polynomials.

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1. Introduction

In the original approach to Fourier series, it is convenient to consider the following partial sums for the interval $[-\pi, \pi]$:

$$f_n(y) = \tfrac{1}{2} a_0 + a_1 \cos y + \cdots + a_n \cos(ny) + + b_1 \sin(y) + \cdots + b_n \sin(ny) \tag{1}$$

assuming for a_r, b_r the values:

$$a_m = \tfrac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(mt) dt, \quad \mathrm{b}_r = \tfrac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(rt) dt \tag{2}$$

We investigate what happens if n increases to infinity. From (1) and (2) we obtain:

$$f_n(y) = \int_{-\pi}^{\pi} f(t) \underset{D}{K_n}(t-y) dt \tag{3}$$

With the Dirichlet kernel [1-3]:

$$\underset{D}{K_n}(t-y) = \frac{1}{2\pi} \frac{\sin\left[\left(n+\frac{1}{2}\right)(t-y)\right]}{\sin\left(\dfrac{t-y}{2}\right)} \tag{4}$$

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Then we hope that with *n* increasing to infinity, $f_n(y)$ approaches $f(y)$ with an error which can be made arbitrarily small. This requires a very strong focusing power of $\underset{D}{K_n}(t-y)$, that is, we would like to have the strict property:

$$\lim_{n\to\infty} \underset{D}{K_n}(t-y) = \delta(t-y) \tag{5}$$

However, Eq. (4) simulates a Dirac delta only until certain approximation, then the convergence:

$$\lim_{n\to\infty} f_n(y) = f(y) \tag{6}$$

has to be restricted to a definite class of functions $f(y)$ which are conveniently smooth to counteract the insufficient focusing power of $\underset{D}{K_n}(t-y)$; the corresponding restrictions on $f(y)$ are the known Dirichlet conditions [1-3] for infinite convergent Fourier series.

From Eq. (4) we see that $\underset{D}{K_n}(\theta)$ is an even function. Here we consider it for $\theta \in [0,\pi]$:

$$\underset{D}{K_n}(\theta) = \frac{1}{2\pi}\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)} \tag{7}$$

thus

$$\begin{aligned} &\underset{D}{K_0}(\theta) = \frac{1}{2\pi}, \quad \underset{D}{K_1}(\theta) = \frac{1}{2\pi}(1+2\cos\theta), \quad \underset{D}{K_2}(\theta) = \frac{1}{2\pi}(-1+2\cos\theta+4\cos^2\theta), \\ &\underset{D}{K_3}(\theta) = \frac{1}{2\pi}(-1-4\cos\theta+4\cos^2\theta+8\cos^3\theta), \quad \text{etc.} \end{aligned} \tag{8}$$

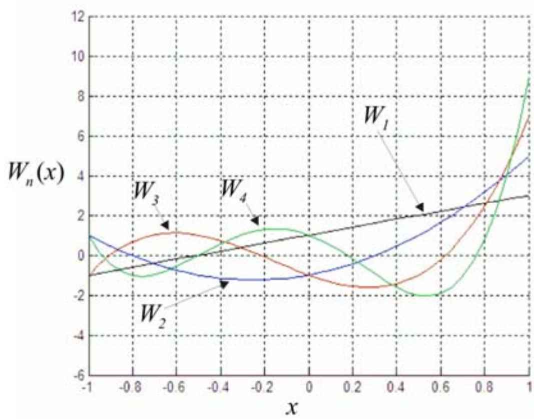

Fig. 1. Some fourth-kind Chebyshev polynomials.

It is then natural to introduce the polynomials:

$$W_n(x) = W_n(\cos\theta) = 2\pi \underset{D}{K_n}(\theta), \quad x \in [-1,1] \tag{9}$$

which were named "fourth-kind Chebyshev polynomials" by Gautschi [4,5]. We thus have:

$$\begin{aligned} &W_0(x) = 1, \quad W_1(x) = 2x+1, \quad W_2(x) = 4x^2 + 2x - 1, \\ &W_3(x) = 8x^3 + 4x^2 - 4x - 1, \quad W_4(x) = 16x^4 + 8x^3 - 12x^2 - 4x + 1, \quad \text{etc.} \end{aligned} \tag{10}$$

These are shown in Fig. 1. In the next section we exhibit a set of associated polynomials to Fejér kernel [1-3].

