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Nearlattices Whose Sets of Principal *n*-ideals Form Relatively Normal Nearlattices

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Abstract

We generalize several results of relatively normal nearlattices in terms of *n*-ideals. We introduce the notion of relative *n*-annihilators in a nearlattice and include some interesting results on this. Several characterizations of the set of principal *n*-ideals $P_n(S)$ are given which forms a relatively normal nearlattice in terms of relative *n*-annihilators. It is shown that $P_n(S)$ is relatively normal if and only if for any two incomparable prime *n*-ideals *P* and *Q*, $P \vee Q$=L.

Keywords: Relatively normal nearlattice; Relative *n*-annihilator; Incomparable prime *n*-ideals.

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1. Introduction

Relative annihilators in lattices and semi-lattices have been studied by many authors including Mandelker [1] and Varlet [2]. Cornish [3] has used the annihilators in studying relative normal lattices. On the other hand, relative annihilators in nearlattices have been studied by Noor and Islam [4]. Recently Noor and Ali [5] have studied the relative *n*-annihilators in a lattice *L* for a fixed element $n \in L$

In this paper we have introduced the notion of relative *n*-annihilators in a nearlattice. Then with the help of relative *n*-annihilators we have studied those $P_n(S)$ which are relatively normal.

2. Preliminaries

A *nearlattice* is a meet semi lattice together with the property that any two elements possessing a common upper bound, have a supremum. A nearlattice *S* is distributive if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in S$ provided $y \vee z$ exists.

For a fixed element $n \in S$, a convex sub nearlattice containing n is called an n-ideal. The concept of n-ideals is a kind of generalization of ideals and filters of a nearlattice. Details on nearlattices and n-ideals in both lattices and nearlattices can be found in refs. [6-9].

An element n of a nearlattice S is called a *standard element* if for all $t, x, y \in S$

$$t \wedge ((x \wedge y) \vee (x \wedge n)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge n)$$

Element n is called *neutral* if

i) it is standard and

ii) $n \wedge ((t \wedge x) \vee (t \wedge y)) = (n \wedge t \wedge x) \vee (n \wedge t \wedge y)$ for all $t, x, y \in S$.

An element n of a nearlattice S is called a *medial element* if $m(x, n, y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$ exists for all $x, y \in S$.

Element n is called an *upper element* of S if $x \vee n$ exists for every $x \in S$ Of course, every upper element is medial.

An element n of a nearlattice S is called a *central element* if it is upper, neutral and complemented in each interval containing it.

For a medial element n, an n-ideal P of a nearlattice S is called a *prime n-ideal* if $P \neq S$ and $m(x, n, y) \in P$ $(x, y \in S)$ implies either $x \in P$ *or* $y \in P$.

The set of all n-ideals of a nearlattice S is denoted by $I_n(S)$ which is an algebraic lattice. For two n-ideals I and J of a nearlattice S, the set theoretic intersection is their infimum. Moreover, when n is standard and medial, then $I \cap J = \{m(i, n, j): i \in I, j \in J\}$. According to [7], the supremum is defined by $I \vee J = \{x : i \wedge j \leq x \leq i_1 \vee j_1\}$, for some $i, i_1 \in I$ and $j, j_1 \in J$ provided $i_1 \vee j_1$ exists.

An n-ideal generated by a finite number of elements $a_1, a_2, \ldots, a_m$ is called a *finitely generated n-ideal*, denoted by $< a_1, a_2, \ldots, a_m >_n$. Following [8],

$$< a_1, a_2, \ldots, a_m >_n = \{y \in S: a_1 \wedge \ldots \wedge a_m \wedge n \leq y = (y \wedge a_1) \vee \ldots \vee (y \wedge a_m) \vee (y \wedge n)\},$$

provided S is distributive.

When S is a lattice, $< a_1, a_2, \ldots, a_m >_n$ is the interval $[a_1 \wedge \ldots \wedge a_m \wedge n, a_1 \vee \ldots \vee a_m \vee n]$.

The set of finitely generated n-ideals is denoted by $F_n(S)$ which is again a nearlattice. An n-ideal generated by a single element a is called a *principal n-ideal*, denoted by $< a >_n$. The set of principal n-ideals is denoted by $P_n(S)$.

By [8] we know that

$< a >_n \cap < b >_n = < m(a, n, b) >_n$ when n is standard and medial.

Thus $P_n(S)$ is a semi lattice when n is medial and standard. Moreover by [8] it is a nearlattice if n is neutral and upper.

