

Ranking Fuzzy Numbers with Mean Value of Points and Comparative Reviews

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Abstract

Even though various ranking fuzzy numbers approaches have been presented so far, none of them can always provide satisfactory results in every situation. Some are counterintuitive, and some are inconsistent for the same circumstances. To overcome the issues mentioned above, we present a new technique for ranking fuzzy numbers using the mean value of the two points. The points are taken on the left and the right membership functions (reference functions) such that they divide the respective membership function in the same ratio $m:n$. The proposed technique determines the order of the fuzzy numbers significantly. To use this technique, the membership functions need not be normal and linear.

Keywords: Ranking fuzzy numbers; Mean value of points; Membership functions; Ratio.

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1. Introduction

The importance of prioritizing fuzzy numbers in practice cannot be overstated because the best choice concept is entirely based on their comparison. Therefore, how to make a preference for fuzzy numbers is a major issue. Jain [1] and Dubois and Prade [2] presented the notion of fuzzy numbers and their preference. Many authors have developed fuzzy ranking algorithms to resolve the issue of comparing fuzzy numbers, which yield an ultimately ordered collection or ranking. These approaches include the simple, tricky, and intricate techniques from a single fuzzy number attribute to a set of fuzzy numbers attribute. Some ranking algorithms require the membership function to be normal; however, the normality restriction on the membership function is inadequate in many approaches. Many scholars have recently investigated various techniques for ranking fuzzy numbers, and these approaches are used to study a lot of their applications in various fields such as Multi-Criteria Decision-Making (MCDM), Risk Analysis (RA), Data Analysis (DA), Control, Optimization, and so on. In the existing techniques, authors have presented either an index or a function based on different constituents related to the fuzzy numbers in different cases. Liou and Wang [3] introduced an indexing technique based on the integral value that considers decision makers' attitudes concerning specific purposes. Cheng [4] proposed the distance technique, while Chu and Tsao [5] presented

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the area method for ranking fuzzy numbers. Abbasbandy and Asady [6] proposed an approach to sign distance by using the parametric form of the fuzzy number. Asady and Zendehnam [7] presented a method of distance minimization for ranking fuzzy numbers. Wang and Lee [8] suggested a revision in Chu and Tsao [5] based on the importance of the degree of the representative location of fuzzy numbers on the real line. Abbasbandy and Hajjari [9] defined the magnitude of the trapezoidal fuzzy numbers for the ranking. Nasseri *et al.* [10] presented a method using the fuzzy number's parametric form and the angle between the reference functions to rank the fuzzy numbers. Yu and Dat [11] proposed an improved method for ranking fuzzy numbers with integral values to overcome the shortcomings of Liou and Wang [3]. Chutia and Chutia [12] presented a technique based on value and ambiguity with their defuzzifiers at different heights. Nguyen [13] defined a unified index by multiplying two discriminatory components of a fuzzy number and presented comparative reviews. Chi and Yu [14] proposed ranking generalized fuzzy numbers using centroid and ranking index. Using the ordered weighted averaging technique, Jiang *et al.* [15] studied fuzzy risk analysis based on ranking generalized fuzzy numbers. Mao [16] discussed the ranking of fuzzy numbers using weighted distance. Wang [17] presented relative preference relation-based ranking triangular interval-valued fuzzy numbers. Dombi and Jonas [18] introduced a probability-based fuzzy relation to comparing fuzzy numbers with trapezoidal membership functions. Sen *et al.* [19] suggested a new approach to similarity measure for generalized trapezoidal fuzzy numbers and its application to fuzzy risk analysis. Barazandeh and Ghazanfari [20] proposed a new method for ranking generalized fuzzy numbers, which takes the fuzzy numbers' left and right heights into account. Patra [21] proposed a fuzzy risk analysis method based on ranking generalized trapezoidal fuzzy numbers by considering their mean, area, and perimeter. Prasad and Sinha [22] presented ranking fuzzy numbers using the unified integral value that multiplies two different discriminatory components of the fuzzy number.

The ranking of fuzzy numbers using linear membership functions has been studied extensively in the literature. There are a few methods for ranking fuzzy numbers with nonlinear membership functions, leaving a wide range of scope for further research and studies. In the proposed approach, coordinate points that divide the respective membership functions in the same ratio $m:n$ are obtained based on the formulae for the length of the curve in terms of integral. The mean value of the points is taken as the mean of the abscissa of the two points. The mean value of the points would be the mean value for the fuzzy number and used as the ranking value. The weight of the fuzzy numbers is also taken into account for ranking when the mean values for the two fuzzy numbers are the same. The suggested technique shows noticeable consistency, intuitiveness, and computational easiness. The advantages of the suggested technique are illustrated through numerical examples and comparisons with the published approaches.

Apart from the introduction, the rest of the paper is divided into the following five sections. Preliminaries concerned with the proposed approach are presented in Section 2. The proposed mean value for the fuzzy number and the ranking algorithm is described in

section 3. In Section 4, the mean value for the generalized trapezoidal fuzzy number is derived in simplified form after a quick overview of the basic notion; some attributes of the mean are discussed with proof. Section 5 presents the comparative studies and investigations with some prevailing ranking approaches, done with several model fuzzy numbers that existed in the literature. Conclusions finish the last section 6.

2. Preliminaries

The generalized L-R type fuzzy number and its image are briefly discussed in this section, with references to Liou and Wang [3].

2.1. Generalized L-R type fuzzy number

A fuzzy subset A of the real line R with membership function $f_A(x)$ which satisfies the following conditions for $a, b, c, d \in R, (a \leq b \leq c \leq d)$ is known as a generalized L-R type fuzzy number.

- (i) $f_A(x)$ is a piece-wise continuous function from R to the closed interval $[0, \omega]$ where ω is constant and $0 \leq \omega \leq 1$.
- (ii) $f_A(x) = 0$, for all $x \in]-\infty, a]$.
- (iii) $f_A(x)$ is strictly increasing on $[a, b]$.
- (iv) $f_A(x) = \omega$, for all $x \in [b, c]$.
- (v) $f_A(x)$ is strictly decreasing on $[c, d]$.
- (vi) $f_A(x) = 0$, for all $x \in [d, \infty[$.

