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On the Hydromagnetic Stability of a Rotating Fluid Annulus

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Abstract

The linear stability of a rotating fluid in the annulus between two concentric cylinders is investigated in the presence of a magnetic field which is azimuthal as well as in axial direction. Several results of MHD stability have been derived by using the inner product method. It is shown that when the swirl velocity component is large, the hydromagnetic effects become small compared with those due to swirl. The presence of a velocity field and imposed magnetic field will lead to the basic state to more stability.

Keywords: Hydromagnetic Stability; Rotating Fluid.

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1. Introduction

Howard and Gupta [1] investigated the stability of inviscid flows between two concentric cylinders which have an axial velocity component depending only on r in addition to a swirl velocity component in the direction of increasing azimuthal angle θ. Acheson [2] studied the hydromagnetic instability of a uniformly rotating fluid in the annular region between two concentric infinitely long cylinders. Following that analysis, a detailed hydromagnetic instability arising out of such a configuration has been simplified to yield a series of stability conditions. It has been shown that regardless of the magnetic field profiles, any unstable disturbance must have the ratio between angular velocity with the phase velocity in the azimuthal direction as negative and hence must propagate against the basic rotation. Zhang and Busse [3] investigated the instability of an electrically conducting fluid of magnetic diffusivity and viscosity in a rapidly rotating sphere when toroidal magnetic field is present. Liu *et al.* [4] examined the stability of an azimuthal base flow for both axisymmetric and plane-polar disturbances for an electrically conducting fluid confined between stationary, concentric, infinitely-long cylinders. When an axial magnetic field is applied, the interaction between the radial electric current and

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the magnetic field gives rise to an azimuthal electromagnetic body force which drives an azimuthal velocity and infinitesimal axisymmetric disturbances to an instability in the flow. Goodman and Ji [5] investigated axisymmetric stability of viscous resistive magnetized Couette flow, with emphasis on flow that would be hydrodynamically stable according to Rayleigh's criterion: opposing gradient of angular velocity and specific angular momentum. In this regime, magnetorotational instabilities may occur. In studies of magnetic Taylor-Couette flow in the presence of an imposed axial magnetic field, Willis and Barenghi [6] find that values of the imposed magnetic field which alters only slightly, the transition from circular-Couette flow to Taylor-vortex can shift the transition from Taylor-vortex flow to wavy modes by a substantial amount. Deka and Gupta [7] have analyzed linear stability of wide-gap MHD dissipative Couette flow of an incompressible electrically conducting fluid between two rotating concentric circular cylinders when a uniform axial magnetic field is present. Rajaee and Shoki [8] considered the case when a transition layer exists between two fluids, and both density and magnetic field change across this layer. The numerical calculations show that while the increase of the Mach number and compressibility have a destabilizing influence, the increases in magnetic field strength and density provide a stabilizing effect. Jasmine [9] investigated stability of radial flow subjected to a radial magnetic field. The stability condition derived is shown to remain valid even when the local velocity is not entirely radial, and that the magnetic field exerts a stabilizing effect on the flow. We have extended this work when both velocity components and magnetic field components are azimuthal as well as axial.

In this presentation, we consider a non-dissipative fluid rotating uniformly in the annular region between two infinitely long cylinders. The objective is to investigate the stability of MHD flow with velocity components for an incompressible fluid permeated by a magnetic field, where the components are along (r, θ, z) directions in cylindrical polar coordinate system, n is positive and A, B are constants. The magnetic lines of force are in general twisted by non-axisymmetric disturbances to the basic flow, whereas the axisymmetric disturbances only bend but do not twist the lines of force, and when the swirl velocity component r is large, the hydromagnetic effects become small compared with those due to swirl. These ideas have led us to investigate the MHD stability with respect to non-axisymmetric disturbances to the basic flow. The presence of velocity field in the basic state may cause more stability. Several results of MHD stability have been derived by using the inner product method.

2. Mathematical Formulation

Let us consider the basic flow $(0, V_{\theta}, V_z)$ of an incompressible, inviscid and perfectly conducting fluid between two concentric cylinders of radii R_1 and R_2 permeated by a magnetic field $(0, A/r^n, B/r^n)$. The governing equations are:

$$\frac{\delta\vec{V}}{\delta t}+(\vec{V}.\nabla)\vec{V}=-\frac{1}{\rho}\nabla P+\frac{\mu}{4\pi\rho}(\nabla\times\vec{H})\times\vec{H}, \tag{1a}$$

$$\nabla.\vec{V} = 0, \tag{1b}$$

$$\frac{\delta\vec{H}}{\delta t} = \nabla \times (\vec{V} \times \vec{H}), \tag{1c}$$

where $\vec{V}$ is the velocity vector, t, P, ρ, μ, and $\vec{H}$ represent time, pressure, density, magnetic permeability, and magnetic field, respectively.

