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Quasi-total Graphs with Crossing Numbers

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Abstract

We establish here necessary and sufficient conditions for quasi-total graphs to have crossing numbers k (k = 1, 2 or 3).

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1. Introduction

All graphs considered here are finite, undirected and without loops or multiple lines. We use the terminology of Harary [1]. A graph is planar if it can be drawn on the plane in such a way that no two of its lines intersect. The crossing number $Cr(G)$ of a graph G is the minimum number of pair wise intersections of its lines when G is drawn in the plane. Obviously, $Cr(G)=0$ if and only if G is planar. A graph G has crossing number 1, if $Cr(G)=1$.

The quasi-total graph $P(G)$ of a graph G is the graph whose point set is $V(G)$ U $X(G)$ and two points are adjacent if and only if they correspond to two non adjacent points of G or to two adjacent lines of G or one is a point and other is a line incident with it in G. This concept was introduced in [2].

The following will be useful in the proof of our results.

Remark **1** [3]**.** For any connected graph G, the middle graph $M(G)$ is a spanning subgraph of $P(G)$.

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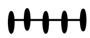
Theorem A [4]**.** The quasi-total graph $P(G)$ of a graph G is planar if and only if G is of order$\leq$ 4.
Theorem B [5]**.** If a graph G has at least one non-cutpoint of degree 4, then Cr(M(G))$\geq$3.
Theorem C [5]**.** Every non-planar graph has a middle graph with crossing number at least 8.

2. Main Results

Fig. 1 shows the connected graph (a) and its quasi-total graph (b) with one crossing. As the graph in Fig. 1(a) is connected, Theorem 1 and Theorem 2 in the paper [1] are incorrect. Theorem 1 of [1] states that the quasi-total graph $P(G)$ of a connected graph G never has crossing number 1 and Theorem 2 of [1] states that the quasi-total graph $P(G)$ of a connected graph G has crossing number 2 if and only if (1) or (2) or (3) holds.

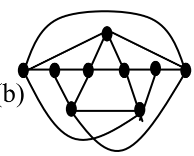
1) G is a path with 5 points
2) G is a path of length two together with two end lines adjoined to some endpoint.
3) G is a path of length two together with a triangle adjoined to some endpoint.

G:



(a)

P(*G*)*:*



(b)

Fig. 1

Theorem 1 gives necessary and sufficient condition for the quasi-total graph of a graph with crossing number one.

Theorem 1. The quasi-total graph $P(G)$ of a graph G has crossing number 1 if and only if G is of order 5 such that every connected component of G is either a path or a triangle.
Proof. Suppose G is a graph satisfying the above condition. Then by Theorem A, $P(G)$ has crossing number at least 1. We now show that its crossing number is at most 1. Now the graphs satisfying the condition of the theorem are shown in Fig 2. Then it is easy to see that in an optimal drawing of $P(G)$, it has crossing number one.

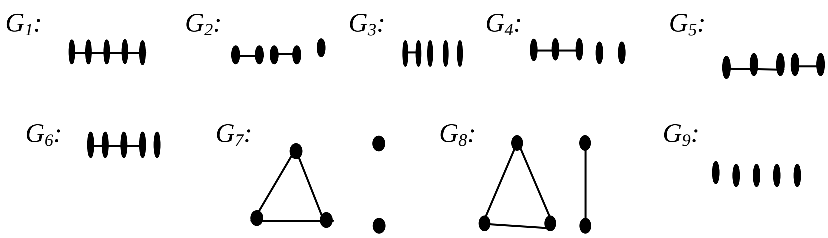


Fig. 2

Conversely, suppose $P(G)$ has crossing number 1. Assume G is a graph with at most 4 points. Then obviously $P(G)$ is planar, a contradiction.

Suppose G is a graph with 6 points and assume $\Delta(G) \leq 2$.Then the graphs satisfying the condition are shown in Fig. 3. Then it is easy to see that in an optimal drawing of $P(G)$, it has crossing number more than 1, which contradicts the hypothesis.

