JSR Publications

Available Online

J. Sci. Res. **1** (3), 558-562 (2009)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

Short Communication

Study of Convex Sublattices of a Lattice by a New Approach

R. M. H. Rahman

Department of Mathematics, Dinajpur Govt. College, Dinajpur, Bangladesh

Received 12 May 2009, accepted in final revised form 12 July 2009

Abstract

The set of all convex sublattices CS(L) of a lattice L have been studied by a new approach. Introducing a new partial ordering relation " ≤ " it is shown that CS(L) is a lattice. Moreover L and CS(L) are in the same equational class. A number of properties of (CS(L); ≤) has also been included.

Keywords: Convex sublattices; Standard element; Neutral element; Congruence.

DOI: 10.3329/jsr.v1i3.2484 J. Sci. Res. **1** (3), 558-562 (2009)

Convex sublattices of a lattice have been studied by many authors including Koh [1-2]. Set of all convex sublattices of a lattice L is denoted by CS (L). By K. M. Koh [2] CS (L) with the empty set is a lattice. On the other hand standard convex sublattices of a lattice L have been studied by Fried and Schmidt [3]. Recently Lavanya and Bhatta [4] have introduced a new partial ordering relation on CS(L), under which CS(L) is a lattice. Moreover L and CS(L) are in the same equational class. On CS(L), they defined the partial order " ≤ " as follows:

For A, B ∈ CS(L), A ≤ B if and only if "for every a ∈ A there exists a b ∈ B, such that a ≤ b and for every b∈ B there exists an a ∈ A, Such that b ≥ a." It is easy to see that '≤' is clearly a partial order and (CS(L); ≤) forms a lattice, where for A, B ∈ CS(L),

$$\text{Inf}\,\{A, B\} = A \mathbin{\dot\wedge} B = \langle\{a\wedge b \mid a\in A,\ b\in B\}\rangle = \{x\in L \mid a\wedge b\le x\le a_1\wedge b_1 \text{ for some } a, a_1\in A \text{ and } b, b_1\in B\}$$

$$\text{Sup}\,\{A, B\} = A \mathbin{\dot\vee} B = \langle\{a\vee b \mid a\in A,\ b\in B\}\rangle = \{x\in L \mid a\vee b\le x\le a_1\vee b_1 \text{ for some } a, a_1\in A \text{ and } b, b_1\in B\}$$

and for any non-empty subset H of L, $\langle H \rangle$ denotes the convex sublattice generated by H. Note that $A \dot{\wedge} B$ and $A \dot{\vee} B$ have also been studied by J. Nieminen [5],where the author studied the distributive and neutral sublattices.

In this paper we studied the structure of CS(L) with this new approach and then include some properties of (CS(L); ≤). We have also given a nice characterization of a standard element of CS(L).

We start with the construction of (CS(L); ≤) of a lattice L of Fig. 1.

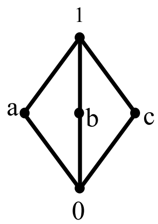

Fig. 1.

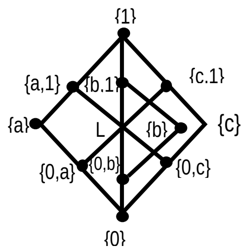

Fig. 2.

It is easy to check that Fig. 2 represents the lattice " (CS(L); ≤)**"**. Now we include some properties of " (CS(L); ≤)". We know that for any congruence of a lattice L, each congruence class is an element of CS(L). We have the following results:

Theorem 1. *For any Congruence* Θ *of a lattice* L, $[a]\Theta \le [b]\Theta$ in $\frac{L}{\Theta}$ *if and only if* $[a]\Theta \le [b]\Theta$ *in* CS(L*). In other words, the quotient lattice* $\frac{L}{\Theta}$ *is a subposet of* (CS(L);≤) *but* $\frac{L}{\Theta}$ *is not necessarily a sublattice of* CS(L).

