JSR Publications

Available Online

J. Sci. Res. **14** (1), 45-65 (2022)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

On Some Values of the Sandor-Smarandache Function

S. M. S. Islam[1*], A. A. K. Majumdar[2]

[1]Hajee Mohammad Danesh Science & Technology University, Dinajpur 5200, Bangladesh

[2]Beppu-shi Oaza Tsurumi, Renace Beppu #205, Beppu-shi 874-0842, Japan

Received 12 April 2021, accepted in final revised form 8 October 2021

Abstract

The Sandor-Smarandache function, due to Sandor, has drawn the attention of the researchers soon after its introduction. The new Smarandache-type arithmetic function, denoted by *SS* (*n*), involves binomial coefficients. Sandor found *SS*(*n*) when *n* (≥ 3) is an odd integer. It has been shown that the function has a simple form even when *n* is even and not divisible by 3. In earlier papers, some closed-form expressions of *SS*(*n*) have been derived for particular cases of *n*. Still, some unexplored results are needed to settle down the function. So, this study finds more forms of *SS*(*n*), starting from the function *SS*(24*m*). Particular attention has been focused on the functions *SS*(120*m*), *SS*(840*m*), *SS*(9240*m*) and *SS*(120120*m*).

Keywords: Sandor-Smarandache function; Binomial coefficient; Diophantine equation; Smarandache function; Pseudo-Smarandache function.

doi: http://dx.doi.org/10.3329/jsr.v14i1.52946 J. Sci. Res. **14** (1), 45-65 (2022)

1. Introduction

In the late 1970s, the Romanian-American mathematician, Florentin Smarandache, proposed a new arithmetic function called the Smarandache function after him and is denoted by *S*(*n*). Since then, more Smarandache-type arithmetic functions have been introduced in the mathematical literature. These functions are different from the traditional arithmetic functions of number theory in many respects. Because of their special properties, they drew the attention of different researchers. One of the most recent Smarandache-type functions is the Sandor-Smarandache function, posed by Sandor [1], and is denoted by *SS* (*n*). The function is defined as follows:

$$SS(n) = max\left\{k : 1 \le k \le n-2, n \text{ divides } \binom{n}{k}\right\}, n \ge 5, \tag{1.1}$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}, 0 \le k \le n,$ and by convention,

$SS(1) = 1, SS(2) = 1, SS(3) = 1, SS(4) = 1, SS(6) = 1.$

Recall that the binomial coefficients are all integers [2], Theorem 73]. The following equivalent form of the binomial coefficients *C*(*n*, *k*) would be used throughout this paper:

* *Corresponding author*: sislam.math@gmail.com

$$C(n,k) \equiv \binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}, 0 \le k \le n.$$

Then, the problem is reformed as follows: Given any integer n (≥ 7), find the minimum integer k such that $k!$ divides $(n-1)(n-2)\ldots(n-k+1)$, where $1 \leq k \leq n-2$. With this minimum k, $SS(n) = n - k$. In Islam, Gunarto, and Majumdar [3], the following results have been proved.

Lemma 1.1: $SS(n) = n - 2$ if and only if n (≥ 7) is an odd integer.

Lemma 1.2: $SS(n) = n - 3$ if and only if n is even and is not divisible by 3.

Lemma 1.3: $SS(12m) = 12m - 5$ for any integer m (≥ 1), not divisible by 5.

Lemma 1.4: $SS(30m) = 30m - 7$, where m (≥ 1) be an integer not divisible by 7 with $m \neq 4\alpha + 3$ (for any $\alpha \geq 0$) and $m \neq 2(6\beta + 5)$ (for any $\beta \geq 0$).

From Lemma 1.1 and Lemma 1.2, it is seen that $SS(n)$ has a simple form when n is odd or when n is even but not divisible by 3. Thus, the problem of finding $SS(n)$ when n is even and divisible by 3 remains a challenging problem. Majumdar [4] considered functions of the form $SS(p+1)$, where p is an odd prime. Later, the problem was studied extensively by Islam, and Majumdar [5], who derived expressions of $SS(2mp)$, $SS(6mp)$, $SS(60mp)$, and $SS(420mp)$, where p is an odd prime and m is any (positive) integer. Recently, Islam *et al.* [3] derived expressions of $SS(6t)$, $SS(12t)$, $SS(18t)$, $SS(42t)$, $SS(30t)$, and $SS(210t)$ for some particular forms of t. So, the forms of $SS(n)$ in other cases remain open, limiting the function SS(n) application.

This paper finds $SS(n)$ for some particular cases of n. This would help unravel more properties of the Sandor-Smarandache function. The relevant background materials are given in Section 2. Section 3 derives the main results of the paper. Lemma 3.1 in Section 3 finds the necessary and sufficient condition such that $SS(n) = n - 4$. Surprisingly, such an n has a simple form, and the finding suggests that concentration must be given to the function $SS(24m)$, where m (≥ 1) is an integer. Thus, starting from $SS(24m)$, the paper derives the expressions of $SS(120m)$, $SS(840m)$, $SS(9240m)$, and $SS(120120m)$. Some remarks are made in Section 4, based on the results found so far. This section contains some interesting equations involving $SS(n)$. The paper concludes with some concluding remarks in Section 5. The findings so far show that the form of $SS(n)$ depends on n, and more specifically, the prime factors of n. Thus, $SS(n)$ has the simplest form when n is odd; the next simplest form is found when n is even but not divisible by 3. The form of $SS(n)$ gets more complicated when the prime factors of n increase.

2. Background Material

This section gives the background material that would be needed in the next section. These are given in the lemmas below. Proofs may be found in Islam *et al*. [3].

Lemma 2.1: (*Fundamental Theorem of Arithmetic*) Let a and b be two (positive) integers with $(a,b) = 1$. Let N be an integer such that a and b each divides N. Then, ab divides N.

An alternative proof of Lemma 2.1 may be found in Olds, Lax and Davidoff [6].

Lemma 2.2: Let A and B be two (positive) integers such that A is divisible by the integer a and B is divisible by the integer b. Then, AB is divisible by ab.
Lemma 2.3: For any integer $a \geq 1$ fixed, $a(a-1)\ldots(a-s+1)$ is divisible by $s!$, where s is an integer with $1 \leq s \leq a$.
Corollary 2.1: For any integer $a \geq 1$ fixed, let $P(a, s) \equiv a(a-1)\ldots(a-s+1)$ for any integer $1 \leq s \leq a$. Then, s divides $(a-1)(a-2)\ldots(a-s+1)$ if and only if s does not divide a.
Corollary 2.2: The product of 9 consecutive (positive) integers is divisible by 3^4.

The paper's main results are derived in Section 3, where the following result would be needed.
Lemma 2.4: Let m, n, and a be any three fixed (but arbitrary) positive integers. Then, the Diophantine equation $mx + ny = a$ has an (integer) solution if and only if a is divisible by $b \equiv (m, n)$. Moreover, if (x_0, y_0) is a solution, then there are an infinite number of solutions, given parametrically by $x = x_0 + (\frac{n}{b})t$, $y = y_0 + (\frac{-m}{b})t$ for any integer t.
Proof: See, for example, Gioia [7].
In applying Lemma 2.4, one has to find the solution of the equation $mx + ny = a$ with minimum x_0 (in the sense that there is no solution x less than x_0). Let (x_0, y_0) be such a solution. Then, if, in particular, $(m, n) = 1$, then the solutions of the equation are given simply by $x = x_0 + nt$, $y = y_0 - mt$, where t is a parameter.
Another result of interest is the following one (see Hardy and Wright [2] for a proof).
Lemma 2.5: (*Dirichlet Theorem*) (*Dirichlet Theorem*) If m and n are two integers with $m > 0$ and $(m, n) = 1$, then there are infinitely many primes of the form $mx + n$, x (> 0) being an integer.
The main results of the paper are given in the following section.

3. Main Results

First the following result is proved.
Lemma 3.1: For some integer n,
$$SS(n) = n - 4 \tag{3.1}$$
if and only if n is of the form $n = 6(4m+3)$ for some integer $m \geq 0$.
Proof: By Lemma 1.1 and Lemma 1.2, any integer n satisfying (3.1) must be even and divisible 3. Now, consider the expression:
$$C(n, n-4) \equiv n\left[\frac{(n-1)(n-2)(n-3)}{2 \times 3 \times 4}\right].$$
Here, the numerator of the term inside the square bracket is divisible by 3 (by Lemma 2.1). Hence, the term inside the square bracket is an integer if and only if 8 divides $(n-2)$. Moreover, such an n must be divisible by 3. Thus,
$$n = 8\alpha + 2 = 3\beta \text{ for some integers } \alpha \geq 1, \beta \geq 2,$$
whose solution is $\alpha = 3m + 2$ for some integer $m \geq 0$. Hence, finally,
$$n = 8(3m+2) + 2 = 6(4m+3),$$
which proves the lemma.
Using Lemma 3.1, the following values are found:

$SS(18) = 14$, $SS(42) = 38$, $SS(66) = 62$, $SS(90) = 86$, $SS(114) = 110$.