The generalized L-R type fuzzy number in Def. 2.1 is conveniently represented as $A = (a, b, c, d; \omega)$, and its membership function $f_A(x)$ is expressed as

$$f_A(x) = \begin{cases} f_A^L(x); & x \in [a, b] \\ \omega; & x \in [b, c] \\ f_A^R(x); & x \in [c, d] \\ 0; & \text{Otherwise} \end{cases}, \quad (1)$$

where $f_A^L(x): [a, b] \rightarrow [0, \omega]$ and $f_A^R(x): [c, d] \rightarrow [0, \omega]$ are known as the left and the right membership functions of the fuzzy number A , respectively. $f_A^L(x)$ is continuous and strictly increasing on $[a, b]$, whereas $f_A^R(x)$ is continuous and strictly decreasing on $[c, d]$.

2.2. Image of a generalized L-R type fuzzy number

The image of a generalized L-R type fuzzy number $A = (a, b, c, d; \omega), 0 \leq \omega \leq 1$ is denoted by A' and defined as $A' = (-d, -c, -b, -a; \omega)$ with its membership function $f_{A'}(x)$ is as follows:

$$f_{A'}(x) = \begin{cases} f_{A'}^L(x); & x \in [-d, -c] \\ \omega; & x \in [-c, -b] \\ f_{A'}^R(x); & x \in [-b, -a] \\ 0; & \text{Otherwise} \end{cases}, \tag{2}$$

Where $f_{A'}^L(x): [-d, -c] \rightarrow [0, \omega]$ and $f_{A'}^R(x): [-b, -a] \rightarrow [0, \omega]$ are known as the left and the right membership functions of A' , respectively. $f_{A'}^L(x)$ is continuous and strictly increasing on $[-d, -c]$, whereas $f_{A'}^R(x)$ is continuous and strictly decreasing on $[-b, -a]$.

3. Mean Value of the Points and Ranking Algorithm

After a brief description of the geometry of the suggested technique, the mean value of the points situated on the membership functions is computed, and the ranking algorithm to discriminate the fuzzy numbers is described.

3.1. Mean value of the points

Let $P_i^L(x_i^L, y_i^L)$ and $P_i^R(x_i^R, y_i^R)$ are the points on the left and the right membership functions of a generalized fuzzy number $A_i = (a_i, b_i, c_i, d_i; \omega_i)$, respectively. The visual depictions of these points are illustrated in Fig. 1.

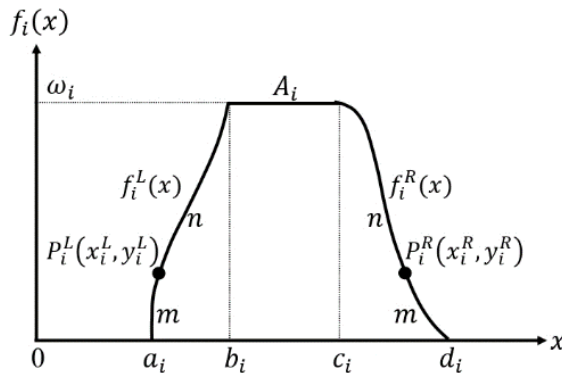


Fig. 1. Visual depictions of the points.

The points divide the corresponding membership functions in the same ratio $m : n$ ($m \neq n$). The coordinates of the points $P_i^L(x_i^L, y_i^L)$ and $P_i^R(x_i^R, y_i^R)$ are obtained by solving the following equations.

$$n \int_{a_i}^{x_i^L} \sqrt{1 + \left(\frac{d f_i^L(x)}{dx}\right)^2} dx = m \int_{x_i^L}^{b_i} \sqrt{1 + \left(\frac{d f_i^L(x)}{dx}\right)^2} dx, \tag{3}$$

$$n \int_0^{y_i^L} \sqrt{1 + \left(\frac{d g_i^L(y)}{dy}\right)^2} dy = m \int_{y_i^L}^{\omega_i} \sqrt{1 + \left(\frac{d g_i^L(y)}{dy}\right)^2} dy, \tag{4}$$

$$m \int_{c_i}^{x_i^R} \sqrt{1 + \left(\frac{d f_i^R(x)}{dx}\right)^2} dx = n \int_{x_i^R}^{d_i} \sqrt{1 + \left(\frac{d f_i^R(x)}{dx}\right)^2} dx, \tag{5}$$

$$m \int_{\omega_i}^{y_i^R} \sqrt{1 + \left(\frac{d g_i^R(y)}{dy}\right)^2} dy = n \int_{y_i^R}^0 \sqrt{1 + \left(\frac{d g_i^R(y)}{dy}\right)^2} dy, \tag{6}$$

where $g_i^L(y)$ and $g_i^R(y)$ are the inverse of the left and the right membership functions $f_i^L(x)$ and $f_i^R(x)$, respectively.

The mean value of the two points $P_i^L(x_i^L, y_i^L)$ and $P_i^R(x_i^R, y_i^R)$ on the left and the right membership functions, respectively, of a generalized L-R type fuzzy number $A_i = (a_i, b_i, c_i, d_i; \omega_i)$ is denoted by $\mathcal{M}(A_i)$ and defined as

$$\mathcal{M}(A_i) = \frac{1}{2}(x_i^L + x_i^R). \tag{7}$$

3.2. Proposed ranking algorithm

Using the mean value, the ranking of the two fuzzy numbers $A_i = (a_i, b_i, c_i, d_i; \omega_i)$ and $A_j = (a_j, b_j, c_j, d_j; \omega_j)$; $i, j = 1, 2, \dots, n$ is defined as follows:

- (i) if $\mathcal{M}(A_i) > \mathcal{M}(A_j)$, then $A_i > A_j$,
- (ii) if $\mathcal{M}(A_i) < \mathcal{M}(A_j)$, then $A_i < A_j$,
- (iii) if $\mathcal{M}(A_i) = \mathcal{M}(A_j)$, then the order of the fuzzy numbers would be as follows:
 - (a) if $\mathcal{M}(A_i). \omega_i > \mathcal{M}(A_j). \omega_j$, then $A_i > A_j$,
 - (b) if $\mathcal{M}(A_i). \omega_i < \mathcal{M}(A_j). \omega_j$, then $A_i < A_j$,
 - (c) if $\mathcal{M}(A_i). \omega_i = \mathcal{M}(A_j). \omega_j$, then $A_i \sim A_j$.

4. Ranking of Generalized Trapezoidal Fuzzy Numbers

In this section, the mean value for the trapezoidal fuzzy numbers is derived in simplified form after a quick overview of their basic notion. The mean value's attributes are stated and proved. Following that, certain observations are labeled as remarks.