Proof: Suppose$[a]\Theta \le [b]\Theta$ in $\frac{L}{\Theta}$, let $s \in [a]\Theta$ then $[s]\Theta = [a]\Theta \le [b]\Theta$ in $\frac{L}{\Theta}$ Thus $[b]\Theta = [b]\Theta \vee [s]\Theta = [b \vee s]\Theta$, this implies that $b \vee s \in [b]\Theta$ and $s \le b \vee s$. On the other hand, let $t \in [b]\Theta$. Then $[a]\Theta \le [b]\Theta = [t]\Theta$ in $\frac{L}{\Theta}$. Thus $[a]\Theta = [a]\Theta \wedge [t]\Theta = [a \wedge t]\Theta$, which implies that $a \wedge t \in [a]\Theta$ and $t \ge a \wedge t$. Therefore, by the definition of '≤' in CS(L), $[a]\Theta \le [b]\Theta$ in CS(L).

Conversely, let $[a]\Theta \le [b]\Theta$ in CS(L). Since $a \in [a]\Theta$ m so there exists $t \in [b]\Theta$ such that $a \le t$. Then $a = a \wedge t \equiv (a \wedge b)\Theta$ and so $[a]\Theta = [a \wedge b]\Theta = [a]\Theta \wedge [b]\Theta$ in $\frac{L}{\Theta}$ This implies $[a]\Theta \le [b]\Theta$ in $\frac{L}{\Theta}$.

To prove the last part, consider the following lattice L in Fig. 3.

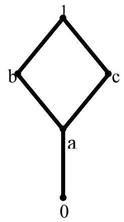

Fig. 3.

Consider the congruence $\Theta=\{0,a\}, \{b\}, \{c\}, \{1\}$, In $\frac{L}{\Theta}$, $[b]\Theta \wedge [c]\Theta = [b \wedge c]\Theta = [a]\Theta = \{0,a\}$. But in CS(L), $[b]\Theta \wedge [c]\Theta = \{a\}$. Therefore $\frac{L}{\Theta}$ is not a sublattice of CS(L). ∎

Theorem 2. *For any* A, B ∈ CS(L), A≤B *if and only if* (A]⊆(B] *and* [A)⊇ [B).

Proof: Suppose A ≤B, let a ∈(A], then a≤ a_1 for some a_1∈A. Since A≤B, so there exists a b_1 ∈ B such that a ≤ b_1 and so a∈(B]. Hence (A] ≤ (B]. Now let b∈[B), then b ≥ b_1 for some b_1 ∈B. Since A ≤ B, so there exists a_1∈A such that b_1≥ a_1. Thus b ≥ a_1 which implies that b∈[A). Hence [A) ⊇ [B).

Conversely, suppose (A] ⊆ [B) and [A) ⊇ [B). Let a∈A, then a∈(A] ⊆ (B]. This implies that a ≤ b for some b∈B. Again for any b∈B, b∈ [B) ⊆ [A) and so b ≥ a for some a∈A. Hence by definition, A ≤ B in CS(L). ∎

For a lattice L, I(L) and D(L) are Lattice of ideals and dual ideals respectively. From the above theorem, we have the following corollary.

Corollary 3. For I, J∈I(L), I ≤ J if and only if I ⊆ J and for D, K∈ D(L), D ≤ K if and only if D ⊇ K. ∎

Theorem 4. *For any lattice* L, I(L*) is a principal ideal generated by* L *in* CS(L) *and* D(L) *is a principal dual ideal generated by* L *in* CS(L).

Proof: By Corollary 3, I(L) is a sublattice of CS(L) with L as its largest element. Now let I∈ I(L) and A ∈ CS(L) with A ≤ I. We need to show that A has the hereditary property. Suppose, x ∈ A and y ≤ x. Since x ∈ A and A ≤ I, so by definition there exists i ∈ I, such that x ≤ i. Since I is an ideal, so y ≤ x ≤ i implies that y∈I. Now A ≤ I implies that there exists an element z∈A, such that y ≥ z. Then z ≤ y ≤ x and so by convexity y∈A. Hence A has the hereditary property and thus A is an ideal, that is, A∈ I(L). Therefore I(L) is an ideal of CS(L) with L as its largest element and so it is a principal ideal generated by L. Similarly, we can show that D(L) is a principal dual ideal generated by L in CS(L). ∎

Observe that in Fig. 2, both I(L) and D(L) are principal ideal and principal dual ideal respectively, in CS(L) generated by L.