In Islam et al. [3], the explicit forms of $SS(6t)$, $SS(12t)$, $SS(18t)$, $SS(30t)$, $SS(42t)$, and $SS(210t)$ have been derived. It might be instructive to examine those results in light of Lemma 3.1. By virtue of Lemma 3.1,

$SS(6t) = 6t - 4$ if and only if $t = 4s + 3$, $s \geq 0$.

Also, $SS(12t) = 12t - 4$ if and only if $2t = 4s + 3$. In this case, $(2, 4) = 2$ does not divide 3, so that, by Lemma 2.4, the equation has no solution. Hence, $SS(12t) \neq 12t - 4$ for any t.

When $n = 18t$, the condition in Lemma 3.1 becomes $3t = 4m + 3$, whose solution is $t = 4s + 1$, $s \geq 0$ (taking into account the fact that $SS(18) = 14$, that is, the result is valid for $s = 0$ as well). Thus,

$SS(18t) = 18t - 4$ if and only if $t = 4s + 1$, $s \geq 0$.

The condition in Lemma 3.1 for $n = 30t$ reduces to $5t = 4m + 3$, whose solution is $t = 4s + 3$, $s \geq 0$. Thus,

$SS(30t) = 30t - 4$ if and only if $t = 4s + 3$, $s \geq 0$.

With $n = 42t$, the condition in Lemma 3.1 reads as $7t = 4m + 3$, with the solution $t = 4s + 1$, $s \geq 0$. Hence,

$SS(42t) = 42t - 4$ if and only if $t = 4s + 1$, $s \geq 0$.

Finally, when $n = 210t$, by Lemma 3.1, $SS(210t) = 210t - 4$ if and only if $35t = 4m + 3$, whose solution is $t = 4s + 1$, $s \geq 0$.

All the conditions above match exactly with those derived, in more detail, in Islam *et al.* [3], employing a different approach.

In view of Lemma 3.1, the function of interest is $SS(24m)$, considered below.

Lemma 3.2: For any integer $m \geq 1$, not divisible by 5, $SS(24m) = 24m - 5$.

Proof: Consider the following expression for $C(24m, 24m - 5)$:

$$C(24m, 24m-5) \equiv (24m)\frac{(24m-1)(24m-2)(24m-3)(24m-4)}{2\times 3\times 4\times 5}$$

$$= 24m\left[\frac{(24m-1)(12m-1)(8m-1)(6m-1)}{5}\right].$$

Now, if 5 does not divide m, by Corollary 2.1, 5 must divide the numerator of the term inside the square bracket. This proves the lemma.

Lemma 3.2 gives the following values :

$SS(24) = 19$, $SS(48) = 43$, $SS(72) = 67$, $SS(96) = 91$, $SS(144) = 139$, $SS(168) = 163$.

After having the values of $SS(24m)$, applying Lemma 1.1. and Lemma 1.2 successively, the values below are obtained :

$$SS(24m + i) = 24m + i - 2 \text{ for } i = 1, 3, \ldots, \tag{3.2}$$

$$SS(24m + 2\nu) = 24m + 2\nu - 3 \text{ for } \nu = 1, 2, 4, 5, 7, \ldots, \tag{3.3}$$

where ν is not divisible by 3. Since $24m + i$ is odd for $i = 1, 3, \ldots$, (3.2) follows from Lemma 1.1, while (3.3) follows from Lemma 1.2. Lemma 3.1 gives the expression of $SS(24m + 18)$. It thus remains to find the expressions of $SS(24m + 6)$ and $SS(24m + 12)$. These are given in the following two propositions.

Proposition 3.1: Let $m \geq 0$ be any integer. Then,

$$SS(24m+6)=\begin{cases} 24m+1, & if \quad m\neq 5s+1,\ s\geq 0 \\ 24m-1, & if \quad m=5s+1,\ s\neq 7t+5,\ t\geq 0 \\ 24m-2, & if \quad m=2(35w+13),\ w\geq 0 \\ 24m-3, & if \quad m=630u+271, \text{ or } m=630v+621,\ u\geq 0,\ v\geq 0 \end{cases}$$

Proof: First, consider the Diophantine equation

$24m+6=5\alpha$ for some integer $\alpha \geq 1$,

which states that 5 divides $24m+6$ for some m. The solution of the equation is $m=5s+1$, $s\geq 0$ being an integer. Now, consider the expression for $C(24m+6, 24m+1)$:

$$(24m+6)[\frac{(24m+5)(24m+4)(24m+3)(24m+2)}{2\times 3\times 4\times 5}]$$

$$=(24m+6)[\frac{(24m+5)(6m+1)(8m+1)(12m+1)}{5}].$$

If $m\neq 5s+1$ (so that $24m+6$ is not divisible by 5), then 5 divides one of the four factors in the numerator so that the term inside the square bracket is an integer.

Next, let $m=5s+1$, $s\geq 0$. The expression

$$(24m+6)[\frac{(24m+5)(6m+1)(8m+1)(12m+1)(24m+1)}{5\times 6}]$$

shows that $SS\ (24m+6)\neq 24m$ for any integer $m\geq 1$. So, consider the expression:

$$(24m+6)\Big[\frac{(24m+5)(6m+1)(8m+1)(12m+1)(24m+1)(24m)}{5\times 6\times 7}\Big]$$

$$=(24m+6)\Big[\frac{(24m+5)(6m+1)(8m+1)(12m+1)(24m+1)(4m)}{5\times 7}\Big].$$

If $24m+6$ is not divisible by 7, then the numerator is divisible by 7, and hence, the term inside the square bracket is an integer. So, consider the Diophantine equation

$24m+6=7a$ for some integer $a\geq 1$,

which states that $24m+6$ is a multiple of 7 for some $m\geq 1$. The solution of the equation is $m = 7b + 5$, $b \geq 0$ being an integer. Note that, the coupled equation $5s + 1 = 7b + 5$ has the solution $s=7t+5$, $t\geq 0$ being an integer, so that, $m=5(7t+5)+1=35t+26$, $t\geq 0$.

So, let $m=35t+26$, $t\geq 0$ (so that $24m+6$ is a multiple of 35). Consider the expression:

$$(24m+6)[\frac{(24m+5)(6m+1)(8m+1)(12m+1)(24m+1)(4m)(24m-1)}{5\times 7\times 8}].$$

If m (and hence t) is even, then the term inside the square bracket is an integer.

On the other hand, if t is odd, then $SS(24m + 6) \leq 24m - 3$. Letting $t = 2c + 1$ for some integer $c\geq 0$, we have $m=35(2c+1)+26=70c+61$. Now, consider the expression

$$(24m+6)\Big[\frac{(24m+5)(6m+1)(8m+1)(12m+1)(24m+1)m(24m-1)(24m-2)}{2\times 5\times 7\times 9}\Big]$$

$$=(24m+6)\Big[\frac{(24m+5)(6m+1)(8m+1)(12m+1)(24m+1)m(24m-1)(12m-1)}{5\times 7\times 9}\Big].$$

Clearly, the term inside the square bracket is an integer if 9 divides either m or $8m + 1$, giving rise to the Diophantine equations

$m = 9x$ for some integer $x \geq 1$, $8m + 1 = 9y$ for some integer $y \geq 1$.

The solution of the second equation is $m = 9\beta + 1$, $\beta \geq 0$ being an integer.

The solution of the combined Diophantine equation $70c + 61 = 9x$ is $c = 9u + 8$, $u \geq 0$ being an integer, so that $m = 70(9u + 8) + 61 = 630u + 621$. The solution of the combined equation $70c + 61 = 9\beta + 1$ is $c = 3(3\delta v + 1)$, so that $m = 210(3v + 1) + 61 = 630v + 271$, $v \geq 0$ being an integer.

Note that, Proposition 3.1 is valid for $m = 0$ as well.