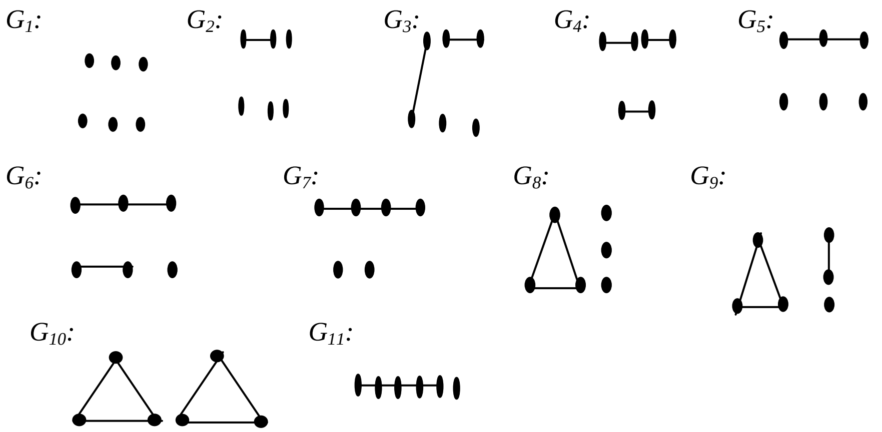


Fig. 3

Suppose G is a graph with 5 points and assume $\Delta(G) \leq 4$. We consider the following cases.

Case 1. Suppose G is non-planar. Then by Theorem C and Remark 1, $P(G)$ has crossing number at least 8, a contradiction.

Case 2. Suppose G has at least one non-cutpoint of degree 4. Then by Theorem B and Remark 1, $Cr(P(G)) \geq 3$, a contradiction.

In all the above cases we have a contradiction. This proves that $\Delta(G) \leq 3$.

Suppose G is a graph with 5 points and assume $\Delta(G)=3$. Then the graphs satisfying these conditions are shown in Fig. 4. Then it is easy to see that $Cr(P(G)) \geq 3$. Thus $\Delta(G) \leq 2$.

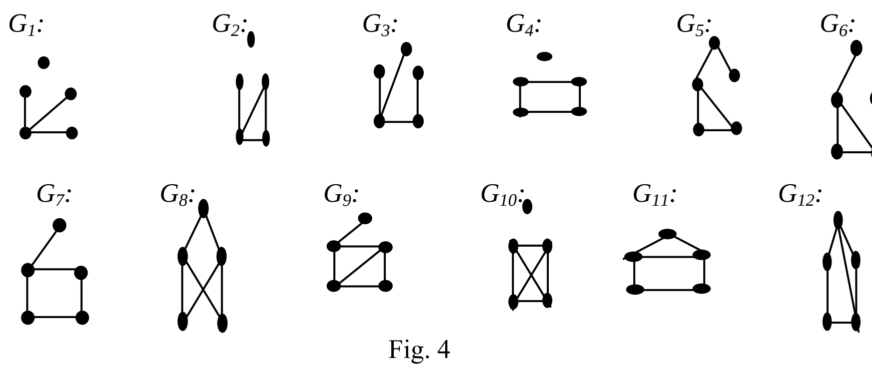


Fig. 4

Now we consider the following cases:

Case 1. Suppose *G* is connected. Then *G* is either a path or a cycle. Suppose *G* is a cycle. Then *P*(*G*) has 3 crossings, a contradiction. Thus *G* is a path of length four.
Case 2. Suppose *G* is disconnected. Then every connected component of *G* is either a path or a triangle. From the above cases, we conclude that *G* satisfies the condition. This completes the proof. □

Corollary 1.1. The quasi-total graph *P*(*G*) of a connected graph *G* has crossing number 1 if and only if *G* is P_5.

Theorem 2 gives necessary and sufficient condition for the quasi-total graph of a graph with crossing number two.

Theorem 2. The quasi-total graph *P*(*G*) of a graph *G* has crossing number 2 if and only if *G* holds either (1) or (2).

1. *G* is a connected graph of order 5 having a unique cut point of degree 2 and 3 respectively.
2. *G* is a disconnected graph of order 5 having an isolated point such that the connected component of *G* has a unique cut point of degree 3.