The perturbed velocity is $\vec{V} = (u_r, V_\theta(r) + u_\theta, V_z(r) + u_z)$. The perturbed magnetic field and the total presure (hydrodynamic and hydromagnetic) are respectively taken as $\vec{H} = (h_r, A/r^n + h_\theta, B/r^n + h_z)$ and $P/\rho = p_0/\rho + p$, where p_0 is the unperturbed total pressure. Analysing the disturbances into normal modes, we seek solution of the foregoing equations whose dependence on t, θ, z is given by $e^{i(ct+m\theta+kz)}$, where c = complex number, m = an integer, and k = real number. We linearize the equtions in the usual way and seek solutions in which all perturbation quantities ϕ may be written as

$$\phi = \hat{\phi}(r)\, e^{i(ct+m\theta+kz)}. \tag{2}$$

The linearized momentum equations are

$$i\sigma u_r - 2\frac{V_\theta}{r}u_\theta - \mu^/[\psi(r)h_r - \frac{2Ah_\theta}{r^{n+1}}] = -\frac{dp}{dr}, \tag{3a}$$

$$i\sigma u_\theta + \frac{1}{r}\frac{d}{dr}(rV_\theta)u_r - \mu^/[\psi(r)h_\theta + \frac{(1-n)A}{r^{n+1}}h_r] = -\frac{imp}{r}, \tag{3b}$$

$$i\sigma u_z + \frac{d}{dr}(V_z)u_r - \mu^/[\psi(r)h_z - \frac{nB}{r^{n+1}}h_r] = -ikp, \tag{3c}$$

The linearized magnetic induction equations are

$$i\sigma h_r - \psi(r)u_r = 0, \tag{4a}$$

$$i\sigma h_\theta - \frac{A(n+1)}{r^{n+1}}u_r - r\frac{d}{dr}\left(\frac{V_\theta}{r}\right)h_r - \psi(r)u_\theta = 0, \tag{4b}$$

$$i\sigma h_z - \frac{d}{dr}(V_z)h_r - \psi(r)u_z + \frac{d}{dr}\left(\frac{A}{r^n}\right) = 0, \tag{4c}$$

where $\sigma = c + \frac{mV_\theta}{r} + kV_z$, $\psi(r) = \frac{1}{r^{n+1}}(Am + Bkr)$, and $\mu^/ = \frac{\mu}{4\pi\rho}$,

The equation of continuity becomes

$$\frac{du_r}{dr} + \frac{u_r}{r} + \frac{imu_\theta}{r} + iku_\theta = 0. \tag{5}$$

Following Ganguly and Gupta [10], who consider the variables ξ_r, ξ_θ, ξ_z related to u_r, u_θ, u_z, we now introduce Lagrangian displacement vector $\vec{\xi} = (\xi_r, \xi_\theta, \xi_z)$

$$u_r = i\sigma\xi_r, \quad u_\theta = i\sigma\xi_\theta - r\xi_r \frac{d}{dr}\left(\frac{V_\theta}{r}\right), \quad ,u_z = i\sigma\xi_z - \xi_r \frac{d}{dr}(V_z). \tag{6}$$

It can be readily shown that consistent with the meaning of $\vec{\xi}$ as the Lagrangian displacement vector, the solenoidal character of $\vec{u} = (u_r, u_\theta, u_z)$ implies the solenoidal character of $\vec{\xi}$, thus

$$\frac{d\xi_r}{dr} + \frac{\xi_r}{r} + \frac{im\xi_\theta}{r} + ik\xi_\theta = 0. \tag{7}$$

In terms of the variables ξ_r, ξ_θ, and ξ_z, the Eqs. (4a-4c) become

$$h_r = \psi(r)\xi_r, \tag{8a}$$

$$h_\theta = \psi(r)\xi_\theta + \frac{A(n+1)}{r^{n+1}}\xi_r, \tag{8b}$$

$$h_z = \psi(r)\xi_z + \frac{Bn}{r^{n+1}}\xi_r. \tag{8c}$$

Substituting Eq. (6) and Eq. (8a-8c) in Eq. (3a-3c), we get

$$\left[\sigma^2 - 2V_\theta \frac{d}{dr}\left(\frac{V_\theta}{r}\right) - \frac{\mu^/}{r^{2n+2}}(Am + Bkr)^2 - \frac{2\mu^/ A^2(n+1)}{r^{2n+2}}\right]\xi_r$$
$$+ 2i\left[\sigma\frac{V_\theta}{r} - \frac{\mu^/ A}{r^{2n+2}}(Am + Bkr)\right]\xi_\theta - \frac{dp}{dr} = 0, \tag{9a}$$