Proposition 3.2: Let $m \geq 0$ be any integer. Then,

$$SS(24m+12) = \begin{cases} 24m+7, & if \quad m \neq 5s+2,\ s \geq 0 \\ 24m+6, & if \quad m = 3(5u+4),\ u \geq 0 \\ 24m+5, & if \quad m = 5s+2,\ s \geq 0,\ m \neq 3(t+1),\ t \geq 0,\ m \neq 7v+3,\ v \geq 0 \\ 24m+4, & if \quad m = 140w+17,\ w \geq 0,\ m \neq 3(t+1),\ t \geq 0 \end{cases}$$

Proof: Consider the expression:

$$(24m+12)\left[\frac{(24m+11)(24m+10)(24m+9)(24m+8)}{2\times3\times4\times5}\right]$$

$$= (24m+12)\left[\frac{(24m+11)(24m+10)(8m+3)(3m+1)}{5}\right].$$

Now, consider the Diophantine equation $24m + 12 = 5\alpha$ for some integers α (> 1), which states that 5 divides $24m + 12$ for some m. The equation has the solution $m = 5s + 2$, $s \geq 0$. Therefore, if $m \neq 5s + 2$, then the term inside the square bracket is an integer.

Next, let $m = 5s + 2$, $s \geq 0$. Consider the expression:

$$(24m+12)\left[\frac{(24m+11)(24m+10)(8m+3)(3m+1)(24m+7)}{5\times6}\right]$$

$$= (24m+12)\left[\frac{(24m+11)(12m+5)(8m+3)(3m+1)(24m+7)}{3\times5}\right].$$

Here, the term inside the square bracket is an integer if and only if 3 divides $8m + 3$ (noting that 5 divides one of the five factors in the numerator). Thus, $8m + 3 = 3\beta$ for some integer $\beta \geq 1$. The solution of the equation is $m = 3(t + 1)$, $t \geq 0$ being an integer. The coupled Diophantine equation is $3(t + 1) = 5s + 2$, whose solution is $t = 5u + 3$, so that $m = 3(5u + 3) + 3 = 3(5u + 4)$, $u \geq 0$ being an integer.

Next, let $m = 5s + 2$, $s \geq 0$, $m \neq 3(t + 1)$, $t \geq 0$. Consider the expression:

$$(24m+12)\left[\frac{(24m+11)(12m+5)(8m+3)(3m+1)(24m+7)(24m+6)}{3\times5\times7}\right]$$

$$= (24m+12)\left[\frac{(24m+11)(12m+5)(8m+3)(3m+1)(24m+7)(8m+2)}{5\times7}\right].$$

If 7 does not divide $24m + 12$, then the term inside the square bracket is an integer (since 5 divides one of the factors in the numerator). Thus, the Diophantine equation to be solved is $24m = 7a - 12$, where a (> 0) is an integer. The solution of the equation is $m = 7v + 3$, where $v \geq 0$ is an integer.

Finally, let $m = 5s + 2$, $s \geq 0$, $m = 7v + 3$, $v \geq 0$, $m \neq 3(t + 1)$, $t \geq 0$. In this case, clearly $SS(24m + 12) \leq 24m + 4$. So, consider the expression:

$$(24m+12)\left[\frac{(24m+11)(12m+5)(8m+3)(3m+1)(24m+7)(8m+2)(24m+5)}{5\times7\times8}\right]$$

$$= (24m+12)\left[\frac{(24m+11)(12m+5)(8m+3)(3m+1)(24m+7)(4m+1)(24m+5)}{4\times5\times7}\right].$$

Note that, the term inside the square bracket is an integer if and only if 4 divides $3m + 1$, giving rise to the Diophantine equation $3m + 1 = 4x$ (for some integer $x \geq 1$), whose solution is $m = 4y + 1$, $y \geq 0$ being an integer. Now, the solution of the coupled Diophantine equation $5s + 2 = 7v + 3$ is $s = 7z + 3$, so that $m = 5(7z + 3) + 2 = 35z + 17$ for some integer $z \geq 0$. Next, the combined Diophantine equation is $35z + 17 = 4y + 1$, whose solution is $z = 4w$, so that finally, $m = 140w + 17$.

Some of the values obtained using Propositions 3.1 are listed below:

$SS(54) = 49$, $SS(78) = 73$, $SS(102) = 97$, $SS(126) = 121$, $SS(174) = 169$,
$SS(30) = 23$, $SS(150) = 143$, $SS(270) = 263$, $SS(390) = 383$, $SS(510) = 503$,
$SS(630) = 622$, $SS(2310) = 2302$, $SS(3990) = 3982$,
$SS(6510) = 6501$, $SS(14910) = 14901$, $SS(21630) = 21621$, $SS(30030) = 30021$.

Proposition 3.2 gives the following values:

$SS(12) = 7$, $SS(36) = 31$, $SS(84) = 79$, $SS(108) = 103$, $SS(132) = 127$,
$SS(300) = 294$, $SS(660) = 654$, $SS(1020) = 1014$, $SS(1740) = 1734$,
$SS(60) = 53$, $SS(180) = 173$, $SS(540) = 533$, $SS(780) = 773$,
$SS(420) = 412$, $SS(3780) = 3772$, $SS(10500) = 10492$, $SS(13860) = 13852$.

Lemma 3.2 finds the expression of $SS(24m)$ when m is not a multiple of 5. Thus, the next problem to consider is the function $SS(120m)$. The following result is evident.

Corollary 3.1: $SS(120m) \leq 120m - 6$ for any m (≥ 1).

But the expression

$C(120m, 120m - 6) \equiv$

$$C(120m, 120m-6) \equiv 120m\left[\frac{(120m-1)(120m-2)(120m-3)(120m-4)(120m-5)}{2\times3\times4\times5\times6}\right]$$

$$= 120m\left[\frac{(120m-1)(60m-1)(40m-1)(30m-1)(24m-1)}{6}\right]$$

shows that $SS\ (120m) \neq 120m - 6$ (since none of the five factors in the numerator inside the square bracket is divisible by 2), and hence, $SS\ (120m) \leq 120m - 7$ for any $m \geq 1$. The following lemma proves that the inequality holds with equality sign if and only if m is not divisible by 7.

Lemma 3.3: $SS(120m) = 120m - 7$ for any integer m not divisible by 7.

Proof: Consider the simplified expression for $C(120m, 120m - 7)$:

$C(120m, 120m - 7)$

$$\equiv 120m\left[\frac{(120m-1)(60m-1)(40m-1)(30m-1)(24m-1)(20m-1)}{7}\right].$$

Now, if m is not a multiple of 7, the term inside the square bracket on the right-hand side is divisible by 7.

Using Lemma 3.3, the following values may be obtained :

$SS(120)=113$, $SS(240)=233$, $SS(360)=353$, $SS(480)=473$, $SS(600)=593$.

The functions $SS(129m+6j)$ for $j=1, 2, \ldots$ are given below.

Corollary 3.2: For any integer $m \geq 1$,

(1) $SS(120m+6)=120m+1$, (2) $SS(120m+12)=120m+7$,

(3) $SS(120m+18)=120m+14$, (4) $SS(120m+24)=120m+19$,

(5) $SS(120m+30)=120m+23$, if $m \neq 7s+5$, (6) $SS(120m+36)=120m+31$,

(7) $SS(120m+42)=120m+38$, (8) $SS(120m+48)=120m+43$,

(9) $SS(120m+54)=120m+49$,

(10) $$SS(120m+60)=\begin{cases} 120m+54, & if\ m=3s+2,\ s\geq 0 \\ 120m+53, & if\ m=3s+2,\ s\geq 0,\ m\neq 7t+3,\ t\geq 0 \end{cases}$$

(11) $SS(120m+66)=120m+62$, (12) $SS(120m+72)=120m+67$,

(13) $SS(120m+78)=120m+74$, (14) $SS(120m+84)=120m+79$,

(15) $SS(120m+90)=120m+86$, (16) $SS(120m+96)=120m+91$,

(17) $SS(120m+102)=120m+97$, (18) $SS(120m+108)=120m+103$,

(19) $SS(120m+114)=120m+110$.

Proof: (1) Writing $SS(120m+6)=SS(24(5m)+6)$, by Proposition 3.1,

$$SS(120m+6)=120m+1.$$

Similarly,

$$SS(120m+30)=SS(24(5m+1)+6)=120m+23, \text{ if } m\neq 7s+5,$$
$$SS(120m+54)=SS(24(5m+2)+6)=120m+49,$$
$$SS(120m+78)=SS(24(5m+3)+6)=120m+73,$$
$$SS(120m+102)=SS(24(5m+4)+6)=120m+97,$$

which are respectively parts (5), (9), (13) and (17).

To prove part (2), write $SS(120m+12)=SS(24(5m)+12)$, so that by Proposition 3.2,

$$SS(120m+12)=120m+7.$$

Similarly,

$$SS(120m+36)=SS(24(5m+1)+12)=120m+31,$$
$$SS(120m+84)=SS(24(5m+3)+12)=120m+79,$$
$$SS(120m+108)=SS(24(5m+4)+12)=120m+103,$$

which are parts (6), (14) and (18) respectively. To prove part (10), write

$$SS(120m+60)=SS(24(5m+2)+12).$$

To apply Proposition 3.2, note that the condition therein takes the form
$5m+2=3(5s+4)$ if and only if $m=3s+2$, $s\geq 0$ being an integer.