Since I(L) is a sub lattice of CS(L), we have the following result.

Theorem 5. *The mapping* f: L → CS(L) *defined by* f(a)= (a] *is an embedding. Moreover, an element a is join irreducible in* L *if and only if* f(a) *is join irreducible in* CS(L).

Proof: The mapping f is obviously an embedding of L into CS(L). Now suppose a is join irreducible in L. Let for A, B ∈ CS(L), A $\dot{\vee}$B= f(a)=(a], implies A ≤ (a] and B ≤ (a] in CS(L).Then each x∈A implies x ≤ a, so x∈(a] and hence A⊆ (a]. Similarly B⊆(a]. Since a∈ A$\dot{\vee}$B, so by definition $a_1 \vee b_1 \leq a \leq a_2 \vee b_2$ for some a_1,a_2 ∈A and b_1,b_2 ∈B. Now A, B ⊆ (a] so $a_2,b_2 \leq a$, thus $a = a_2 \vee b_2$.Since a is join irreducible so either $a_2 = a$ or b_2=a. Without loss of generality, suppose a =a_2, then a∈ A. Now we prove that A = (a]. If not, then there exist an element t ∈(a] such that t∉A. Since t∈ (a] =A$\dot{\vee}$B, so there exist p_1,p_2∈A; q_1,q_2 ∈ B, such that $p_1 \vee q_1 \leq t \leq p_2 \vee q_2$ this implies $p_1 \leq t \leq a$ and so by convexity t∈ A, which is a contradiction. Therefore A= (a]. Similarly, by considering a =b_2 we can show that B= (a], therefore f(a) =(a] is join irreducible in CS(L).

Conversely, suppose f(a) is join irreducible in CS(L). Let a=b∨c in L, then (a]=(b]∨(c] =(b] $\dot{\vee}$ (c] in CS(L). Since f(a) =(a] is join irreducible in CS(L), so either (b]= (a] or (c]= (a], that is, either b=a or c=a. Therefore a is join irreducible in L. ∎

Since D(L) is also a sub lattice of CS(L) a dual proof of above gives the following result.

Theorem 6. *The mapping* f : L → CS(L) *defined by* f(a)= [a) *is an embedding. Moreover, an element a is meet irreducible in* L *if and only if* f(a) *is meet irreducible in* CS(L). ∎

The following theorem is due to S. Lavanya and S. P. Bhatta [4]This gives a clear idea on the structure of (CS(L);≤).

Theorem 7. *For any lattice L the map* f: CS(L) → I(L)×D(L) *defined by for any* X∈ CS(L), f(x)=((X],[X)) *is an imbedding. In fact,* CS(L) *is isomorphic to the sublattice* {(I,D)| I ∈ I(L), D ∈D (L), I ∩D ≠ ϕ} of I(L)×D(L) ∎

We know from Grätzer [6] that the identities of lattices are preserved under the function of sublattices, homomorphic images, direct products, ideal lattices and dual ideal lattices. Also it is easily seen that L can be embedded in CS(L). Therefore, by above theorem we have the following result, which is also mentioned by Lavanya and Bhatta [4].

Corollary 8. CS(L) *satisfies all the identities satisfied by L and conversely* ∎

Thus in particular, a lattice L is distributive (modular) if and only if CS(L) is distributive (modular.)

According to Grätzer [6] an element n of a lattice L is called a standard element if for all x, y ∈ L, x ∧ (y ∨ n)= (x ∧ y) ∨ (x ∧ n) Element n is called a neutral element if

(i) n is standard, and

(ii) $n \wedge (x \vee y) = (n \wedge x) \vee (n \wedge y)$ for all $x, y \in L$.

Since L is the largest element and the smallest element of (I(L); $\subseteq$) and (D(L); $\supseteq$) respectively, so it is a neutral element of both I (L) and D(L). Therefore, by Theorem 7, we have the following result.

Corollary 9. *L is a neutral element of* CS(L) ∎

We conclude the paper with the following characterization of standard elements of CS(L)

Theorem 10. *For a lattice L, a convex sublattice* S *is a standard element of* CS(L) *if and only if for any* $a, b \in L$; $\{a\} \dot{\wedge} (S \dot{\vee} \{b\} = (\{a\} \dot{\wedge} S) \dot{\vee} (\{a\} \dot{\wedge} \{b\})$.