Let S be a nearlattice. For $a, b \in S$, $< a, b > = \{x \in S: x \wedge a \leq b\}$ is called the *annihilator of* a *relative to b*, or simply a *relative annihilator*. It is easy to see that in presence of distributivity, $< a, b >$ is an ideal of S.

Also note that $< a,b > = < a, a \wedge b >$. Again for a, $b \in L$, where L is a lattice we define $< a, b >_d = \{x \in L: x \vee a \geq b\}$, which we call a *dual annihilator of* a *relative to* b or simply a *dual relative annihilator*. In presence of distributivity of L, $< a, b >_d$ is a dual ideal (filter).

For a, $b \in S$ and an upper element $n \in S$, we define,

$$< a, b >^n = \{x \in S: m(a, n, x) \in < b >_n\}$$
$$= \{x \in S: b \wedge n \leq m(a, n, x) \leq b \vee n\}.$$

We call $< a, b >^n$ the *annihilator of a relative to b around the element n* or simply a *relative n-annihilator*. It is easy to see that for all a, $b \in S$, $< a, b >^n$ is always a convex subset containing n. In presence of distributivity, it can easily be seen that $< a, b >^n$ is an n-ideal. If $0 \in S$, then putting $n = 0$, we have, $< a, b >^n = < a, b >$.

For two n-ideals A and B of a nearlattice S,

$< A, B >$ denotes $\{x \in S: m(a, n, x) \in B$ for all $a \in A\}$, when n is a medial element.

In presence of distributivity, clearly $< A, B >$ is an n-ideal. Moreover, we can easily show that $< a, b >^n = << a >_n, < b >_n >$.

A prime n-ideal P of a nearlattice S is called a *minimal prime n-ideal* if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$.

A distributive nearlattice S with 0 is *normal* if every prime ideal of S contains a unique minimal prime ideal. A distributive nearlattice S is *relatively normal* if each interval $[x, y]$ in S $(x, y \in S)$ $x < y$, is normal.

We start the paper with the following result on n-ideals due to [8].

Lemma 1.1 For a central elemenl $n \in S$, $P_n(S) \cong (n]^d \times [n)$.

Following result is also essential for the development of this paper, which is due to [10].

Lemma 1.2 *Let S be a distributive near-lattice with an upper element n and let I , J be two n-ideals of S. Then for any* $x \in I \vee J$, $x \vee n = i \vee j$ *and* $x \wedge n = i' \wedge j'$ *for some* i, $i' \in I$, j, $j' \in J$ *with* $i, j \geq n$ *and* $i', j' \leq n$.

Following result in lattices is due to [5] and can be proved by similar technique in case of nearlattices. This is also a generalization of Lemma 3.6 [3].

Theorem 1.3 *Let S be a distributive nearlattice with an upper element n. Then the following conditions hold.*

(i) $<< x >_n \vee < y >_n, < x >_n > = << y >_n, < x >_n >$.

(ii) $<< x >_n, J > = \vee_{y \in J} << x >_n, < y >_n >$, *the supremum of n-ideals* $<< x >_n, < y >_n >$ *in the lattice of n-ideals of S, for any* $x \in S$ *and any n-ideal J.*

Lemma 1.4 and lemma 1.5 are due to [5]. We prefer to omit the proofs as they are easy to prove.

Lemma 1.4 *Let S be a distributive nearlattice with an upper element n. Suppose a, b, c*∈S.

(i) *If a,b, c ≥ n, then* $<< m(a, n, b) >_n, < c >_n > = << a >_n, < c >_n > \vee << b >_n, < c >_n >$ *is equivalent to* $< a \wedge b, c > = < a, c > \vee < b, c >$.

(ii) *If a, b, c ≤ n, then* $<< m(a, n, b) >_n, < c >_n > = << a >_n, < c >_n > \vee << b >_n, < c >_n >$ *is equivalent to* $< a \vee b, c >_d = < a, c >_d \vee < b, c >_d$. □

Lemma 1.5 *Let S be a distributive nearlattice with an upper element n. Suppose a, b, c*∈S.

(i) *If a, b, c ≥ n and a ∨ b exists, then* $<< c >_n, < a >_n \vee < b >_n > = << c >_n, < a >_n > \vee << c >_n, < b >_n >$ *is equivalent to* $< c, a \vee b > = < c, a > \vee < c, b >$.

(ii) *If a, b, c ≤ n, then* $<< c >_n, < a >_n \vee < b >_n > = << c >_n, < a >_n > \vee << c >_n, < b >_n >$ *is equivalent to* $< c, a \wedge b >_d = < c, a >_d \vee < c, b >_d$.

(iii) *For each x, y*∈*L,* $[x \vee y)^{*d} = [x)^{*d} \vee [y)^{*d}$.