Therefore, $SS(120m+60)=120m+54$, if $m=3s+2$, otherwise

$$SS(120m+60)=53, \text{ if } m\neq 7s+3.$$

Now, since $SS(120m+18)=SS(24(5m)+18)$, by Lemma 3.1,

$$SS(120m+18)=120m+14.$$

Similarly,

$SS(120m+42) = SS(24(5m+1)+18) = 120m+38,$
$SS(120m+66) = SS(24(5m+2)+18) = 120m+62,$
$SS(120m+90) = SS(24(5m+3)+18) = 120m+86,$
$SS(120m+114) = SS(24(5m+4)+18) = 120m+110.$

Thus, parts (3), (7), (11), (15) and (19) are proved. Finally, by Lemma 3.2,

$SS(120m+24) = SS(24(5m+1)) = 120m+19,$
$SS(120m+48) = SS(24(5m+2)) = 120m+43,$
$SS(120m+72) = SS(24(5m+3)) = 120m+67,$
$SS(120m+96) = SS(24(5m+4)) = 120m+91,$

establishing parts (4), (8), (12) and (16) of the corollary.

An immediate implication of Lemma 3.3 is the following

Corollary 3.3: $SS(840m) \leq 840m-8$ for any m (≥ 1).

However, the expression

$$C(840m, 840m-8)$$
$$\equiv 840m\left[\frac{(840m-1)(840m-2)(840m-3)(840m-4)(840m-5)(840m-6)(840m-7)}{2\times3\times4\times5\times6\times7\times8}\right]$$
$$=840m\left[\frac{(840m-1)(420m-1)(280m-1)(210m-1)(168m-1)(140m-1)(120m-1)}{8}\right]$$

shows that SS $(840m) \neq 840m-8$ for any $m \geq 1$ (since each factor in the numerator inside the square bracket is odd, and hence, not divisible by 2). The following lemma gives the condition under which SS $(840m) = 840m-9$.

Lemma 3.4: For any integer $m \geq 1$, $SS(840m) = 840m-9$ if and only if either $m = 9s+1$, $s \geq 0$, or $m = 9t+2$, $t \geq 0$.

Proof: Consider the following simplified expression for $C(840m, 840m-9)$:

$$840m\left[\frac{(840m-1)(420m-1)(280m-1)(210m-1)(168m-1)(140m-1)(120m-1)(105m-1)}{9}\right].$$

Now, the term inside the square bracket is an integer if and only if 9 divides either $280m-1$, or else $140m-1$. The first condition leads to the Diophantine equation

$280m = 9x+1$ for some integer $x \geq 1$,

whose solution is $m = 9s+1$, $s \geq 0$. The second possibility results in the equation

$140m = 9y+1$ for some integer $y \geq 1$,

with the solution $m = 9t+2$, $t \geq 0$.

Some of the values obtained from Lemma 3.4 are the following:

$SS(840) = 831$, $SS(1680) = 1671$, $SS(8400) = 8391$, $SS(9240) = 9231$.

Lemma 3.5: For any integer $m \geq 1$, $SS(840m) = 840m-10$ if and only if $m = 10s+7$, $s \geq 0$ with $s \neq 9u+3$, $u \geq 0$, or $s \neq 9v+4$, $v \geq 0$.

Proof: Consider $C(840m, 840m-10)$, which in simplified form, is

$$840m\left[\frac{(840m-1)(420m-1)(280m-1)(210m-1)(168m-1)(140m-1)(120m-1)(105m-1)(280m-3)}{3\times10}\right].$$

Now, the problem is to find the condition such that the term inside the square bracket is an integer. First, note that exactly one of $280m-1$, $140m-1$, and $280m-3$ is divisible by 3 (by Corollary 2.2). Thus, it is sufficient to find the condition such that the numerator is divisible by 2×5. Clearly, $105m-1$ is even if and only if m is odd. In addition, $168m-1$ must be divisible by 5. Thus, the resulting Diophantine equations are

$105m = 2\alpha + 1$ for some integer $\alpha \geq 1$, $168m = 5\beta + 1$ for some integer $\beta \geq 1$,

with respective solutions

$m = 2a+1$, $m = 5b+2$ for some integers $a \geq 0$, $b \geq 0$.

The combined Diophantine equation is $2a = 5b+1$, with the solution $a = 5s+3$, and hence

$m = 2(5s+3)+1 = 10s+7$, $s \geq 0$ being an integer.

Next, we have to find the conditions such that m does not satisfy either of the two conditions of Lemma 3.4. These lead to the Diophantine equations

$10s+7 = 9x+1$, $10s+7 = 9y+2$,

respectively, with respective solutions

$s = 9u+3$, $u \geq 0$, $s = 9v+4$, $v \geq 0$.

Lemma 3.6: Let the integer m be such that $m \neq 9u+1$, $u \geq 0$, $m \neq 9v+2$, $v \geq 0$, $m \neq 10w+7$, $w \geq 0$. Then,

$SS(840m) = 840m - 11$, if 11 does not divide m.

Proof: The simplified form of $C(840m, 840m-11)$ is

$$840m\left[\frac{(840m-1)(420m-1)(280m-1)(210m-1)(168m-1)(140m-1)(120m-1)(105m-1)(280m-3)(840m-10)}{3\times 10\times 11}\right]$$

$$= 840m\left[\frac{(840m-1)(420m-1)(280m-1)(210m-1)(168m-1)(140m-1)(120m-1)(105m-1)(280m-3)(84m-1)}{3\times 11}\right]$$

Now, one of $280m-1$, $140m-1$, and $280m-3$ must be divisible by 3 (by Corollary 2.2). Also, if 11 does not divide m, then 11 divides the numerator so that the term inside the square bracket is an integer.

Some of the values obtained using Lemma 3.5 and Lemma 3.6 are the following :

$SS(5880) = 5870$, $SS(14280) = 14270$, $SS(22680) = 22670$, $SS(47880) = 47870$,

$SS(2520) = 2509$, $SS(3360) = 3349$, $SS(4200) = 4189$, $SS(5040) = 5029$.

Lemma 3.7: Let the integer m be of any one of the forms $m = 9s+1$, $s \geq 0$, $m = 9t+5$, $t \geq 0$. Then,

$SS(9240m) = 9240m - 9$.

Proof: Consider the following expression for $C(9240m, 9240m-9)$:

$$9240m\left[\frac{(9340m-1)(9240m-2)(9240m-3)(9240m-4)(9240m-5)(9240m-6)(9240m-7)(9240m-8)}{2\times 3\times 4\times 5\times 6\times 7\times 8\times 9}\right],$$

which simplifies to

$$9240m\left[\frac{(9240m-1)(4620m-1)(3080m-1)(2310m-1)(1848m-1)(1540m-1)(1320m-1)(1155m-1)}{9}\right].$$

Now, the term inside the square bracket is an integer if and only 9 divides either $3080m-1$, or $1540m-1$. Thus, one of the following two Diophantine equations must be satisfied :

$3080m = 9\alpha + 1$ for some integer $\alpha \geq 1$, $1540m = 9\beta + 1$ for some integer $\beta \geq 1$.

The solutions of the above equations are respectively

$m = 9t+5$, $m = 9s+1$ for some integers $t \geq 0$, $s \geq 0$.

Lemma 3.8: Let $m = 10s + 7$, $u \geq 0$, with $s \neq 9x + 3$, $x \geq 0$, $s \neq 9y + 7$, $y \geq 0$. Then,

$SS(9240m) = 9240m - 10$.

Proof: $C(9240m, 9240m - 10)$, in simplified form, is

$$9240m[\frac{(9340m-1)(4620m-1)(3080m-1)(2310m-1)(1848m-1)(1540m-1)(1320m-1)(1155m-1)(9240m-9)}{9\times 10}]$$

$$= 9240m[\frac{(9240m-1)(4620m-1)(3080m-1)(2310m-1)(1848m-1)(1540m-1)(1320m-1)(1155m-1)(3080m-3)}{3\times 10}].$$

Now, the numerator of the term inside the square bracket is divisible by 3 (by virtue of Corollary 2.2). Thus, it is sufficient to find the condition such that the numerator is divisible by 10. But this is possible only if $1155m - 1$ is even and $1848m - 1$ is divisible by 5. Thus, the following two coupled Diophantine equations result :

$1155m = 2\alpha + 1$, $1848m = 5\beta + 1$ for some integers α, $\beta\,(\geq 1)$,

with respective solutions

$m = 2u + 1$, $m = 5v + 2$ for some integers $u \geq 0$, $v \geq 0$.