2. Chebyshev-Fejér polynomials

Fejér [5] invented a new method of summing the Fourier series by which he greatly extended the validity of the series. Using the arithmetic means of the partial sums (Eq. 1), instead of the $f_n(y)$ themselves, he could sum series which were divergent. The only condition the function still has to satisfy is the natural restriction that $f(y)$ shall be absolutely integrable.

Then, in the Fejér approach we construct the sequence:

$$\begin{aligned} &g_1(y) = f_0(y), \quad g_2(y) = \tfrac{1}{2}[(f_0(y) + f_1(y)], \quad g_3(y) = \tfrac{1}{3}[(f_0(y) + f_1(y) + f_2(y)], \ldots, \\ &g_n(y) = \tfrac{1}{n}[(f_0(y) + f_1(y) + \cdots + f_{n-1}(y)] \end{aligned} \tag{11}$$

Accepting the expressions (1) and (2), therefore:

$$g_n(y) = \int_{-\pi}^{\pi} f(t) \underset{F}{K_n}(t-y)dt \tag{12}$$

We thus see that Fejér results come about by the fact that his method is related with the following kernel [1-3]:

$$\underset{F}{K_n}(t-y) = \frac{1}{2\pi n} \frac{\sin^2\left[\frac{n}{2}(t-y)\right]}{\sin^2 \dfrac{t-y}{2}} \tag{13}$$

This possesses a strong focusing power, that is, it satisfies (5), then a $f(y)$ absolutely integrable in $[-\pi, \pi]$ guarantees the convergence of $g_n(y)$ towards $f(y)$.

Now we consider the Fejér kernel:

$$\underset{F}{K_n}(\theta) = \frac{1}{2\pi n} \frac{\sin^2\left(n\dfrac{\theta}{2}\right)}{\sin^2 \dfrac{\theta}{2}}, \quad \theta \in [0, \pi] \tag{14}$$

that is:

$$\begin{aligned} &\underset{F}{K_0}(\theta) = 0, \quad \underset{F}{K_1}(\theta) = \frac{1}{2\pi}, \quad \underset{F}{K_2}(\theta) = \frac{1}{2\pi}(1+\cos\theta), \\ &\underset{F}{K_3}(\theta) = \frac{1}{6\pi}(1 + 4\cos\theta + 4\cos^2\theta), \quad \text{etc.} \end{aligned} \tag{15}$$

Then it is natural to introduce the functions:

$$\tilde{W}_n(x) = \tilde{W}_n(\cos\theta) = \frac{2\pi}{n+1} K_{\substack{n+1\\F}}(\theta), \quad x \in [-1,1] \tag{16}$$

We name these "fifth-kind Chebyshev polynomials", which are not explicitly in the literature. Therefore:

$$\begin{aligned} &\tilde{W}_0(x) = 1, \quad \tilde{W}_1(x) = \frac{1}{2}(x+1), \quad \tilde{W}_2(x) = \frac{1}{9}\left(4x^2 + 4x + 1\right), \\ &W_3(x) = \frac{1}{2}\left(x^3 + x^2\right), \quad \tilde{W}_4(x) = \frac{1}{25}\left(16x^4 + 16x^3 - 4x^2 - 4x + 1\right), \ \textit{etc.} \end{aligned} \tag{17}$$

Thus $\tilde{W}_n(1) = 1$, and so on. We plot these in Fig. 2.

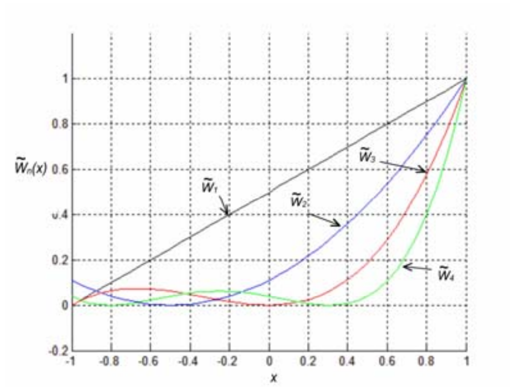

Fig. 2. Some fifth-kind Chebyshev polynomials.

Eqs. (17) are the solutions of the non-homogeneous differential equation:

$$(1-x)\left[(1-x^2)\tilde{W}_n'' - (3x+2)\tilde{W}_n' + (n+1)^2\tilde{W}_n\right] + x\tilde{W}_n = 1. \tag{18}$$

In a forthcoming paper we will consider topics such as recurrence, Rodrigues formula, interpolation properties, orthonormality, generating function, and so on for fifth-kind Chebyshev polynomials introduced in this work.

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