$$\left[\sigma^2 - \frac{\mu^/}{r^{2n\_2}}(Am + Bkr)^2\right]\xi_\theta + 2i\left[-\sigma\frac{V_\theta}{r} + \frac{\mu^/ A}{r^{2n+2}}(Am + Bkr)\right]\xi_r - \frac{imp}{r} = 0, \tag{9b}$$

$$\left[\sigma^2 - \frac{\mu^/}{r^{2n+2}}(Am + Bkr)^2\right]\xi_z - ikp = 0. \tag{9c}$$

Clearly the induced magnetic field $\vec{h} = (h_r, h_\theta, h_z)$ given by (8a-8c) satisfies $\nabla.\vec{h} = 0$ by virtue of (7).

If the fluid is contained between two co-axial cylinders of radii R_1 and R_2, we must require that the radial component of the velocity vanishes for these value of r, thus Eqs. (9a-9c) must be considered together with boundary conditions

$$\xi_r(R_1) = \xi_r(R_2) = 0. \tag{10}$$

We rewrite equations (9a-9c) in the form

$$\left[c^2+2c\delta(r)+\delta^2(r)-2V_\theta\frac{d}{dr}\left(\frac{V_\theta}{r}\right)-\frac{\mu^/}{r^{2n\_2}}\chi^2(r)-\frac{2\mu^/A^2(n+1)}{r^{2n+2}}\right]\xi_r$$
$$+2i\left[c\frac{V_\theta}{r}+\frac{V_\theta}{r}\delta(r)-\frac{\mu^/A}{r^{2n+2}}\chi(r)\right]\xi_\theta=\frac{dp}{dr}, \tag{11a}$$

$$\left[c^2+2c\delta(r)+\delta^2(r)-\frac{\mu^/}{r^{2n\_2}}\chi^2(r)\right]\xi_\theta+2i\left[-c\frac{V_\theta}{r}-\frac{V_\theta}{r}\delta(r)+\frac{\mu^/A}{r^{2n+2}}\chi(r)\right]\xi_r$$
$$=\frac{imp}{r}, \tag{11b}$$

$$\left[c^2+2c\delta(r)+\delta^2(r)-\frac{\mu^/}{r^{2n+2}}\chi^2(r)\right]\xi_z=ikp, \tag{11c}$$

where $\delta(r)=\frac{mV_\theta}{r}+kV_z$, $\chi(r)=Am+Bkr$.

Eqs. (11a-11c) can be written in the matrix form as

$$[c^2M+icG-H]\vec{\xi}=-F_\xi, \tag{12}$$

where $\vec{\xi}=(\xi_r,\xi_\theta,\xi_z)^T$, and

$$\text{M}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \text{iG}=2\begin{pmatrix}\delta(r)&\frac{iV_\theta}{r}&0\\-\frac{iV_\theta}{r}&\delta(r)&0\\0&0&\delta(r)\end{pmatrix},\ \text{H}=\begin{pmatrix}H_{11}&H_{12}&0\\H_{21}&H_{22}&0\\0&0&H_{33}\end{pmatrix},\ F_\xi=\begin{pmatrix}\frac{dp}{dr}\\\frac{imp}{r}\\ikp\end{pmatrix},$$

$$H_{11}=\left[-\delta^2(r)+2V_\theta\frac{d}{dr}\left(\frac{V_\theta}{r}\right)+\frac{\mu^/}{r^{2n\_2}}\chi^2(r)+\frac{2\mu^/A^2(n+1)}{r^{2n+2}}\right],$$

$$H_{12}=2i\left[\frac{\mu^/A}{r^{2n+2}}\chi(r)-\frac{V_\theta}{r}\delta(r)\right],\quad H_{21}=2i\left[\frac{\mu^/A}{r^{2n+2}}\chi(r)-\frac{V_\theta}{r}\delta(r)\right],$$

$$H_{22}=\left[-\delta^2(r)+\frac{\mu^/}{r^{2n\_2}}\chi^2(r)\right],\text{ and } H_{33}=\left[-\delta^2(r)+\frac{\mu^/}{r^{2n\_2}}\chi^2(r)\right].$$