Proof. Suppose *G* holds (1) or (2). Then *G* is of order 5. Therefore by Theorem A, crossing number of *P*(*G*) is at least 1, since *G* has a point of degree 3. Therefore $\Delta(G)=3$. Then by Theorem 1, crossing number of *P*(*G*) is at least 2.
Now the graphs satisfying the above condition are shown in Fig 5. Then it is easy to see that in an optimal drawing of P(G), there are exactly 2 crossings.

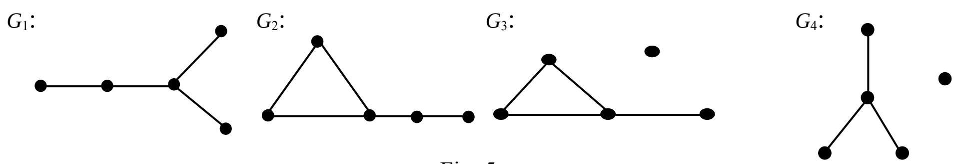


Fig. 5

Conversely, suppose P(G) has crossing number 2. Assume *G* is a graph with at most 4 points. Then by Theorem A, P(G) is planar, a contradiction. Therefore *G* is of order at least 5.
Case 1. Assume *G* is a connected graph of order 5. We consider the following subcases.
Subcase 1.1. Assume *G* is a tree. Suppose $\Delta(G)\leq 2$. Then *G* is a path of length four and by Theorem 1, $Cr(P(G))=1$, a contradiction. From the above cases we conclude that *G* is a path of length three together with an end line adjoined to some non-endpoint.

Subcase 1.2. Assume *G* is not a tree. We consider the following subcases.

Subcase 1.2.1. Suppose $\Delta(G)$=2. Then G is C_5. In an optimal drawing of $P(G)$, it has 3 crossings, a contradiction.
Subcase 1.2.2. Suppose G has at least two points of degree 3. Then it is easy to observe that G is a cycle of length three together with two end lines. Thus $P(G)$ has 3 crossings, a contradiction.
Subcase 1.2.3. Suppose G has exactly one point of degree 3. Then clearly G is a cycle of length four together with an end line adjoined to some point. Then it easy to see that $P(G)$ has at least 3 crossings, again a contradiction. From the above cases we conclude that G is a triangle together with a path of length two adjoined at some point.

Case 2. Assume G is a disconnected graph of order 5. We consider the following subcases.
Subcase 2.1. Assume the connected component of G is a tree. Suppose G has exactly one isolated point with $\Delta(G)\leq 2$. Then clearly G is a path of length three with an isolated point and by Theorem 1, $Cr(P(G))$=1, a contradiction. From the above cases we conclude that G has a unique cut point of degree 3 with one isolated point. That is clearly G is $K_{1,3}$ with exactly one isolated point.
Subcase 2.2. Assume the connected component of G is not a tree. Suppose $\Delta(G)\leq$ 2. Then clearly, G is a triangle with two isolated points and by Theorem 1, $Cr(P(G))$=1, again a contradiction. From the above cases, we conclude that G is a path of length one together with a triangle adjoined to some end point with exactly one isolated point. From the above cases, we conclude that G satisfies the condition. This completes the proof. □

Corollary 2.1. The quasi-total graph $P(G)$ of a connected graph G has crossing number 2 if and only if G is of order 5 having a unique cutpoint of degree 2 and 3 respectively.

We now give a characterization of quasi-total graphs with crossing number 3.

Theorem 3. The quasi-total graph $P(G)$ of a graph G has crossing number 3 if and only if G is either (1) or (2) or (3).

1. G is $K_{1,4}$ or C_5 or a triangle together with two end line adjoined at different points.
2. G has an isolated point and the connected component of G is $K_4 - x$.
3. G is of order 6 such that every connected component of G is either a path or a triangle, except the graph $P_5 \cup K_1$.