(iv) *If* $x \vee y = 1$*, then* $[x)^{*d} \vee [y)^{*d} = L$.

Following result is due to Theorem 2.4 [3]:

Theorem 1.6: *For a distributive lattice with* 0*, the following conditions are equivalent.*

(i) *Any two distinct minimal prime ideals are comaximal,*

(ii) *L is normal,*

(iii) *For any* $x, y \in L, (x \wedge y]^* = (x]^* \vee (y]^*$,

(iv) *For any* $x, y \in L$ *with* $x \wedge y = 0$ *implies* $(x]^* \vee (y]^* = L$.

Moreover, when L has a largest element 1*, then each of the above conditions is equivalent to" for any* $x, y \in L$, $x \wedge y = 0$ *implies* $x_1, y_1 \in L$ *such that* $x \wedge x_1 = y \wedge y_1 = 0$ *and* $x_1 \vee y_1 = 1$*".*

The following result is also due to Theorem 3.7 [3]:

Theorem 1.7. *Let L be a distributive lattice. Let a, b, c be arbitrary elements and A, B be arbitrary ideals. Then the following conditions are equivalent.*

(i) *L is relatively normal.*

(ii) $< a, b > \vee < b, a > = L$.

(iii) $< c, a \vee b > = < c, a > \vee < c, b >$.

(iv) $< (c], A \vee B > = < (c], A > \vee < (c], B >$.

(v) $< a \wedge b, c > = < a, c > \vee < b, c >$.

The following result has been proved by [5] in case of lattices. The idea of dual relative annihilators in nearlattices is not always possible. Since $(n]$ is a sublattice of *S* for each $n \in S$, we have:

Theorem 1.8. *Let a, b, $c \in (n]$ be arbitrary elements and A, B be arbitrary filters on $(n]$. Then the following conditions are equivalent.*

(i) $(n]$ *is relatively normal.*

(ii) $< a, b >_d \vee < b, a >_d = (n]$.

(iii) $< c, a \wedge b >_d = < c, a >_d \vee < c, b >_d$.

(iv) $< [c), A \vee B >_d = < [c), A >_d \vee < [c), B >_d$.

(v) $< a \vee b, c >_d = < a, c >_d \vee < b, c >_d$.

Now we prove our main results of this paper, which are generalizations of Theorem 3.7 [3] and Theorem 5 [1].

Theorem 1.9. *Let n be a central element of a distributive nearlattice. Suppose A, B are two n-ideals of S. Then for all a, b, $c \in S$ the following conditions are equivalent.*

(i) $P_n(S)$ *is relatively normal.*

(ii) $<< a >_n, < b >_n > \vee << b >_n, < a >_n > = S$.

(iii) $<< c >_n, < a >_n \vee < b >_n > = << c >_n, < a >_n > \vee << c >_n, < b >_n >$, *whenever $a \vee b$ exists.*

(iv) $<< c >_n, A \vee B > = << c >_n, A > \vee << c >_n, B >$.

(v) $<< m(a, n, b) >_n, < c >_n > = << a >_n, < c >_n > \vee << b >_n, < c >_n >$.

Proof. (i)⇒(ii). Let $z \in S$. Consider the interval $I = [< a >_n \cap < b >_n \cap < z >_n, < z >_n]$ in $P_n(S)$. Then $< a >_n \cap < b >_n \cap < z >_n$ is the smallest element of the interval I. By (i), I is normal. Then by Theorem 1.6, there exist principal n-ideals $< p >_n, < q >_n \in I$ such that, $< a >_n \cap < z >_n \cap < p >_n = < a >_n \cap < b >_n \cap < z >_n = < b >_n \cap < z >_n \cap < q >_n$ and $< z >_n = < p >_n \vee < q >_n$.

Now, $< a >_n \cap < p >_n = < a >_n \cap < p >_n \cap < z >_n = < a >_n \cap < b >_n \cap < z >_n \subseteq < b >_n$ implies $< p >_n \subseteq << a >_n, < b >_n >$.

Also, $< b >_n \cap < q >_n = < b >_n \cap < z >_n \cap < q >_n = < a >_n \cap < b >_n \cap < z >_n \subseteq < a >_n$ implies $< q >_n \subseteq << b >_n, < a >_n >$

Thus $< z >_n \subseteq << a >_n, < b >_n > \vee << b >_n, < a >_n >$ and so $z \in << a >_n, < b >_n > \vee << b >_n, < a >_n >$.

Hence $<< a >_n, < b >_n > \vee << b >_n, < a >_n > = S$.