Next, the combined Diophantine equation $2u = 5v + 1$ is considered, whose solution is $u = 5s + 3$, so that

$m = 2(5s + 3) + 1 = 10s + 7$, $s \geq 0$ being an integer.

Now, by Lemma 3.7, $m \neq 9a + 1$ and $m \neq 9b + 5$. Thus, the two Diophantine equations below are to be considered:

$10s = 9a - 6$, $10s = 9b - 2$, for some integers $a > 0$, $b > 0$.

The solutions of the above equations are $s = 9x + 3$ and $s = 9y + 7$ respectively.

Lemma 3.7 and Lemma 3.8 give the following values.

$SS(9240) = 9231$, $SS(46200) = 46191$, $SS(92400) = 92391$, $SS(129360) = 129351$,

$SS(64680) = 64670$, $SS(157080) = 157070$, $SS(249480) = 249470$.

The expression

$$9240m[\frac{(9240m-1)(4620m-1)(3080m-1)(2310m-1)(1848m-1)(1540m-1)(1320m-1)(1155m-1)(3080m-3)(9240m-10)}{3\times 10\times 11}]$$

$$= 9240m[\frac{(9240m-1)(4620m-1)(3080m-1)(2310m-1)(1848m-1)(1540m-1)(1320m-1)(1155m-1)(3080m-3)(924m-1)}{3\times 11}]$$

shows that $SS(9240m) \neq 9240m - 11$. The following lemma gives a set of necessary and sufficient conditions such that $SS\,(9240m) = 9240m - 12$.

Lemma 3.9: Let the integer m be of any one of the forms $m = 36u + 15$, $u \geq 0$, $m = 36v + 19$, $v \geq 0$, $m = 36w + 23$, $w \geq 0$, with $m \neq 9a + 1$, $a \geq 0$, $m \neq 9b + 5$, $b \geq 0$, $m \neq 10c + 7$, $c \geq 0$. Then,

$SS(9240m) = 9240m - 12$.

Proof: First, $C(9240m, 9240m - 12)$ is simplified as follows:

$$9240m[\frac{(9240m-1)(4620m-1)(3080m-1)(2310m-1)(1848m-1)(1540m-1)(1320m-1)(1155m-1)(3080m-3)(924m-1)(9240m-11)}{3\times 11\times 12}]$$

$$= 9240m[\frac{(9240m-1)(4620m-1)(3080m-1)(2310m-1)(1848m-1)(1540m-1)(1320m-1)(1155m-1)(3080m-3)(924m-1)(840m-1)}{3\times 12}].$$

Now, we have to find the condition such that the term inside the square bracket is an integer. The first condition is that $1155m - 1$ must be divisible by 4, giving rise to the Diophantine equation

$1155m = 4x + 1$ for some integer $x \geq 0$,
whose solution is $m = 4\nu + 3$ for some integer $\nu \geq 0$.
The second necessary condition is that one of $3080m - 1$, $1540m - 1$ and $3080m - 3$ is divisible by 9. Thus, one of the following three equations must hold true :

$3080m = 9\alpha + 1$, $1540m = 9\beta + 1$, $3080m = 9\gamma + 3$ for some integers $\alpha, \beta, \gamma\,(\geq 1)$,

with respective solutions

$m = 9a + 5$, $m = 9b + 1$, $m = 9c + 6$ for some integers $a \geq 0$, $b \geq 0$, $c \geq 0$.

Now, considering the combined Diophantine equation $4\nu + 3 = 9a + 5$, the solution is found to be $\nu = 9u + 5$, so that

$m = 4(9u + 5) + 3 = 36u + 23$, $u \geq 0$.

Next, the coupled equation to be considered is $4\nu + 3 = 9b + 1$, whose solution is $\nu = 9v + 4$, so that

$m = 4(9v + 4) + 3 = 36v + 19$, $v \geq 0$.

Finally, the solution of the coupled Diophantine equation $4\nu + 3 = 9c + 6$ is $\nu = 9w + 3$, so that

$m = 4(9w + 3) + 3 = 36w + 15$, $w \geq 0$.

Some of the values obtained from Lemma 3.9 are as follows:

$SS(138600) = 138588$, $SS(471240) = 471228$, $SS(1136520) = 1136508$,
$SS(1469160) = 1469148$, $SS(1801800) = 1801788$, $SS(2134440) = 2134428$.

Lemma 3.10: Let the integer m be such that $m \neq 13\alpha$, $\alpha \geq 1$, $m \neq 9a + 1$, $a \geq 0$, $m \neq 9b + 5$, $b \geq 0$, $m \neq 10c + 7$, $c \geq 0$, $m \neq 36u + 15$, $u \geq 0$, $m \neq 36v + 19$, $v \geq 0$, $m \neq 36w + 23$, $w \geq 0$. Then,

$SS(9240m) = 9240m - 13.$

Proof: The proof is evident from the simplified expression of $C(9240m, 9240m - 13)$:

$$9240m\left[\frac{(9240m-1)(4620m-1)(3080m-1)(2310m-1)(1848m-1)(1540m-1)(1320m-1)(1155m-1)(3080m-3)(924m-1)(840m-1)(9240m-12)}{3\times 12\times 13}\right]$$

$$= 9240m\left[\frac{(9240m-1)(4620m-1)(3080m-1)(2310m-1)(1848m-1)(1540m-1)(1320m-1)(1155m-1)(3080m-3)(924m-1)(840m-1)(770m-1)}{3\times 13}\right],$$

since, by assumption, the numerator of the term inside the square bracket is divisible by 13.

Lemma 3.10 gives the following values :

$SS(18480) = 18467$, $SS(27720) = 27707$, $SS(36960) = 36947$, $SS(55440) = 55427$.

Next, the function to study is $SS\,(120120m)$, $m \geq 1$ being any integer. Note that

$SS(120120m) \leq 120120m - 9$ for all $m \geq 1$.

Lemma 3.11: Let the integer m be of the form $m = 9s + 7$, $s \geq 0$, or $m = 9t + 8$, $t \geq 0$. Then,

$SS(120120m) = 120120m - 9.$

Proof: Consider the following expression for $C(120120m, 120120m - 9)$:

$$120120m\left[\frac{(120120m-1)(120120m-2)(120120m-3)(120120m-4)(120120m-5)(120120m-6)(120120m-7)(120120m-8)}{2\times 3\times 4\times 5\times 6\times 7\times 8\times 9}\right]$$

$$= 120120m\left[\frac{(120120m-1)(60060m-1)(40040m-1)(30030m-1)(24024m-1)(20020m-1)(17160m-1)(15015m-1)}{9}\right].$$

Now, the term inside the square bracket is an integer if and only 9 divides either $20020m - 1$, or $40040m - 1$, which give rise to the following two Diophantine equations respectively:

$20020m = 9\alpha + 1$, $40040m = 9\beta + 1$ for some integers $\alpha \geq 0$, $\beta \geq 0$.

The above equations have the following solutions respectively :

$m = 9s + 7$, $m = 9t + 8$ for some integers $s \geq 0$, $t \geq 0$.

Lemma 3.12: Let $m = 10s + 9$, $s \geq 0$, with $s \neq 9x + 7$, $x \geq 0$, $s \neq 9y + 8$, $y \geq 0$. Then,

$SS(120120m) = 120120m - 10$.

Proof: $C(120120m, 120120m - 10)$, in simplified form, is

$$120120m[\frac{(120120m-1)(60060m-1)(40040m-1)(30030m-1)(24024m-1)(20020m-1)(17160m-1)(15015m-1)(120120m-9)}{9\times 10}]$$

$$=120120m[\frac{(120120m-1)(60060m-1)(40040m-1)(30030m-1)(24024m-1)(20020m-1)(17160m-1)(15015m-1)(40040m-3)}{3\times 10}].$$

Note that, the numerator of the term inside the square bracket is divisible by 3 (by virtue of Corollary 2.2). Thus, the term inside the square bracket is an integer if and only if $15015m - 1$ is even and $24024m - 1$ is divisible by 5. That is,

$15015m = 2\alpha + 1$, $24024m = 5\beta + 1$ for some integers $\alpha, \beta\ (\geq 1)$,

with the solutions

$m = 2u + 1$, $m = 5v + 4$ for some integers $u \geq 0$, $v \geq 0$.

respectively. Now, the combined Diophantine equation is $2u = 5v + 3$ whose solution is $u = 5s + 4$, so that

$m = 2(5s + 4) + 1 = 10s + 9$, $s \geq 0$ being an integer.