4.1. Trapezoidal fuzzy numbers

A generalized L-R type fuzzy number $A_i = (a_i, b_i, c_i, d_i; \omega_i)$ is said to be a generalized trapezoidal fuzzy number, if its membership function $f_i(x)$ is given by

$$f_i(x) = \begin{cases} f_i^L(x) = \omega_i \frac{x-a_i}{b_i-a_i}; & x \in [a_i, b_i] \\ \omega_i; & x \in [b_i, c_i] \\ f_i^R(x) = \omega_i \frac{x-d_i}{c_i-d_i}; & x \in [c_i, d_i] \\ 0; & \text{Otherwise} \end{cases}, \tag{8}$$

For convenience, the trapezoidal fuzzy number is also denoted as $A_i = (a_i, b_i, c_i, d_i; \omega_i)$.

In the case of generalized trapezoidal fuzzy number $A_i = (a_i, b_i, c_i, d_i; \omega_i)$, the points $P_i^L(x_i^L, y_i^L)$ and $P_i^R(x_i^R, y_i^R)$ on the respective left and right membership functions, which divide them in the same ratio $m : n$ are obtained as follows:

$$x_i^L = \frac{mb_i+na_i}{m+n}; y_i^L = \frac{m\omega_i}{m+n}, \tag{9}$$

$$x_i^R = \frac{mc_i+nd_i}{m+n}; y_i^R = \frac{m\omega_i}{m+n}. \tag{10}$$

Hence, from Eq. (7), the mean value $\mathcal{M}(A_i)$ for the generalized trapezoidal fuzzy number $A_i = (a_i, b_i, c_i, d_i; \omega_i)$ is given by

$$\mathcal{M}(A_i) = \frac{1}{2} \left(\frac{mb_i+na_i}{m+n} + \frac{mc_i+nd_i}{m+n} \right). \tag{11}$$

Again let $P_i^{L'}(x_i^{L'}, y_i^{L'})$ and $P_i^{R'}(x_i^{R'}, y_i^{R'})$ are the points on the left and the right membership functions of the partnered image $A_i' = (-d_i, -c_i, -b_i, -a_i; \omega_i)$ of the generalized fuzzy number $A_i = (a_i, b_i, c_i, d_i; \omega_i)$ and they divide the respective membership functions in the ratio $m : n$, then we have

$$x_i^{L'} = \frac{m(-c_i)+n(-d_i)}{m+n}; y_i^{L'} = \frac{m\omega_i}{m+n}, \tag{12}$$

$$x_i^{R'} = \frac{m(-b_i)+n(-a_i)}{m+n}; y_i^{R'} = \frac{m\omega_i}{m+n}. \tag{13}$$

Hence, from Eq. (7), the mean value for the image $A_i' = (-d_i, -c_i, -b_i, -a_i; \omega_i)$ is given by,

$$\mathcal{M}(A_i') = \frac{1}{2} \left\{ \frac{m(-c_i)+n(-d_i)}{m+n} + \frac{m(-b_i)+n(-a_i)}{m+n} \right\}. \tag{14}$$

4.2. Arithmetic operations

Any two generalized trapezoidal fuzzy numbers $A_i = (a_i, b_i, c_i, d_i; \omega_i)$ and $A_j = (a_j, b_j, c_j, d_j; \omega_j)$, $0 \leq \omega_i, \omega_j \leq 1$ have the following arithmetic operations:

(i) Addition

$$A_i \oplus A_j = (a_i, b_i, c_i, d_i; \omega_i) \oplus (a_j, b_j, c_j, d_j; \omega_j), \\ = (a_i + a_j, b_i + b_j, c_i + c_j, d_i + d_j; \min\{\omega_i, \omega_j\}).$$

(ii) Subtraction

$$A_i \ominus A_j = (a_i, b_i, c_i, d_i; \omega_i) \ominus (a_j, b_j, c_j, d_j; \omega_j), \\ = (a_i - a_j, b_i - b_j, c_i - c_j, d_i - d_j; \min\{\omega_i, \omega_j\}).$$

(iii) Multiplication

$$A_i \otimes A_j = (a_i, b_i, c_i, d_i; \omega_i) \otimes (a_j, b_j, c_j, d_j; \omega_j), \\ = (a_i \times a_j, b_i \times b_j, c_i \times c_j, d_i \times d_j; \min\{\omega_i, \omega_j\}).$$

(iv) Division

$$A_i \oslash A_j = (a_i, b_i, c_i, d_i; \omega_i) \oslash (a_j, b_j, c_j, d_j; \omega_j), \\ = \left(\frac{a_i}{a_j}, \frac{b_i}{b_j}, \frac{c_i}{c_j}, \frac{d_i}{d_j}; \min\{\omega_i, \omega_j\} \right).$$

(v) Multiplication by a scalar 'k'

$$kA_i = \begin{cases} (ka_i, kb_i, kc_i, kd_i; \omega_i); & \text{if } k \geq 0 \\ (kd_i, kc_i, kb_i, ka_i; \omega_i); & \text{if } k < 0 \end{cases}.$$

4.3. Mean value's attributes

Property 1. The mean value of the points is a well-defined function (real-valued).

Proof: Let E be the set of all generalized trapezoidal fuzzy numbers and A_i and $A_j \in E, i, j \in \overline{1, n}$. We let $A_i \sim A_j$, then

$$\begin{aligned} A_i \sim A_j &\Rightarrow (a_i, b_i, c_i, d_i; \omega_i) \sim (a_j, b_j, c_j, d_j; \omega_j), \\ &\Rightarrow (a_i, b_i, c_i, d_i; \omega_i) \ominus (a_j, b_j, c_j, d_j; \omega_j) \sim (0, 0, 0, 0; \min\{\omega_i, \omega_j\}), \\ &\Rightarrow (a_i - a_j, b_i - b_j, c_i - c_j, d_i - d_j; \min\{\omega_i, \omega_j\}) \sim (0, 0, 0, 0; \min\{\omega_i, \omega_j\}), \\ &\Rightarrow (n(a_i - a_j), m(b_i - b_j), m(c_i - c_j), n(d_i - d_j); \min\{\omega_i, \omega_j\}) \sim \\ &\hspace{15em} (0, 0, 0, 0; \min\{\omega_i, \omega_j\}), \\ &\Rightarrow n(a_i - a_j) + m(b_i - b_j) + m(c_i - c_j) + n(d_i - d_j) = 0, \\ &\Rightarrow \left(\frac{mb_i + na_i}{m+n} + \frac{mc_i + nd_i}{m+n}\right) - \left(\frac{mb_j + na_j}{m+n} + \frac{mc_j + nd_j}{m+n}\right) = 0, \\ &\Rightarrow \mathcal{M}(A_i) = \mathcal{M}(A_j), \\ &\Rightarrow \text{The mean value of the points is a well-defined function.} \end{aligned}$$