(ii)⇒(iii). Suppose (ii) holds and $a \vee b$ exists. For (iii), R.H.S. $\subseteq$ L.H.S. is obvious. Now, let $z \in << c >_n, < a >_n \vee < b >_n >$. Then $z \vee n \in << c >_n, < a >_n \vee < b >_n >$ and $m(z \vee n, n, c) \in < a >_n \vee < b >_n$.

That is, $m(z \vee n, n, c) \in [a \wedge b \wedge n, a \vee b \vee n]$. This implies $(z \vee n) \wedge (c \vee n) \leq a \vee b \vee n$. Now, by (ii), $z \vee n \in << a >_n, < b >_n > \vee << b >_n, < a >_n >$. So by Lemma 1.2, $z \vee n = r \vee t$ for some $r \in << a >_n, < b >_n >$ and $t \in << b >_n, < a >_n >$, $r, t \geq n$. Then $b \wedge n = m(r, n, a) = r \wedge (a \vee n) \leq b \vee n$.

Hence, $r \wedge (c \vee n) = r \wedge (z \vee n) \wedge (c \vee n) \leq r \wedge (a \vee b \vee n) = (r \wedge (a \vee n)) \vee (r \wedge (b \vee n)) \leq (b \vee n)$. This implies $r \in << c >_n, < b >_n >$. Similarly, $t \in << c >_n, < a >_n >$. Hence $z \vee n \in << c >_n, < a >_n > \vee << c >_n, < b >_n >$.

Again, $z \in << c >_n, < a >_n \vee < b >_n >$ implies $z \wedge n \in << c >_n, < a >_n \vee < b >_n >$. A dual calculation of the above shows, $z \wedge n \in << c >_n, < a >_n > \vee << c >_n, < b >_n >$. Thus by convexity, $z \in << c >_n, < a >_n > \vee << c >_n, < b >_n >$ and so L.H.S. $\subseteq$ R.H.S. Hence (iii) holds.

(iii)⇒(iv). Suppose (iii) holds. In (iv), R.H.S. $\subseteq$ L.H.S. is obvious.

Now let $x \in << c >_n, A \vee B >$. Then $x \vee n \in << c >_n, A \vee B >$. Thus $m(x \vee n, n, c) \in A \vee B$. Now $m(x \vee n, n, c) = (x \vee n) \wedge (n \vee c) \geq n$ implies $m(x \vee n, n, c) \in (A \vee B) \cap [n)$. Hence by Theorem 1.3(ii), $x \vee n \in << c >_n, (A \cap [n)) \vee (B \cap [n)) > = \vee_{r \in (A \cap [n)) \vee (B \cap [n))} << c >_n, < r >_n >$. But by Lemma 1.2, $r \in (A \cap [n)) \vee (B \cap [n))$ implies $r = s \vee t$ for some $s \in A$, $t \in B$ and $s, t \geq n$. Then by (iii),

$$<< c >_n, < r >_n > = << c >_n, < s \vee t >_n > = << c >_n, < s >_n \vee < t >_n >$$
$$= << c >_n, < s >_n > \vee << c >_n, < t >_n > \subseteq << c >_n, A > \vee << c >_n, B >$$

Hence $x \vee n \in << c >_n, A > \vee << c >_n, B >$. Also $x \in << c >_n, A \vee B >$ implies $x \wedge n \in << c >_n, A \vee B >$.

Since $m(x \wedge n, n, c) = (x \wedge n) \vee (x \wedge c) \leq n$, so $x \wedge n \in << c >_n, (A \vee B) \cap (n] >$. Then, by Theorem 1.3(ii),

$$x \wedge n \in << c >_n, (A \cap (n]) \vee (B \cap (n]) > = \vee_{i \in (A \cap (n]) \vee (B \cap (n])} << c >_n, < i >_n.$$

Again, using Lemma 1.2, we see that $i = p \wedge q$ where $p \in A$, $q \in B$ and $p, q \leq n$. Then by (iii),

$$<< c >_n, < i >_n > = << c >_n, < p \wedge q >_n > = << c >_n, < p >_n \vee < q >_n >$$
$$= << c >_n, < p >_n > \vee << c >_n, < q >_n > \subseteq << c >_n, A > \vee << c >_n, B >$$

Hence $x \wedge n \in << c >_n, A > \vee << c >_n, B >$. Therefore, by convexity, $x \in << c >_n, A > \vee << c >_n, B >$ and so L.H.S. $\subseteq$ R.H.S. Thus (iv) holds.

(iv)⇒(iii) is trivial.

(ii)⇒(v). Suppose (ii) holds. In (v), R.H.S. $\subseteq$ L.H.S. is obvious.