Proof. Suppose G is a graph satisfying (1) or (2) or (3). Then by Theorem 1 and Theorem 2, $P(G)$ has crossing number at least 3. We now show that its crossing number is at most 3.
Suppose G satisfies condition (1). Then the graphs satisfying the condition (1) are shown in Fig. 6. Then $P(G)$ has crossing number 3.

Suppose G satisfies condition (2). Then the graphs satisfying the condition (2) are shown in Fig. 7. Then $P(G)$ has crossing number 3.
Suppose G satisfies condition (3). Then the graphs satisfying the condition (3) are shown in Fig. 3. Then $P(G)$ has crossing number 3.

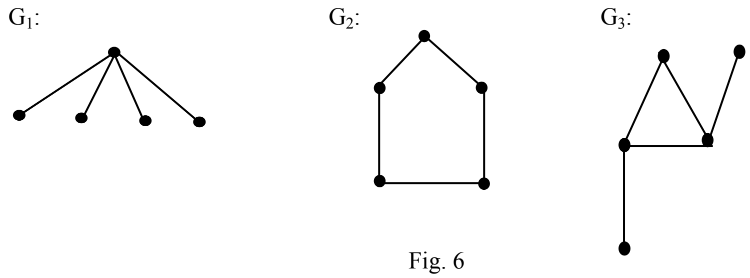


Fig. 6

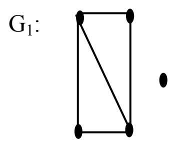


Fig. 7

Conversely, suppose the quasi-total graph $P(G)$ of a graph G has crossing number 3.Then it is non-planar. By Theorem A, G is a graph with at least 5 points.

Case 1. Assume G is a connected graph of order 5. We consider the following subcases,
Subcase 1.1. Assume G is a tree. We consider the following subcases
Subcase 1.1.1. Suppose $\Delta(G) \le 2$. Then G is a path and by Theorem 1, $Cr(P(G))=1$, a contradiction.
Subcase 1.1.2. Suppose $\Delta(G)=3$. Then clearly G is a path of length three together with an end line adjoined to some non-end point. Then by Theorem 2, $Cr(P(G))=2$,again a contradiction.
Subcase 1.2. Assume G is not a tree. We consider the following subcases.
Subcase 1.2.1. Suppose G has exactly one point of degree 3. Then G is a cycle of length four together with an end line adjoined to some point. Then $Cr(P(G))=4$, a contradiction.
Subcase 1.2.2. Suppose G has exactly one point of degree 3. Then G is a triangle together with a path of length 2 adjoined at some point. Then it is easy to see that the crossing number of $P(G)$ is at least 4, again a contradiction.
Subcase 1.2.3. Suppose $\Delta(G) \le 4$. Then clearly G has at least 2 points of degree 2 and one point of degree 3, 4 and 1. Then, $P(G)$ has crossing number 4. From the above cases we conclude that G holds (1).

Case 2. Assume G is a disconnected graph of order five. We consider the following subcases.

Subcase 2.1. Assume the connected component of G is a tree. Then every connected component of G is a path. Then by Theorem 1 and Theorem 2, $Cr(P(G)) \leq 2$, a contradiction.

Subcase 2.2. Assume the connected component of G is not a tree. We consider the following subcases.

Subcase 2.2.1. Suppose G has at least 3 points of degree 2 and one point of degree 3 and 1 with an isolated point. Then $P(G)$ has crossing number 4.

Subcase 2.2.2. Suppose G has at least two points of degree 3 and 2 and also one point is of degree 1. Then $P(G)$ has crossing number 4. From the above cases, we conclude that G holds (2).

Case 3. Assuming G is a disconnected graph of order 6, we consider the following subcases.

Subcase 3.1. If we assume the connected component of G is a tree, then every connected component of G is a path.

Subcase 3.2. If we assume the connected component of G is not a tree, then every connected component of G is a triangle. From the above cases, we conclude that G holds (3). This completes the proof.

□

3. Conclusion

We establish here necessary and sufficient conditions for quasi-total graphs to have crossing numbers k (k = 1, 2 or 3). We further find necessary and sufficient conditions for quasi-total graphs to have forbidden sub graphs for crossing numbers k (k = 1, 2 or 3).

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