Property 2. The mean value is linear in the set E of all the generalized trapezoidal fuzzy numbers. *i.e.* for any two fuzzy numbers A_i and $A_j \in E, i, j \in \overline{1, n}$, and for any scalar quantity k ,

- (i) $\mathcal{M}(A_i \oplus A_j) = \mathcal{M}(A_i) + \mathcal{M}(A_j)$,
- (ii) $\mathcal{M}(kA_i) = k\mathcal{M}(A_i)$.

Proof. We have $A_i = (a_i, b_i, c_i, d_i; \omega_i)$ and $A_j = (a_j, b_j, c_j, d_j; \omega_j)$,

- (i) By the addition of two fuzzy numbers, we have

$$\begin{aligned} A_i \oplus A_j &= (a_i, b_i, c_i, d_i; \omega_i) \oplus (a_j, b_j, c_j, d_j; \omega_j), \\ &= (a_i + a_j, b_i + b_j, c_i + c_j, d_i + d_j; \min\{\omega_i, \omega_j\}), \\ \mathcal{M}(A_i \oplus A_j) &= \frac{1}{2} \left\{ \frac{m(b_i + b_j) + n(a_i + a_j)}{m+n} + \frac{m(c_i + c_j) + n(d_i + d_j)}{m+n} \right\}, \quad \text{by Eq. (11)} \\ &= \frac{1}{2} \left(\frac{mb_i + na_i}{m+n} + \frac{mc_i + nd_i}{m+n} \right) + \frac{1}{2} \left(\frac{mb_j + na_j}{m+n} + \frac{mc_j + nd_j}{m+n} \right), \\ &= \mathcal{M}(A_i) + \mathcal{M}(A_j). \end{aligned}$$

- (ii) By scalar multiplication of a fuzzy number, we have

$$\begin{aligned} kA_i &= (ka_i, kb_i, kc_i, kd_i; \omega_i), \\ \mathcal{M}(kA_i) &= \frac{1}{2} \left(\frac{mkb_i + nka_i}{m+n} + \frac{mkc_i + nkd_i}{m+n} \right), \quad \text{by Eq. (11)} \\ &= \frac{1}{2} k \left(\frac{mb_i + na_i}{m+n} + \frac{mc_i + nd_i}{m+n} \right), \\ &= k\mathcal{M}(A_i) \hspace{10em} \text{Hence, proved.} \end{aligned}$$

Property 3. If $A_i = (a_i, b_i, c_i, d_i; \omega_i)$ are the trapezoidal fuzzy numbers and $A'_i = (-d_i, -c_i, -b_i, -a_i; \omega_i)$ are their associated images, then,

- (i) $x_i^L = -x_i^{R'}$ and $x_i^R = -x_i^{L'}$,

- (ii) $\mathcal{M}(A_i) = -\mathcal{M}(A'_i)$,
- (iii) $\mathcal{M}(A_i) \geq \mathcal{M}(A_j) \Leftrightarrow \mathcal{M}(A'_i) \leq \mathcal{M}(A'_j)$,
- (iv) $\mathcal{M}(A_i) < \mathcal{M}(A_j) \Leftrightarrow \mathcal{M}(A'_i) > \mathcal{M}(A'_j)$.

Proof.

(i) From Eq. (9), (10) and Eq. (12), (13), we have

$$x_i^L = \frac{mb_i+na_i}{m+n} = -\frac{m(-b_i)+n(-a_i)}{m+n} = -x_i^{R'}$$

and $x_i^R = \frac{mc_i+nd_i}{m+n} = -\frac{m(-c_i)+n(-d_i)}{m+n} = -x_i^{L'}$.

(ii) From Eq. (11), we have

$$\begin{aligned} \mathcal{M}(A_i) &= \frac{1}{2} \left(\frac{mb_i+na_i}{m+n} + \frac{mc_i+nd_i}{m+n} \right), \\ &= -\frac{1}{2} \left[\frac{m(-b_i)+n(-a_i)}{m+n} + \frac{m(-c_i)+n(-d_i)}{m+n} \right], \\ &= -\frac{1}{2} [x_i^{R'} + x_i^{L'}], \\ &= -\frac{1}{2} [x_i^{L'} + x_i^{R'}], \\ &= -\mathcal{M}(A'_i). \end{aligned}$$

(iii) Let $\mathcal{M}(A_i) \geq \mathcal{M}(A_j) \Leftrightarrow -\mathcal{M}(A'_i) \geq -\mathcal{M}(A'_j)$, by (ii)
 $\Leftrightarrow \mathcal{M}(A'_i) \leq \mathcal{M}(A'_j)$.

(iv) Let $\mathcal{M}(A_i) < \mathcal{M}(A_j) \Leftrightarrow -\mathcal{M}(A'_i) < -\mathcal{M}(A'_j)$, by (ii)
 $\Leftrightarrow \mathcal{M}(A'_i) > \mathcal{M}(A'_j)$.

Hence, proved.

Remark 1. Let $A_k = (a_k, b_k, c_k, d_k; \omega_k)$, $k = \overline{1, n}$ are the trapezoidal fuzzy numbers and $A'_k = (-d_k, -c_k, -b_k, -a_k; \omega_k)$ are their respective images. Then, by Prop. 3 and the ranking algorithm in subsection 3.2, the following statements can be made for the pairwise comparison of fuzzy numbers A_i, A_j and their respective images A'_i, A'_j for $i, j \in k$.

- (i) $A_i > A_j$ if and only if $A'_i < A'_j$,
- (ii) $A_i < A_j$ if and only if $A'_i > A'_j$,
- (iii) $A_i \sim A_j$ if and only if $A'_i \sim A'_j$.