Here M, iG, and H are independent of c. According to Barston [11], the inner product can be defined as

$$<\xi,\eta>=\int_{R_1}^{R_2}(\bar{\xi}_r\eta_r+\bar{\xi}_\theta\eta_\theta+\bar{\xi}_z\eta_z)rdr, \tag{13}$$

where $\bar{\xi}$ is the complex conjugate of ξ. Taking inner product with ξ

$$<\xi,(c^2M+icG-H)\xi>=<\xi,F_\xi>, \tag{14}$$

where

$$<\xi,F_\xi>=\int_{R_1}^{R_2}p[(\frac{d}{dr}\bar{\xi}_r+\frac{\bar{\xi}_r}{r}+\frac{im\bar{\xi}_\theta}{r}+ik\bar{\xi}_z)]rdr=-\int_{R_1}^{R_2}p[\overline{(\frac{d}{dr}\xi_r+\frac{\xi_r}{r}+\frac{im\xi_\theta}{r}+ik\xi_z)}]rdr=0$$

The above equation leads to

$$c^2<\xi,M\xi>+c<\xi,iG\xi>-<\xi,H\xi>=0, \tag{15}$$

where $<\xi,M\xi>=\int_{R_1}^{R_2}[|\xi_r|^2+|\xi_\theta|^2+|\xi_z|^2]rdr$

Hence, $<\xi,M\xi>$ is real. We shall now show that $<\xi,iG\xi>$ and $<\xi,H\xi>$ are also real.

$$<\xi,iG\xi>=2\int_{R_1}^{R_2}\delta(r)[(|\xi_r|^2+|\xi_\theta|^2+|\xi_z|^2)+\frac{iV_\theta}{r}(\xi_\theta\bar{\xi}_r-\xi_r\bar{\xi}_\theta)]rdr \tag{16}$$

where $\xi_\theta\bar{\xi}_r-\xi_r\bar{\xi}_\theta=iR_3$, R_3 being real. Hence, $<\xi,iG\xi>$ is real.

$$\xi,H\xi>=2\int_{R_1}^{R_2}[(\delta^2(r)-\frac{\mu^/}{r^{2n+2}}\chi^2(r))(|\xi_r|^2+|\xi_\theta|^2+|\xi_z|^2)-2(2V_\theta\frac{d}{dr}(\frac{V_\theta}{r})+\frac{\mu^/A^2(n+1)}{r^{2n+2}})|\xi_r|^2$$

$$+2i(\frac{V_\theta}{r}\delta(r)-\frac{\mu^/A}{r^{2n+2}}\chi(r))(\xi_\theta\bar{\xi}_r-\xi_r\bar{\xi}_\theta)]rdr. \tag{17}$$

Hence, $<\xi,H\xi>$ is real. Therefore, the equation

$$c^2<\xi,M\xi>-c<\xi,iG\xi>-<\xi,H\xi>=0, \tag{18}$$

is a quadratic equation in c with real co-efficients. Its roots are

$$c=\frac{-<\xi,iG\xi>\pm D^{\frac{1}{2}}}{2<\xi,M\xi>}, \tag{19}$$

where $D = (<\xi, iG\xi>)^2 + 4 <\xi, M\xi><\xi, H\xi>$.

3. Conclusion

We investigated the stability of MHD flow when both velocity components and magnetic field components are azimuthal as well as axial. The observations are:

(a) If $<\xi, H\xi>$ 0, then D is a positive real quantity. The motion is accordingly oscillatory. Again if $(<\xi, H\xi>) < 0$, but $(<\xi, iG\xi>)^2 \geq 4(<\xi, M\xi>)\left|<\xi, H\xi>\right|$, then D is also real and the motion is oscillatory.

(b) But, if the inequality does not hold then

$D = (<\xi, iG\xi>)^2 + 4(<\xi, M\xi><\xi, H\xi> < 0$. That is $c = \dfrac{-<\xi, iG\xi> \pm i\lambda^{\frac{1}{2}}}{2<\xi, M\xi>}$, where $D = -\lambda$, and λ being positive. Therefore, $c = c_r \pm ic_i$, where

$$c_r = -\frac{<\xi, iG\xi>}{2<\xi, M\xi>}, \quad \text{and} \quad c_i = \frac{\lambda^{\frac{1}{2}}}{2<\xi, M\xi>}.$$

The motion will be stable if $c_i > 0$, and the motion will be unstable if $c_i < 0$.

We thus demonstrate that for a velocity field and imposed magnetic field as mentioned above, the basic state will lead to more stability.

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