Since, by Lemma 3.11, $m \neq 9a + 7$ and $m \neq 9b + 8$, the two Diophantine equations below are to be considered :

$10s = 9a - 2$, $10s = 9b - 1$, for some integers $a > 0$, $b > 0$.

The solutions of the above equations are $s = 9x + 7$ and $s = 9y + 8$ respectively.

Lemma 3.11 and Lemma 3.12 give the following values.

$SS(840840) = 840831$, $SS(960960) = 960951$, $SS(1921920) = 1921911$.

$SS(1081080) = 1081070$, $SS(2282280) = 2282270$, $SS(3483480) = 3483470$.

The expression

$$120120m[\frac{(120120m-1)(60060m-1)(40040m-1)(30030m-1)(24024m-1)(20020m-1)(17160m-1)(15015m-1)(40040m-3)(120120m-10)}{3\times 10\times 11}]$$

$$=120120m[\frac{(120120m-1)(60060m-1)(40040m-1)(30030m-1)(24024m-1)(20020m-1)(17160m-1)(15015m-1)(40040m-3)(12012m-1)}{3\times 11}]$$

shows that $SS(1241240m) \neq 1241240m - 11$. However, the following lemma can be proved.

Lemma 3.13: Let $m = 36s + 15$, $s \geq 0$ with $m \neq 9a + 7$, $a \geq 0$, $m \neq 9b + 8$, $b \geq 0$, $m \neq 5c + 4$, $c \geq 0$. Then,

$SS(120120m) = 1201200m - 12$.

Proof: Consider $C(9240m, 9240m - 12)$, which is simplified as follows:

$$120120m[\frac{(120120m-1)(60060m-1)(40040m-1)(30030m-1)(24024m-1)(20020m-1)(17160m-1)(15015m-1)(40040m-3)(12012m-1)(120120m-11)}{3\times 11\times 12}]$$

$$=120120m[\frac{(120120m-1)(60060m-1)(40040m-1)(30030m-1)(24024m-1)(20020m-1)(17160m-1)(15015m-1)(40040m-3)(12012m-1)(10920m-1)}{3\times 12}].$$

Now, the problem is to find the condition such that the term inside the square bracket is an integer. The first condition is that $15015m-1$ must be divisible by 4, so that

$$15015m = 4x+1 \text{ for some integer } x \geq 1.$$

The solution of the above equation is $m = 4\nu + 3$ for any integer $\nu \geq 0$.
In addition, $40040m-3$ must be divisible by 9, so that

$$40040m = 9\alpha + 3 \text{ for some integers } \alpha\ (\geq 1),$$

whose solution is

$$m = 9a+6 \text{ for some integer } a \geq 0.$$

Now, the solution of the combined Diophantine equation $4\nu + 3 = 9a + 6$ is $\nu = 9s + 3$, so that

$$m = 4(9s+3)+3 = 36s+15,\ s \geq 0.$$

Some of the values obtained from Lemma 3.13 are as follows :

$$SS(1801800) = 1801788,\ SS(6126120) = 6126108,\ SS(10450440) = 10450428.$$

Lemma 3.14: $SS(120120m) \neq 120120m-13$ for any $m \geq 1$.
Proof: The proof is evident from the following expression of $C(9240m, 9240m-13)$:

$$120120m\left[\frac{(120120m-1)(60060m-1)(40040m-1)(30030m-1)(24024m-1)(20020m-1)(17160m-1)(15015m-1)(40040m-3)(12012m-1)(10920m-1)(120120m-12)}{3\times12\times13}\right]$$

$$=120120m\left[\frac{(120120m-1)(60060m-1)(40040m-1)(30030m-1)(24024m-1)(20020m-1)(17160m-1)(15015m-1)(40040m-3)(12012m-1)(10920m-1)(10010m-1)}{3\times13}\right],$$

since the numerator of the term inside the square bracket is not divisible by 13.

4. Some Remarks

Exploiting the properties of the function $SS\ (n)$ found so far, interesting results may be derived, some of which are given below.
Lemma 4.1: There is an infinite number of integer n such that $\frac{SS(n)}{SS(n+1)} \geq 1$.
Proof: Let $n = 6s-1$ for any $s \geq 1$. Then, $SS(n) = n-2$ (by Lemma 1.1), $SS(n+1) \leq n-3$ (by Corollary 1.1), so that

$$\frac{SS(n)}{SS(n+1)} \geq \frac{n-2}{n-3} > 1.$$

Next, let $n = 6s+1$ for any $s \geq 1$. Then, $SS(n) = n-2$ and (by Lemma 1.2), $SS(n+1) = n-2$, so that

$$\frac{SS(n)}{SS(n+1)} = 1.$$

Note that, in either case, there is an infinite number of n with the given property.
Consider the sequence

$$\frac{SS(5)}{SS(6)}, \frac{SS(11)}{SS(12)}, \frac{SS(17)}{SS(18)}, \frac{SS(23)}{SS(24)}, \frac{SS(29)}{SS(30)}, \frac{SS(35)}{SS(36)}, \dots,$$

that is,

$$\frac{3}{1}, \frac{9}{7}, \frac{15}{14}, \frac{21}{19}, \frac{27}{23}, \frac{33}{31}, \dots,$$

where each term is greater than 1. Also, note that

$SS(7) = 5 = SS(8)$, $SS(13) = 11 = SS(14)$, $SS(19) = 17 = SS(20)$, $SS(25) = 23 = SS(26)$.

It may be mentioned here that, let $n = 6t + 3$ for some integer $t \geq 1$. Then, $SS(n) = n - 2$. Now, since $n + 1 = 6s + 4$ is even, and is not divisible by 3, by Lemma 1.2, $SS(n+1) = n - 2$, so that

$$\frac{SS(n)}{SS(n+1)} = 1.$$

Thus, for example,

$SS(9) = 7 = SS(10)$, $SS(15) = 13 = SS(16)$, $SS(21) = 19 = SS(22)$, $SS(27) = 25 = SS(28)$.

Lemma 4.2: There is an infinite number of integer n such that $\frac{SS(n)}{SS(n+1)} < 1$.

Proof: Let $n = 6s$ for any $s \geq 0$. Then, $SS(n) \leq n - 4$, $SS(n+1) = n - 1$. Therefore,

$$\frac{SS(n)}{SS(n+1)} \leq \frac{n-4}{n-1} < 1.$$

Now, since there is an infinite number of n of the given form, the lemma is proved.

In the sequence

$$\frac{SS(6)}{SS(7)}, \frac{SS(12)}{SS(13)}, \frac{SS(18)}{SS(19)}, \frac{SS(24)}{SS(25)}, \frac{SS(30)}{SS(31)}, \frac{SS(36)}{SS(37)}, \ldots,$$

that is,

$$\frac{1}{5}, \frac{7}{11}, \frac{14}{17}, \frac{19}{23}, \frac{23}{29}, \frac{31}{35}, \ldots,$$

each term is less than 1.

Lemma 4.1 and Lemma 4.2 together show no regular pattern in the behavior of the function $SS(n)$. Moreover,

$$\frac{1}{5} < \frac{7}{11} < \frac{14}{17} < \frac{19}{23}, \text{ but } \frac{19}{23} > \frac{23}{29}.$$

Thus, the sequence $\left\{\frac{SS(n)}{SS(n+1)}\right\}$ is not monotonic.

Lemma 4.3: Let $m = 2s + 1$, $s \geq 0$. Then there is an integer n such that $SS(n) = m$.

Proof: Let $n = 2s + 3$, $s \geq 1$. Then, by Lemma 1.1, $SS(n) = m$.

Lemma 4.3 shows that the odd (positive) integers (including 1) are in the range of the function $SS(.)$.

It has been proved in Islam *et al.* [3] that, there is no solution to the equation $SS(n + 1) = SS(n) + 1$. However, the following results can be proved.

Lemma 4.4: The equation

$$SS(n+1) = SS(n) + 2 \qquad (4.1)$$

has an infinite number of solutions.

Proof: Let the Diophantine equation (4.1) be satisfied. Then, n must be even. Otherwise, n is odd, so that

$SS(n+1) \leq n - 2$, $SS(n) = n - 2$,

violating the equation (4.1). Hence, n must be even with

$SS(n+1) = n-1$, $SS(n) \leq n-3$.

Thus, (4.1) is satisfied if and only if n is such that $SS(n) = n-3$. Then, by Lemma 1.2, n is not divisible by 3.

Let $n = 6m+2$, $m \geq 1$. Then, by Lemma 1.2 and Lemma 1.1,

$SS(n) = n-3$, $SS(n+1) = n-1$,

so that $SS(n+1) = SS(n) + 2$. Since there is an infinite number of integer n of the given form, the lemma is proved.