Remark 2. A generalized trapezoidal fuzzy number $A_i = (a_i, b_i, c_i, d_i; \omega_i)$, $i = \overline{1, n}$ reduces to a generalized triangular fuzzy number if $b_i = c_i$, represented by $A_i = (a_i, b_i, b_i, d_i; \omega_i)$. The membership function $f_i(x)$ of the triangular fuzzy number is described as

$$f_i(x) = \begin{cases} f_i^L(x) = \omega_i \frac{x-a_i}{b_i-a_i}; & x \in [a_i, b_i] \\ \omega_i; & x = b_i \\ f_i^R(x) = \omega_i \frac{x-d_i}{b_i-d_i}; & x \in [b_i, d_i] \\ 0; & otherwise \end{cases}, \tag{15}$$

5. Comparative Numerical Examples

This section compares the ranking results of the proposed approach with those of several representative approaches using several fuzzy-number examples from the literature that are common for a wide range of comparative studies. According to Remark 1, the ranking values of the images do not need to be presented in the comparative tables for their ranking. In examples 5.1 to 5.5, the detailed explanations of existing approaches in contrast to the proposed approach are subsequently described. For mean value computations, the ratio $m: n = 2: 3$ is utilized throughout the numerical studies.

Example 5.1. Consider the following sets of triangular and trapezoidal fuzzy numbers, taken from Nasseri *et al.* [10].

Set 1: $A_1 = (0.4, 0.5, 0.5, 1; 1)$, $A_2 = (0.4, 0.7, 0.7, 1; 1)$, $A_3 = (0.4, 0.9, 0.9, 1; 1)$.

Set 2: $A_1 = (0.3, 0.4, 0.7, 0.9; 1)$, $A_2 = (0.3, 0.7, 0.7, 0.9; 1)$, $A_3 = (0.5, 0.7, 0.7, 0.9; 1)$.

Set 3: $A_1 = (0.3, 0.5, 0.5, 0.7; 1)$, $A_2 = (0.3, 0.5, 0.8, 0.9; 1)$, $A_3 = (0.3, 0.5, 0.5, 0.9; 1)$.

Set 4: $A_1 = (0, 0.4, 0.7, 0.8; 1)$, $A_2 = (0.2, 0.5, 0.5, 0.9; 1)$, $A_3 = (0.1, 0.6, 0.6, 0.8; 1)$.

Figs. 2 to 5 are the visual representations of the membership functions of the fuzzy numbers in the above four sets of Ex. 5.1. Using formulae in Eq. (11), the mean values of the points for the fuzzy numbers are obtained and displayed in Table 1. The detailed discussions for this example are as below.

Set 1: The fuzzy numbers A_1 , A_2 and A_3 in set-1 are the approximation of 0.5, 0.7, and 0.9, respectively, and their left-right spreads are the same. Therefore, the intuitive ranking will be $A_1 < A_2 < A_3$. From Table 1, the ranking outcome of our proposed method is the same as the intuitive ranking results. Other methods [4-7,9,10,21] in the table also demonstrate the same ranking results $A_1 < A_2 < A_3$. Hence, the proposed method can be used reliably to rank the fuzzy numbers.

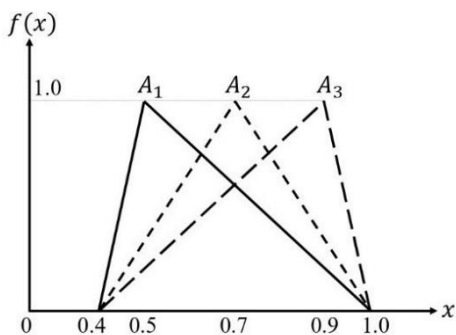


Fig. 2. Fuzzy numbers in set 1 of Example 5.1.

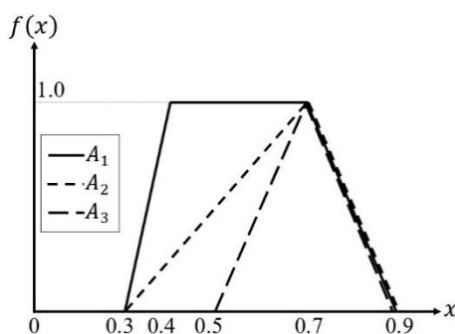


Fig. 3. Fuzzy numbers in set 2 of Example 5.1.

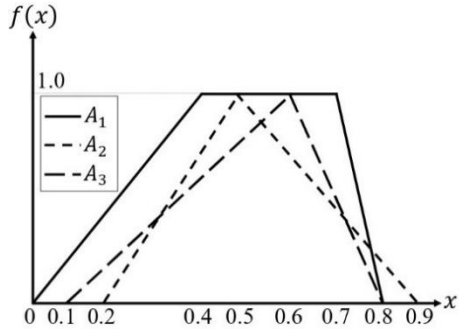
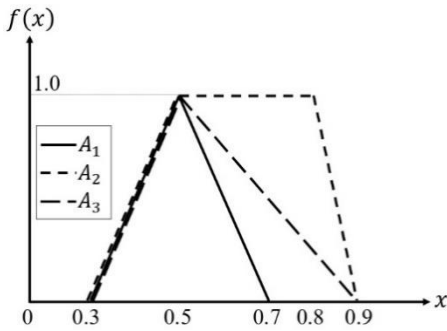


Fig. 4. Fuzzy numbers in set 3 of Example 5.1. Fig. 5. Fuzzy numbers in set 4 of Example 5.1.

Table 1. Comparative ranking orders of the fuzzy numbers in Ex. 5.1.

