JSR Publications

Available Online

J. Sci. Res. **3** (2), 291-301 (2011)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

On Study of Vertex Labeling of Graph Operations

M. A. Rajan[1*], V. Lokesha[2], and K. M. Niranjan[3]

[1]Innovation Labs, Tata Consultancy Services Limited, Bangalore, India

[2]Department of Mathematics, Acharya Institute of Technology, Bangalore -560090, India

[3]Department of Mathematics, S.J.M.I.T, Chitradurga-577 502, India

Received 10 August 2010, accepted in final revised form 23 March 2011

Abstract

Vertices of the graphs are labeled from the set of natural numbers from 1 to the order of the given graph. Vertex adjacency label set (AVLS) is the set of ordered pair of vertices and its corresponding label of the graph. A notion of vertex adjacency label number (VALN) is introduced in this paper. For each VLS, VLN of graph is the sum of labels of all the adjacent pairs of the vertices of the graph. $\aleph$ is the maximum number among all the VALNs of the different labeling of the graph and the corresponding VALS is defined as maximal vertex adjacency label set $\mathrm{MVALS}_{\aleph}$. In this paper $\aleph$ for different graph operations are discussed.

Keywords: Subdivision; Graph labeling; Direct sum; Direct product.

doi:10.3329/jsr.v3i26222 J. Sci. Res. **3** (2), 291-301 (2011)

1. Introduction

The vertex natural labeling of graphs is introduced in ref. [1-3]. Research in the graph enumeration and graph labeling started way back in 1857 by Arthur Cayley. Graph enumeration is defined as counting number of different graphs of particular type, subgraphs, etc. with graph variants (the number of vertices and edges of the graph). Labeling of graph is assigning labels to the vertices or edges of a graph. Most graph labeling concepts trace their origins to labeling presented by Alex Rosa [1]. Some of graph labeling methods are graceful labeling, harmonious labeling, and coloring of graphs. For the detailed survey on graph labeling see ref. [4]. Graph labeling and enumeration finds the application in chemical graph theory, social networking and computer networking. For example, Cayley demonstrated that the number of different trees of n vertices is analogues to number of isomers of the saturated hydrocarbon with n carbon items C_nH_{2n+2}. More such applications can be found in ref. [1]. This paper is an

[*] *Correponding author*: rajan1729@gmail.com

extension of our previous work [5], in which vertex labeling for graph operations are studied. The following definitions are used abundantly in this paper.

Vertex natural labeling of a graph: A vertex natural labeling of G is a mapping function l which assigns each vertex v of G, an unique number $l(u)$ from the set of natural numbers $N = \{1,2,3,......, p\}$. That is all the vertices are having distinct labels from 1 to p. Thus l is bijective. So there are $p!$ sets of different labeling of a given graph. Each such label set is called vertex adjacency label set (VALS). Vertex adjacency labeling number (VALN) for each VALS is defined as the sum of labels of all the adjacent pairs of the vertices of the graph, which is given by $\sum_{for\ each\ edge(u,v)\in E(G)} l(u)+l(v)$. Let the set $\{VALS_1, VALS_2,........., VALS_{p!}\}$ be all the VALSs of the given graph $G(p,q)$ and the set $\{VALN_1, VALN_2,........., VALN_{p!}\}$ be set of corresponding VALNs. Then $\aleph(G) = Max(VALN_1, VALN_2,........., VALN_{p!})$ is the maximum vertex label number of a given graph $G(p,q)$ and the corresponding VALS is called maximal vertex adjacency label set(MAVLS). Similarly vertex non adjacency labeling number (VNALN) for each VALS is defined as sum of labels of all non adjacent pairs of the vertices of the graph, which is given by $\sum_{for\ each\ edge(u,v)\notin E(G)} l(u)+l(v)$.

Subdivision of a graph: The subdivision of graph of a graph G denoted by $S(G)$ is a graph obtained from G by replacing each of its edge by two series edges by introducing a new vertex into each edge of the G. See the ref. [7] for more details.

Direct Sum of two graphs: Direct sum of two graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ is denoted by $G_1 \oplus G_2$ where $V_1 \cap V_2 = \phi$ is the graph $G(V,E)$ for which $V = V_1 + V_2$ and $E = E_1 + E_2$.

Direct product of two graphs: The direct product of two graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ is denoted by $G_1 \otimes G_2$ is the graph obtained from $G_1 \oplus G_2$ by joining every vertex of $G_1(V_1,E_1)$ with every vertex of $G_2(V_2,E_2)$. Total number of links in complete product of two graphs is $|E_1| + |E_2| + |V_1|\,|V_2|$.

This work is an extension of our paper [5] in which $\aleph$ is computed for various graph operation which includes subdivision, direct sum and product operations. The paper is organized into several sections. In section 2, important results and observations which are useful for deriving the new results are described. In section 3, new problems are stated and results are provided for these problems. Conclusions are given in section 4.