Proof: Suppose, S is standard in (CS(L); $\le$). Then of course the given condition holds. Conversely, suppose the given condition holds for any $a, b \in S$. We have to show that

$A \dot{\wedge} (S \dot{\vee} B) = (A \dot{\wedge} S) \dot{\vee} (A \dot{\wedge} B)$ for any $A, B \in$ CS(L). Since (CS(L); $\dot{\wedge}, \dot{\vee}$) is a lattice, so clearly $(A \dot{\wedge} S) \dot{\vee} (A \dot{\wedge} B) \le A \dot{\wedge} (S \dot{\vee} B)$. For the reverse inequality, let $x \in A \dot{\wedge} (S \dot{\vee} B)$. Then $x \le a_1 \wedge t_1$ for some $a_1 \in A$ and 7$t_1 \in S \dot{\vee} B$. Now $t_1 \in S \dot{\vee} B$ implies that $t_1 \le s_1 \vee b_1$ for some $s_1 \in S$ and $b_1 \in B$. Then $x \le a_1 \wedge (s_1 \vee b_1) = y$ (say). But $y = a_1 \wedge (s_1 \vee b_1) \in \{a_1\} \dot{\wedge} (S \dot{\vee} \{b_1\}) = (\{a_1\} \dot{\wedge} S) \dot{\vee} (\{a_1\} \dot{\wedge} \{b_1\})$ (using the given condition) $\subseteq (A \dot{\wedge} S) \dot{\vee} (A \dot{\wedge} B)$. In other words, there exists an element $y \in (A \dot{\wedge} S) \dot{\vee} (A \dot{\wedge} B)$ with $x \le y$. Now let $p \in (A \dot{\wedge} S) \dot{\vee} (A \dot{\wedge} B)$. Then $p \ge c_1 \vee d_1$ for some $c_1 \in A \dot{\wedge} S$ and $d_1 \in A \dot{\wedge} B$. Now $c_1 \in A \dot{\wedge} S$ implies $c_1 \ge a_2 \wedge s_2$ and $d_1 \in A \dot{\wedge} B$ implies $d_1 \ge a_3 \wedge b_3$ for some $a_2, a_3 \in A$, $s_2 \in S$ and $b_3 \in B$. Thus, $p \ge (a_2 \wedge a_3 \wedge s_3) \vee (a_2 \wedge a_3 \wedge b_3) \in (a' \wedge s_3) \vee (a' \wedge b_3)$ where $a' = a_2 \wedge a_3$. But $(a' \wedge s_3) \vee (a' \wedge b_3) \in (\{a'\} \dot{\wedge} S) \dot{\vee} (\{a'\} \dot{\wedge} B) = \{a'\} \dot{\wedge} (S \dot{\vee} B)$ (by the given condition) $\subseteq A \dot{\wedge} (S \dot{\vee} B)$.That is, for $p \in (A \dot{\wedge} S) \dot{\vee} (A \dot{\wedge} B)$, there exists $q = (a' \wedge s_3) \vee (a' \wedge b_3) \in A \dot{\wedge} (S \dot{\vee} B)$ with $p \ge q$.

Therefore, $A \dot{\wedge} (S \dot{\vee} B) \le (A \dot{\wedge} S) \dot{\vee} (A \dot{\wedge} B)$ and so $A \dot{\wedge} (S \dot{\vee} B) = (A \dot{\wedge} S) \dot{\vee} (A \dot{\wedge} B)$

∎

References

1. K. M. Koh, Nanta Math. **6** (1), 55 (1973).
2. K. M Koh, Nanta Math. **5**, 18 (1972).
3. E. Fried and E. T. Schmidt, Algebra Universalis 5, 203 (1975). doi:10.1007/BF02485254
4. S. Lavanya and S. P. Bhatta, Algebra Universalis **35**, 63 (1996). doi:10.1007/BF01190969
5. J. Nieminen, Commentari Mathematical Universalis Stancti Paulie **33** (1), 87 (1984).
6. G. Grätzer, General lattice theory (Birkhäuser Verlag, Basel, 1978).