It may be mentioned here that, taking $n = 6m+4$, $m \geq 1$, by Lemma 1.2 and Lemma 1.1,

$SS(n+1) = n-1$, $SS(n) = n-3$,

so that $SS(n+1) = SS(n) + 2$.

Lemma 4.5: The equation $SS(n+1) = SS(n) + 3$ has an infinite number of solutions.

Proof: Let $n = 24m+18$, where $m \geq 0$ is an integer. By Lemma 3.1 and Lemma 1.1,

$SS(n) = n-4$, $SS(n+1) = n-1$,

so that $SS(n+1) = SS(n) + 3$. Note that, there is an infinite number of integer n of the given form.

Lemma 4.6: The equation $SS(n+1) = SS(n) + 4$ has an infinite number of solutions.

Proof: Let $n = 12m$, $m \geq 1$ is an integer not divisible by 5. Then, by Lemma 1.3 and Lemma 1.1,

$SS(n) = n-5$, $SS(n+1) = n-1$.

The lemma is established with such an infinite number of n.

Lemma 4.7: The equation $SS(n+1) = SS(n) + 5$ has an infinite number of solutions.

Proof: Let $n = 24m+12$, $m = 3(5s+4)$, $s \geq 0$. By Proposition 3.2 and Lemma 1.1,

$SS(n) = n-6$, $SS(n+1) = n-1$.

Thus, the given equation is satisfied with this n. Obviously, there is an infinite number of such n.

Lemma 4.8: The equation $SS(n+1) = SS(n) + 6$ has an infinite number of solutions.

Proof: Let m (≥ 1) be an integer not divisible by 7 with $m \neq 4\alpha + 3$ (for any integer $\alpha \geq 0$) and $m \neq 2(6\beta + 5)$ (for any $\beta \geq 0$). Let $n = 30m$. Then, by Lemma 1.4 and Lemma 1.1,

$SS(n) = n-7$, $SS(n+1) = n-1$.

This n proves the lemma.

The next two lemmas prove that each of the three equations, $SS(n+1) = SS(n) - 1$, $SS(n+1) = SS(n) - 2$, $SS(n+1) = SS(n) - 3$, and $SS(n+1) = SS(n) - 4$ is satisfied for an infinite number of n.

Lemma 4.9: The equation

$$SS(n+1) = SS(n) - 1 \tag{4.2}$$

has an infinite number of solutions.

Proof: First, observe that any integer n satisfying (4.2) must be odd. The proof is as follows : Let n be even, so that

$SS(n) \leq n-3$, $SS(n+1) = n-1$,

violating the equation (4.2). Hence, n must be odd with $SS(n) = n-2$, $SS(n+1) = n-3$.

So, let $n = 24m+17$, $m \geq 0$. Then, by Lemma 3.1,

$SS(n+1) = n-3 = SS(n) - 1$.

Since there is an infinite number of integer n of the given form, the lemma is proved.
The proof of Lemma 3.1 shows that the necessary and sufficient condition that the relationship $SS(n+1) = SS(n) - 1$ holds is that $n = 24m + 17$, $m \geq 0$.

Lemma 4.10: The equation $SS(n+1) = SS(n) - 2$ has an infinite number of solutions.

Proof: With $n = 24m - 1$, $m \geq 1$ being an integer not divisible by 5, by Lemma 3.2,

$$SS(n+1) = n - 4 = SS(n) - 2.$$

The lemma is proved by noting that there is an infinite number of such n.

Lemma 4.11: The equation $SS(n+1) = SS(n) - 3$ has an infinite number of solutions.

Proof: Let $n = 24m + 11$, $m = 3(5s + 4)$, $s \geq 0$. By Proposition 3.2 and Lemma 1.1,

$$SS(n) = n - 2,\ SS(n+1) = n - 5.$$

Thus, the given equation is satisfied with this n, which is infinite in number.

Lemma 4.12: The equation $SS(n+1) = SS(n) - 4$ has an infinite number of solutions.

Proof: Let $n = 30m - 1$, $m \geq 1$ being an integer not divisible by 7, with $m \neq 4\alpha + 3$ (for any integer $\alpha \geq 0$) and $m \neq 2(6\beta + 5)$ (for any integer $\beta \geq 0$). Then, by Lemma 1.4,

$$SS(n+1) = n - 6.$$

Since (by Lemma 1.1), $SS\ (n) = n - 2$, the lemma is established with this n.

Lemma 4.13: The equation $SS(n+2) = SS(n)$ has an infinite number of solutions.

Proof: Let $n = 24m - 2$, $m \geq 1$ being an integer not divisible by 5. Then, by Lemma 3.2,

$$SS(n+2) = n - 3.$$

This, coupled with Lemma 1.2, proves the desired result.

Lemma 4.14: The following results hold :

1. $SS(n) \neq 2$ for any integer $n \geq 1$,
2. $SS(n) \neq 3$ for any integer $n \geq 8$,
3. $SS\ (n) \neq 4$ for any integer $n \geq 10$,
4. $SS\ (n) \neq 5$ for any integer $n \geq 12$,
5. $SS\ (n) \neq 6$ for any integer $n \geq 14$.

Proof: The proof is by part 1 only, the proof being similar in other cases. So, let $SS(n) = 2$ for some integer n. Then, by definition,

$$\frac{(n-1)(n-2)\ ...\ 4\times 3}{2\times 3\times\ ...\ (n-3)(n-2)} \text{ is an integer,} \quad 1$$

and $\frac{(n-1)(n-2)\ ...\ 4}{2\times 3\times\ ...\ (n-3)}$ is not an integer. But this is self-contradictory for $n \geq 6$, since, so $\frac{3}{n-2} < 1$, , so that

$$\frac{(n-1)(n-2)\ ...\ \times 5\times 4\times 3}{2\times 3\times\ ...\ \times(n-3)(n-2)} < \frac{(n-1)(n-2)\ ...\ \times 5\times 4}{2\times 3\times\ ...\ \times(n-3)}.$$

The case $n < 6$ can easily be checked.

The following two lemmas involve $SS\ (n)$ and the divisor function $d(.)$.

Lemma 4.15: The inequality $SS(n) + d(SS(n)) < n$ has an infinite number of solutions.

Proof: Let $n = p + 3$, where $p \geq 5$ is a prime. Then, $SS(n) = p$, $d(SS(n)) = 2$, so that the given inequality is satisfied with this n.

The following results involve the two functions *SS* (n) and $S(n)$, where $S(n)$ is the Smarandache function, introduced by Smarandache [8], and is defined as follows :

$S(n) = \min\{m : n \text{ divides } m!\}$.

Several researchers have studied the function, including Majumdar [9, 10], which also include summaries of other major research works. For a recent survey on $S(n)$, the reader is referred to Huaning [11].

Lemma 4.16: The equation $SS(n) = S(n) - 2$ has an infinite number of solutions.

Proof: For any prime $p \geq 3$, $SS(p) = p - 2$ and $S(p) = p$ (by Lemma 3.1.1 in Majumdar [9]). Thus, the given equation is satisfied when $n = p$.

Lemma 4.17: The equation $SS(n) = 2S(n) - 3$ possesses an infinite number of solutions.

Proof: For any prime $p \geq 5$, $SS(2p) = 2p - 3$, $S(2p) = p$ (by Lemma 3.1.3 in Majumdar [9]). Thus, the given equation is satisfied with this n.

The lemmas below involve the two functions, *SS* (n) and $Z(n)$, where $Z(n)$ is the pseudo-Smarandache function, introduced by Kashihara [12], and defined as follows :

$Z(n) = \min\{m : n \text{ divides } \frac{m(m+1)}{2}\}$.

For details on $Z(n)$, the readers are referred to Majumdar [9, 10], which also provides summaries of the findings of the other researchers. A recent survey on $Z(n)$ is given in Huaning [13].

Lemma 4.18: The equation $SS(n) = 2Z(n) - 3$ has an infinite number of solutions.

Proof: Let the prime $p \geq 3$ be of the form $p = 4x - 1$, $x \geq 1$. Then, by virtue of Lemma 1.2, $SS(2p) = 2p - 3$, while by Corollary 4.2.1 in Majumdar [9], $Z(2p) = p$. Thus, the given equation is satisfied when $n = 2p$. The proof is completed by noting that there is an infinite number of primes of the given form.

Lemma 4.19: The equation $SS(n) = 2Z(n) + 7$ possesses an infinite number of solutions.