Author	Fuzzy Number	The ranking order of the fuzzy numbers in the set			
		Set 1	Set 2	Set 3	Set 4
Cheng [4] (Distance)	A_1	0.7901	0.7594	0.7071	0.7015
	A_2	0.8602	0.8150	0.8025	0.7257
	A_3	0.9269	0.8602	0.7458	0.7242
Ranking order		$A_1 < A_2 < A_3$	$A_1 < A_2 < A_3$	$A_1 < A_3 < A_2$	$A_1 < A_3 < A_2$
Abbasbandy and Asady [6] $P = 1$	A_1	1.20	1.15	1.00	0.95
	A_2	1.40	1.30	1.25	1.05
	A_3	1.60	1.40	1.10	1.05
Ranking order		$A_1 < A_2 < A_3$	$A_1 < A_2 < A_3$	$A_1 < A_3 < A_2$	$A_1 < A_2 \sim A_3$
$P = 2$	A_1	0.8869	0.8756	0.7257	0.7853
	A_2	1.0198	0.9522	0.9416	0.7958
	A_3	1.1605	1.0033	0.8165	0.7979
Ranking order		$A_1 < A_2 < A_3$	$A_1 < A_2 < A_3$	$A_1 < A_3 < A_2$	$A_1 < A_2 < A_3$
Asady and Zendehnam [7]	A_1	0.60	0.575	0.50	0.475
	A_2	0.70	0.65	0.625	0.525
	A_3	0.80	0.70	0.55	0.525
Ranking order		$A_1 < A_2 < A_3$	$A_1 < A_2 < A_3$	$A_1 < A_3 < A_2$	$A_1 < A_2 \sim A_3$
Abbasbandy and Hajjari [9]	A_1	0.5333	0.5583	0.5001	0.5250
	A_2	0.70	0.6834	0.6417	0.5083
	A_3	0.8667	0.70	0.5167	0.5750
Ranking order		$A_1 < A_2 < A_3$	$A_1 < A_2 < A_3$	$A_1 < A_3 < A_2$	$A_2 < A_1 < A_3$
Nasseri <i>et al.</i> [10]	A_1	1.6228	1.6281	1.4617	1.3935
	A_2	1.8174	1.7189	1.7281	1.4414
	A_3	2.0228	1.8615	1.5189	1.4447
Ranking order		$A_1 < A_2 < A_3$	$A_1 < A_2 < A_3$	$A_1 < A_3 < A_2$	$A_1 < A_2 < A_3$
K. Patra [21]	A_1	0.60	0.575	0.185	0.475
	A_2	0.70	0.40	0.625	0.296
	A_3	0.80	0.26	0.338	0.298
Ranking order		$A_1 < A_2 < A_3$	$A_3 < A_2 < A_1$	$A_1 < A_3 < A_2$	$A_2 < A_3 < A_1$
Proposed Approach	A_1	0.62	0.58	0.50	0.46
	A_2	0.70	0.64	0.62	0.53
	A_3	0.78	0.70	0.56	0.51
Ranking order		$A_1 < A_2 < A_3$	$A_1 < A_2 < A_3$	$A_1 < A_3 < A_2$	$A_1 < A_3 < A_2$

Set 2: The right spreads of all the three fuzzy numbers in set-2 are the same; therefore, account for the left spreads of A_1 , A_2 and A_3 and their approximate values, the logical ranking outcome will be $A_1 < A_2 < A_3$. From Table 1, the ranking results of the proposed technique are the same as the intuitive outcome. Other methods [4-7,9,10] also produce the same ranking results $A_1 < A_2 < A_3$ but K. Patra [21] demonstrate different ranking outcome as $A_3 < A_2 < A_1$. Hence, the proposed ranking approach has intuitive discrimination strength.

Set 3: The left spreads of all the three fuzzy numbers in set-3 are the same; therefore, on account the right spreads of the fuzzy numbers A_1 , A_2 and A_3 in set-3 and their approximate values, the intuitive and logical ranking results will be $A_1 < A_3 < A_2$. From Table 1, the ranking preferences of the proposed approach are the same as the intuitive outcome. Other methods [4-7,9,10] and Patra [21] also demonstrate the same ranking results $A_1 < A_3 < A_2$. Hence, the proposed method shows strong intuitive discrimination power again.

Set 4: On account of left-right spreads and the approximate values of the fuzzy numbers in set-4, the intuition is not as clear as in the previous examples to guess their preference. From the Table 1, the proposed method yields the ranking result $A_1 < A_3 < A_2$, consistent with Cheng (distance) [4]. Other methods, Abbasbandy and Asady [6], Asady and Zendehnam [7], Abbasbandy and Hajjari [9], Nasseri et al. [10], and K. Patra [21] demonstrate ranking orders differently.

Example 5.2. Consider the following two triangular and three trapezoidal fuzzy numbers, given in Liou and Wang [3], Cheng [4], and Nasseri et al. [10].

$$A_1 = (3, 5, 5, 7; 1), \quad A_2 = (3, 5, 5, 7; 0.8),$$

$$B_1 = (5, 7, 9, 10; 1), \quad B_2 = (6, 7, 9, 10; 0.6), \quad B_3 = (7, 8, 9, 10; 0.4).$$

Fig. 6. presents the visual representation of the membership functions of these fuzzy numbers. From Fig. 6, we can see that the two fuzzy numbers A_1 and A_2 are symmetrical about the line $x = 5$ and have the same support but different weights. Therefore, based on the weight of the fuzzy numbers, the intuitive preference will be $A_1 > A_2$. From Table 2, the ranking results of the proposed method are the same as the intuitive outcome $A_1 > A_2$. The other three methods Wang and Lee [8], Nasseri et al. [10], and Patra [21] demonstrate the same ranking result as $A_1 > A_2$, whereas, [6,7,9] are failed to discriminate and yields $A_1 \sim A_2$.

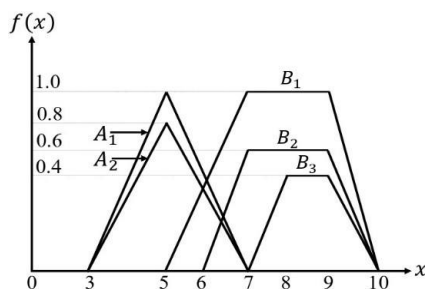


Fig. 6. Visual representation of the fuzzy numbers of Ex. 5.2.

Table 2. Comparative ranking order of the fuzzy numbers of Ex. 5.2.

Author	Ranking value of the fuzzy number					Ranking order
	A_1	A_2	B_1	B_2	B_3	
Abbasbandy and Asady [6] $P = 1$	10.00	10.00	15.50	16.00	17.00	$A_1 \sim A_2; B_1 < B_2 < B_3$
$P = 2$	7.26	7.26	11.26	11.52	12.11	$A_1 \sim A_2; B_1 < B_2 < B_3$
Asady and Zendehnam [7]	5.00	5.00	7.75	8.00	8.50	$A_1 \sim A_2; B_1 < B_2 < B_3$
Abbasbandy <i>et al.</i> [9]	5.00	5.00	7.917	8.00	8.50	$A_1 \sim A_2; B_1 < B_2 < B_3$
Wang and Lee [8]	0.50	0.40	7.714	8.00	8.5	$A_1 > A_2; B_1 < B_2 < B_3$
Nasserri <i>et al.</i> [10]	9.70	9.66	15.34	15.84	16.80	$A_1 > A_2; B_1 < B_2 < B_3$
K. Patra [21]	5.00	4.90	7.75	5.36	2.81	$A_1 > A_2; B_1 > B_2 > B_3$
Proposed method	5.00	4.00	7.70	8.00	8.50	$A_1 > A_2; B_1 < B_2 < B_3$

From Fig. 6, we also see that the fuzzy numbers B_1, B_2 and B_3 have different supports and weights. Wang and Lee [8] improved Chu and Tsao's area method and presented a revised technique, suggesting that the importance of the degree of representative location is higher than the average height and demonstrated the ranking results as $B_1 < B_2 < B_3$. From Table 2, the ranking outcome of the proposed method is consistent with Wang and Lee [8]. The other methods, [6, 7, 9, 10] are also congruent with [8]. K. Patra's method [21] is inconsistent with the others and yields $B_1 > B_2 > B_3$.