Now let $z \in << m(a, n, b) >_n, < c >_n >$ which implies $z \vee n \in << m(a, n, b) >_n, < c >_n >$. By (ii), $z \vee n \in << a >_n, < b >_n > \vee << b >_n, < a >_n >$. Then by Theorem 1.2, $z \vee n = x \vee y$ for some $x \in << a >_n, < b >_n >$ and $y \in << b >_n, < a >_n >$ and $x, y \geq n$.
Thus, $< x >_n \cap < a >_n \subseteq < b >_n$ and so $< x >_n \cap < a >_n = < x >_n \cap < a >_n \cap < b >_n \subseteq < z \vee n >_n \cap < a >_n \cap < b >_n = < z \vee n >_n \cap < m(a, n, b) >_n \subseteq < c >_n$. This implies $x \in << a >_n, < c >_n >$.

Similarly, $y \in << b >_n, < c >_n >$ and so $z \vee n \in << a >_n, < c >_n > \vee << b >_n, < c >_n >$.

Similarly, a dual calculation of above shows that $z \wedge n \in << a >_n, < c >_n > \vee << b >_n, < c >_n >$. Thus by convexity, $z \in << a >_n, < c >_n > \vee << b >_n, < c >_n >$ and so L.H.S. $\subseteq$ R.H.S. Hence (v) holds.

(v)⇒(i). Suppose (v) holds. Let $a, b, c \geq n$.

By (v), $<< m(a, n, b) >_n, < c >_n > = << a >_n, < c >_n > \vee << b >_n, < c >_n>$. But by Lemma 1.5(i), this is equivalent to $< a \wedge b, c > = < a, c > \vee < b,c>$. Then by Theorem 1.7, this shows that $[n)$ is relatively normal. Similarly, for $a, b, c \leq n$, using Lemma 1.5(ii) and Theorem 1.8, we find that (n] is relatively normal. Therefore by Lemma 1.1, $P_n(S)$ is relatively normal.

Finally we need to prove that **(iii)⇒(i).**

Suppose (iii) holds. Let $a, b, c \in S \cap [n)$. By (iii), $<< c >_n, < a >_n \vee < b >_n > = << c >_n, < a >_n \vee << c >_n, < b >_n$, whenever $a \vee b$ exists. But by Lemma 1.6(i), this is equivalent to $< c, a \vee b > = < c, a > \vee < c, b >$. Then by Theorem 1.7, this shows that $[n)$ is relatively normal.

Similarly, for $a, b, c \leq n$, using the Lemma 1.6(ii) and Theorem 1.8, we find that $(n]$ is relatively normal. Therefore by Lemma 1.1, $P_n(S)$ is relatively normal.

We conclude the paper with the following result which is a generalization of a result in [11].

Theorem 1.10. *Let S be a distributive nearlattice. If n is central in S, then the following conditions are equivalent.*

(i) *$P_n(S)$ is relatively normal.*

(ii) *Any two incomparable prime n-ideals P and Q, $P \vee Q = S$.*

Proof. (i)⇒(ii). Suppose (i) holds. Let P and Q be two incomparable prime n-ideals of S. Then there exist $a, b \in S$ such that $a \in P\text{-}Q$ and $b \in Q\text{-}P$.

Then $< a >_n \subseteq P\text{-}Q$ and $< b >_n \subseteq Q\text{-}P$. Since by (i), $P_n(S)$ is relatively normal, so by Theorem 1.9, $<< a >_n, < b >_n > \vee << b >_n, < a >_n > = S$.

But as P, Q are prime, so it is easy to see that $<< a >_n, < b >_n > \subseteq Q$ and $<< b >_n, < a >_n > \subseteq P$.

Therefore, $S \subseteq P \vee Q$ and so $P \vee Q = S$. Thus (ii) holds.

(ii)⇒(i). Suppose (ii) holds. Let P_1 and Q_1 be two incomparable prime ideals of $[n)$. Then by [12], there exist two incomparable prime ideals P and Q of S such that $P_1 = P \cap [n)$ and $Q_1 = Q \cap [n)$. Since $n \in P_1$ and $n \in Q_1$, so P and Q are in fact two incomparable prime n-ideals of S. Then by (ii), $P \vee Q = S$.

Therefore, $P_1 \vee Q_1 = (P \vee Q) \cap [n) = S \cap [n) = [n)$. Thus by [11], $[n)$ is relatively normal.

Similarly, considering two prime filters of $(n]$ and proceeding as above and using the dual result of Theorem 3.5 [3] we find that $(n]$ is relatively normal. Therefore, by Lemma 1.1, $P_n(S)$ is relatively normal. □

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