Proof: For any integer $k \geq 3$, $SS(2^k) = 2^k - 3$, and $Z(2^k) = 2^{k+1} - 1$ (by Lemma 4.2.2 in Majumdar [9]). Thus, $n = 2^k$ satisfies the given equation.

Lemma 4.20: The equation $SS(n) = 3Z(n) + 1$ admits an infinite number of solutions.

Proof: Let p be a prime of the form $p = 3y + 1$, $y \geq 2$. Note that there is an infinite number of prime of this form (by Lemma 2.5). Now, by Lemma 1.1, $SS(3p) = 3p - 2$ and by Corollary 4.2.1 in Majumdar [9], $Z(3p) = p - 1$. Thus, the given equation is satisfied when $n = 3p$.

Given the function $SS(n)$, let $SS^{(k)}(n)$ be the k-fold composition of $SS(n)$ with itself, that is,

$$SS^{(k)}(n) = \underbrace{SS \circ SS \circ \ldots \circ SS}_{k-fold}(n),$$

defined by

$$SS^{(k)}(n) = \underbrace{SS(SS(\ldots SS))}_{k-fold}(n).$$

Then the following lemmas can be proved.

Lemma 4.21: Let $n \geq 1$ be odd. Then, there is an integer k (> 0) such that $SS^{(k)}(n) = 1$.

Proof: For definiteness, let $n = 2m + 1$ for some integer $m \geq 0$. Then, by Lemma 1.1,

$SS(n) = 2m - 1.$

We now want to show that

$SS^{(j)}(n) = 2m - (2j - 1)$ for all j with $1 \le j \le m$. (4.3)

The proof is by induction on j. The result is clearly true for $j = 1$. Now, assuming its validity for some j (so that $SS^{(j)}(n) = 2m - (2j - 1)$), we get

$SS^{(j+1)}(n) = SS(2m - (2j - 1)) = [2m - (2j - 1)] - 2 = 2m - [(2(j + 1) - 1]$,

which shows that the result is true for $j + 1$ as well. With $j = m$ in (4.3), we get $SS^{(m)}(n) = 1$.

Lemma 4.22: Let n be an even integer not divisible by 3. Then, there exists an integer k (> 0) such that $SS^{(k)}(n) = 1$.

Proof: For such an n, $SS(n) = n - 3$ is odd, and consequently, the result follows by virtue of Lemma 4.21.

Lemma 4.23: Let $n = 24m + 18$, $m \ge 0$ being an integer. Then, there exists an integer k ($>$ 0) such that $SS^{(k)}(n) = 1$.

Proof: For such an n, by Lemma 3.1, $SS(n)$ = n – 4 is even but not divisible by 3. Therefore, by Lemma 1.2, $SS^{(2)}(n)$ = n – 7 which is odd. The rest now follows by Lemma 4.21.

Lemma 4.24: There exists an integer k (> 0) such that $SS^{(k)}(24m) = 1$ for any integer $m \ge 1$.

Proof: By Lemma 3.2, $SS(24m) = 24m - 5$ is odd. Then, the result follows by Lemma 4.21.

Lemma 4.25: There exists an integer k (> 0) such that $SS^{(k)}(24m + 6) = 1$ for any integer $m \ge 1$.

Proof: By Proposition 3.1, $SS(24m + 6)$ is odd. Then, the result follows by Lemma 5.1.

Lemma 4.26: There exists an integer k (> 0) such that $SS^{(k)}(24m + 12) = 1$ for any integer $m \ge 1$.

Proof: By Proposition 3.2, $SS(24m + 12) = 24m + 6$, if $m = 3(5s + 4)$, s ≥ 0 being any integer. But, by Proposition 3.1, $SS^{(2)}(24m + 12)$ is odd. In the remaining cases, $SS(24m + 12)$ is odd. Therefore, applying Lemma 4.21, the result follows immediately.

Lemma 4.27: There exists an integer k (> 0) such that $SS^{(k)}(120m) = 1$.

Proof: If m is not divisible by 7, then by Lemma 3.3, $SS(120m) = 120m - 7$ is odd, and the result follows immediately from Lemma 4.21. Next, consider SS (480m). If the conditions in Lemma 3.4 are satisfied, then $SS(840m) = 840m - 9$ is odd, while if the conditions in Lemma 3.6 are satisfied, $SS(840m) = 840m - 11$ is odd, and in either case, we may apply Lemma 4.21 to get the desired result. On the other hand, $SS(840m) = 840m - 10$ under the conditions of Lemma 3.5. In this case, Lemma 1.2 is applicable, which gives $SS^{(2)}(840m) = 840m - 13$. Now, Lemma 4.21 may be applied to get the desired result.

Lemma 4.28: There exists an integer k (> 0) such that $SS^{(k)}(9240m) = 1$, if 13 does not divide m.

Proof: The proof is similar when m satisfies the conditions of Lemma 3.7 or Lemma 3.8 or Lemma 3.10, and the details are omitted here. When the conditions of Lemma 3.9 are satisfied, then $SS(9240m) = 9240m - 12$. Now, consider the expression below:

$$(9240m - 12)\left[\frac{(9240m - 13)(9240m - 14)(9240m - 15)(9240m - 16)}{2 \times 3 \times 4 \times 5}\right].$$

Thus, $SS(9240m-12)=9240m-5$. Therefore, if the conditions of Lemma 3.9 are satisfied, then $SS^{(2)}(9240m) = 9240m - 17$, which, together with Lemma 4.21, gives the desired result.

5. Conclusion

The Sandor-Smarandache function, $SS(n)$, defined through the equation (1.1), is a recently introduced Smarandache-type arithmetic function. In earlier studies, it has been shown that $SS(n)$ is not multiplicative and is neither increasing nor decreasing. Clearly, the function is not bijective.

It has been found that $SS(n)$ has a simple form when n is odd or when n is even but not divisible by 3. This paper derives the expressions of $SS(n)$, starting with $SS(24m)$, $m \geq 1$. Lemma 3.2 gives the expression of $SS(24m)$ when m is not a multiple of 5. It has been observed that $SS(n)$ depends on the form of n. For example, it has been shown that $SS(24m) \neq 24m-6$ for any $m \geq 1$, though earlier finding proves that $SS(60m) = 60m-6$ if $m = 6s+5$, $s \geq 0$. Lemma 3.3 derives $SS(120m)$ when 7 does not divide m. Again, though for any $m \geq 1$, $SS(840m) \neq 840m-8$, it is found $SS(210m) = 210m-8$ if $m = 8s+3$, $s \geq 0$, or if $m = 2(8t+1)$, $t \geq 0$. Lemma 3.4 and Lemma 3.5 deal with the function $SS(840m)$, while $SS(9240m)$ has been treated in the subsequent four lemmas, from Lemma 3.7 through Lemma 3.10. We observe that the forms of $SS(n)$ depends on the prime factors of n: $SS(n)$ has the simplest form when 2 does not divide n, irrespective of other prime factors of n, and the expression of $SS(n)$ gets more and more complicated when the prime factors 3, 5, 7, 11, … are included in this order in n. We conclude the paper with the following conjecture indicating the new research direction.

Conjecture 5.1: There is an integer k such that $SS^{(k)}(n) = 1$ for any integer $n \geq 1$.

References

1. J. Sandor, Smarandache Notions J. **12**, 247 (2001).
2. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 5th Edition (Oxford University Press, UK, 2002).
3. S. M. S. Islam, H. Gunarto, and A. A. K. Majumdar, J. Sci. Res. **13**, 439 (2021). https://doi.org/10.3329/jsr.v13i2.49965
4. A. A. K. Majumdar, GANIT: J. Bangl. Math. Soc. **39**, 15 (2019). http://doi.org/10.3329/ganit.v39i0.44161
5. S. M. S. Islam and A. A. K. Majumdar, J. Sci. Res. **13**, 73 (2021). https://doi.org/10.3329/jsr.v13i1.47721
6. C. D. Olds, A. Lax, and G. Davidoff, The Geometry of Numbers (The Mathematical Association of America, USA, 2000). https://doi.org/10.5948/UPO9780883859551
7. A. A. Gioia, The Theory of Numbers (Dover Publications Inc., NY, 2001).
8. F. Smarandache, An. Univ. Timisoara, Ser. St. Mat. **18**, 79 (1980).
9. A. A. K. Majumdar, Wandering in the World of Smarandache Numbers (ProQuest Information and Learning, USA 2010).
10. A. A. K. Majumdar, Smarandache Numbers Revisited (Pons Publishing House, Belgium 2018).
11. L. Huaning, Scientia Magna **12**, 132 (2017).

12. K. Kasihara, Comments and Topics on Smarandache Notions and Problems (Erhus University Press, USA 1996).
13. L. Huaning, Scientia Magna **12**, 145 (2017).