Example 5.3. Consider a pair of fuzzy triangular numbers $A_1 = (1, 4, 4, 5)$ and $A_2 = (2, 3, 3, 6)$ which are congruent and overlapped as visualized in Fig. 7. Fuzzy numbers are taken from Nguyen [13]. Their respective images $A'_1 = (-5, -4, -4, -1)$ and $A'_2 = (-6, -3, -3, -2)$ are on the left of the membership axis. It is unclear for intuition to distinguish these fuzzy numbers due to overlapping after flipping and sliding.

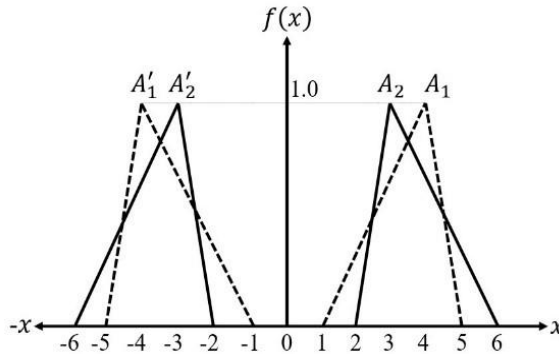


Fig. 7. Visual representation of the fuzzy numbers and their images of Ex. 5.3.

Table 3. Comparative ranking results of the fuzzy numbers in Ex. 5.3.

Author	Ranking value of the fuzzy number				Ranking order
	A_1	A_2	A'_1	A'_2	
Abbasbandy and Asady [6] $P = 1$	7.00	7.00	-7.00	-7.00	$A'_1 \sim A'_2 < A_2 \sim A_1$
$P = 2$	5.2281	5.2281	-5.2281	-5.2281	$A'_1 \sim A'_2 < A_2 \sim A_1$
Asady and Zendehnam [7]	3.50	3.50	-3.50	-3.50	$A'_1 \sim A'_2 < A_2 \sim A_1$
Abbasbandy and Hajjari [9]	3.8334	3.1667	-3.8334	-3.1667	$A'_1 < A'_2 < A_2 < A_1$
Nasseri et al. [10]	6.7764	6.7764	-7.2237	-7.2237	$A'_2 \sim A'_1 < A_1 \sim A_2$
Yu and Dat [11] (Me)	3.4495	3.5506	-3.4495	-3.5506	$A'_2 < A'_1 < A_1 < A_2$
Nguyen [13] $\lambda = 0.5$	11.667	12.833	-11.667	-12.833	$A'_2 < A'_1 < A_1 < A_2$
K. Patra [21]	3.50	3.50	-3.50	-3.50	$A'_1 \sim A'_2 < A_2 \sim A_1$
Proposed Method	3.40	3.60	-3.40	-3.60	$A'_2 < A'_1 < A_1 < A_2$

Using formulae in Eq. (11), the ranking values of both the fuzzy triangular numbers are obtained and displayed in Table 3. The ranking outcomes of our proposed method is $A'_2 < A'_1 < A_1 < A_2$, consistent with the approaches of Yu and Dat [11], and Nguyen [13]. Abbasbandy and Hajjari [9] is inconsistent with the proposed approach and yields $A'_1 < A'_2 < A_2 < A_1$. However, approaches [6,7,10,21] are failed to infer any preference. Hence, the proposed approach can rank the fuzzy numbers and their images in an unclear situation for intuition.

Example 5.4. Consider a triangular fuzzy number $A_1 = (1, 5, 5, 7; 1)$ and a trapezoidal fuzzy number $A_2 = (1, 3, 5, 9; 1)$, The two fuzzy numbers are taken intuitively. Fig. 8. presents the visual representation of the two fuzzy numbers and their associated images $A'_1 = (-7, -5, -5, -1; 1)$ and $A'_2 = (-9, -5, -3, -1; 1)$. Although, the fuzzy numbers $A_1 = (1, 5, 5, 7; 1)$ and $A_2 = (1, 3, 5, 9; 1)$ have different core, and different right spread still, there is a blurred situation for intuition to distinguish them due to left and right overlapping of $A_2 = (1, 3, 5, 9; 1)$ over $A_1 = (1, 5, 5, 7; 1)$. Using formulae in Eq. (11), the ranking scores of these two fuzzy numbers are obtained and displayed in Table 4. The two fuzzy numbers' detailed discussions of comparative ranking results are described below.

The ranking outcome of our proposed method is $A'_2 < A'_1 < A_1 < A_2$, consistent with the Abbasbandy and Asady [6] for ($P = 2$) and a recent approach of Patra [21]. Abbasbandy and Hajjari [9] and Chutia and Chutia [12] distinguish the fuzzy numbers differently and yield a result $A'_1 < A'_2 < A_2 < A_1$ for moderate decision-making attitude ($\alpha = 0.5$). However, other approaches such as Abbasbandy and Asady [6] for ($P = 1$), Asady and Zendehnam [7] and Nasseri et al. [10] are failed to show any preference and yield result $A'_1 \sim A'_2 < A_2 \sim A_1$. As a result, the proposed method can rank fuzzy numbers and their images in an unclear situation for intuition.

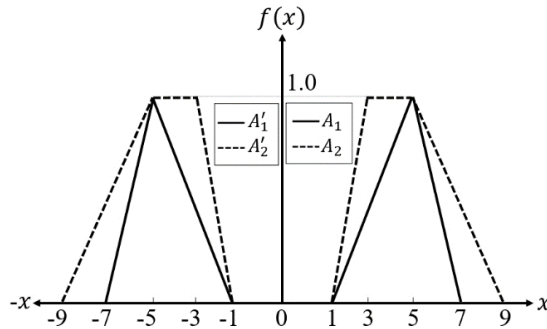


Fig. 8. Visual representation of the fuzzy numbers and their images of Ex. 5.4.

Table 4. Comparative ranking results of the fuzzy numbers of Ex. 5.4.

Author	Ranking value of the fuzzy number				Ranking order
	A_1	A_2	A'_1	A'_2	
Abbasbandy and Asady [6] $P = 1$	9.00	9.00	-9.00	-9.00	$A'_1 \sim A'_2 < A_2 \sim A_1$
$P = 2$	6.83	7.39	-6.83	-7.39	$A'_2 < A'_1 < A_1 < A_2$
Asady and Zendehnam [7]	4.50	4.50	-4.50	-4.50	$A'_1 \sim A'_2 < A_2 \sim A_1$
Abbasbandy and Hajjari [9]	4.83	4.17	-4.83	-4.17	$A'_1 < A'_2 < A_2 < A_1$
Nasseri <i>et al.</i> [10]	8.62	8.62	-9.38	-9.38	$A'_2 \sim A'_1 < A_1 \sim A_2$
Chutia and Chutia [12], $\alpha = 0.5$	3.58	3.17	-3.58	-3.17	$A'_1 < A'_2 < A_2 < A_1$
K. Patra [21]	2.04	4.50	-2.04	-4.50	$A'_2 < A'_1 < A_1 < A_2$
Proposed Method	4.40	4.60	-4.40	-4.60	$A'_2 < A'_1 < A_1 < A_2$

Example 5.5. Considering a triangular fuzzy number $A_1 = (1, 2, 2, 5; 1)$ and a general fuzzy number $A_2 = (1, 2, 2, 4; 1)$ with non-linear membership function $f_{A_2}(x)$, given by

$$f_{A_2}(x) = \begin{cases} f_{A_2}^L(x) = \sqrt{1 - (x - 2)^2} & ; \quad 1 \leq x \leq 2 \\ f_{A_2}^R(x) = \sqrt{1 - \frac{1}{4}(x - 2)^2} & ; \quad 2 \leq x \leq 4 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

taken from Liou and Wang [3]. The visual representation of their membership functions is shown in Fig. 9.

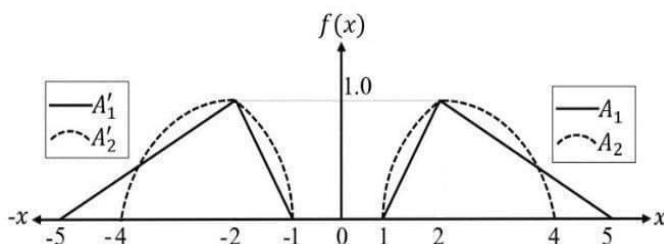


Fig. 9. Visual representation of the fuzzy numbers and their images of Ex. 5.5.

The intuition judgment realizes on $A_1 > A_2$ ($A'_1 < A'_2$) based on the right spreads. For the fuzzy number $A_2 = (1, 2, 2, 4; 1)$, using Eq. (3) and Eq. (5), we have

$$3 \int_1^{x^L} \sqrt{1 + \left(\frac{d f_{A_2}^L(x)}{dx}\right)^2} dx = 2 \int_{x^L}^2 \sqrt{1 + \left(\frac{d f_{A_2}^L(x)}{dx}\right)^2} dx, \tag{16}$$

and $2 \int_2^{x^R} \sqrt{1 + \left(\frac{d f_{A_2}^R(x)}{dx}\right)^2} dx = 3 \int_{x^R}^4 \sqrt{1 + \left(\frac{d f_{A_2}^R(x)}{dx}\right)^2} dx.$ (17)

Solving equations (16) and (17), we get $x^L = 1.1910$ and $x^R = 3.4114$. Substituting these values of x^L and x^R in Eq. (7), the mean value $\mathcal{M}(A_2)$ for the fuzzy number A_2 is obtained and displayed in Table 5. Using the formulae in Eq. (11), the ranking value of the fuzzy number A_1 is obtained and displayed in Table 5. On account of Remark 1, the ranking outcome of our proposed approach is found as $A_1 > A_2$ ($A'_1 < A'_2$) which is in support of intuitive perception. We have considered index approaches from the literature to compare and validate the proposed method's results. From Table 5, we find that the ranking results of the proposed approach coincide with the neutral decision of Liou and Wang [3] and Nguyen [13]. The ranking results of Chutia and Chutia [12] is inconsistent with the proposed approach for all values of the indicator of optimism in the interval $[0, 1]$. Patra [21] also coincides with the proposed approach. Hence, the proposed method is also consistent in discriminating the fuzzy numbers with nonlinear membership functions.

Table 5. Comparative ranking results of the fuzzy numbers in Ex. 5.5.

Author	Ranking value of the fuzzy number				Ranking order
	A_1	A_2	A'_1	A'_2	
Liou and Wang [3]	2.50	2.40	-2.50	-2.40	$A'_1 < A'_2 < A_2 < A_1$
Nguyen [13]	6.67	5.80	-6.67	-5.80	$A'_1 < A'_2 < A_2 < A_1$
Chutia and Chutia [12] $\alpha = 0.5$	1.667	1.7165	-1.667	-1.7165	$A'_2 < A'_1 < A_1 < A_2$
Patra [21]	2.122	1.835	-2.122	-1.835	$A'_1 < A'_2 < A_2 < A_1$
Proposed Method	2.60	2.30	-2.60	-2.30	$A'_1 < A'_2 < A_2 < A_1$

6. Conclusion

The ranking of fuzzy numbers is hindered by inconsistency, counter-intuitiveness, and computational complexity. This paper defines the mean value for the fuzzy number as a ranking tool to reduce this dizziness. According to comparative studies and investigations, the ranking function shows noticeable ranking benefits regarding consistency, intuitive support, and computational easiness. It has four advantages in ordering the fuzzy numbers according to theoretical proofs and comparative reviews. To begin with, the ranking results support human perception. Secondly, it ensures that computation is simple regardless of the type of fuzzy numbers. Thirdly, the proposed method can overcome the limitations of the other methods arising from the compensation of areas. Fourthly, the proposed method gives a justified ranking preference to rank images of the fuzzy numbers. These properties are important in various fields such as Multi-Criteria Decision-Making, Risk Analysis, Data Analysis, and Optimization Techniques.

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