2. Known Results

Most of the graph theory definitions are found in literature [6-8]. We recall some essential results which are required in a sequel from ref. [5].

i. For complete graph, K_p, any order vertex natural label of graphs gives $\aleph$. That is all the p! Vertex labeling of graphs yields same $\aleph$ value.

ii. For a $G(p,r)$ is *r*-regular graph, any order vertex natural label of graphs gives $\aleph$.

iii. For any graph *G*, by assigning the labels from the label set $\{p, p-1,....,1\}$ to the VALS, $\{v_1, v_2,....v_p\}$ respectively, where $d_1 \geq d_2 \geq .... >= d_p$ one of the MVALS and corresponding MVALN is $\aleph(G)$.

iv. For any complete graph with K_p, $\aleph(K_p) = \frac{(p-1)p(p+1)}{2}$.

v. If $G = K_{m,n}$ is a complete bipartite graph with $m \geq n$, then

$$\aleph\left(K_{m,n}\right) = \frac{1}{2}\left[m(m + n)(m + n + 1) - (m - n)m(m + 1)\right].$$

vi. If $G(p,r)$ is a *r*-regular graph , then

(a). $\aleph(G(p,r)) = r\frac{p(p+1)}{2}$.

(b). $\aleph\left(\overline{G(p,r)}\right) = (p-(1+r))\frac{p(p+1)}{2}$.

(c). $\aleph(G(p,r)) + \aleph\left(\overline{G(p,r)}\right) = \aleph(K_p)$.

vii. Let $G(p,q)$ is a graph, whose vertices have degree either *m* or *n*. Let p_m vertices have degree m and p_n vertices have degree *n*. Then

$$\aleph(G(p,q)) = \begin{cases} m\binom{p+1}{2} & for\ m = n \\ m\binom{p+1}{2} - (m-n)\binom{p_n+1}{2} & for\ m \succ n \\ n\binom{p+1}{2} - (n-m)\binom{p_m+1}{2} & for\ m \prec n \end{cases}.$$

3. Results

Theorem 1: Let $H = G - v$ is a graph obtained from *G*, by removing any vertex *v* from *G*. Let $\{v_1, v_2,....v_p\}$ be the vertices of *G* such that $d_1 \geq d_2 \geq .... >= d_p$ and the new vertex sequence $v_1, v_2,..,\ v_{j-1}, v_{i_{j+1}}$, … , v_p is unaltered. Let $v = v_j$ be the vertex, which has degree d_j to be removed from *G*, then (a). $\aleph(H) \le \aleph(G) - \left\{\sum_{k=1}^{j-1} d_k + jd_j\right\}$.

(b). $\aleph(H) = \aleph(G) - \left\{\sum_{k=1}^{j-1} d_k + jd_j\right\} - \sum_{(v_k,v_j)\in E(G),k>j} p-k+1 - \sum_{(v_k,v_j)\in E(G),k<j} p-k+2$.

Proof: (a). From the data, the subgraph *H* is obtained from *G* by removing the vertex v_{i_j}. Let the vertices of *G* $v_1, v_2,..$, v_{j-1}, v_j , … , v_p are labeled as $p, p-1,.p-2,...p-j, p-j-1...,1$ respectively. So in H, the vertices, $v_1, v_2,..$, $v_{j-1}, v_{i_{j+1}}$, … , v_p are re-labeled as $p, p-1, p-2.,...p-j-1,...,1$ respectively. Thus $\sum_{k=1}^{j-1} d_{i_k}$ has to be reduced from $\aleph(G)$. Clearly the contribution of vertex v_j to the summation $\aleph(H)$ is zero. So jd_j needs to be reduced from $\aleph(G)$ on removing the vertex v_j from G, results in removal of edges associated from it. This implies the contribution of labels to the summation from the adjacent vertices of v_j needs to be removed. Thus

$$\aleph(H) \le \aleph(G) - \left\{\sum_{k=1}^{j-1} d_k + jd_j\right\}.$$

Proof: (b). From the above proof, $\aleph(H) \le \aleph(G) - \left\{\sum_{k=1}^{j-1} d_k + jd_j\right\}$. To reduce it to equality, the *RHS* of the inequality further needs to be subtracted. The vertices which are adjacent to v_j can be of two types: vertices with their labels higher or lower than that of v_j. In the above inequality, for the adjacent vertices with their labels lower than that of v_j only their degree are reduced by 1 and where as for the adjacent vertices with their labels.

Theorem 2: Let $S(K_p)$ is a graph obtained by subdivision operation on K_p, then

(a). $$\aleph(S(K_p)) = (p-1)\binom{\binom{p+1}{2}+1}{2} + (p-3)\binom{\binom{p}{2}+1}{2}.$$

Proof: The vertices of $S(K_p)$ is partitioned into, $V_1 = \{v_1, v_2, ...v_p\}$, $V_2 = \left\{v_{p+1}, v_{p+2}, ...v_{p+\binom{p}{2}}\right\}$ with vertices of sets V_1 and V_2 have degree $p-1$ and 2 respectively. $\Rightarrow m = p-1, n = 2, p_m = p, p_n = p\frac{p-1}{2} = \binom{p}{2}, p = \binom{p+1}{2}$. Substituting these values in result **(VII)**,

$$\aleph(S(K_p)) = (p-1)\binom{\binom{p+1}{2}+1}{2} + (p-3)\binom{\binom{p}{2}+1}{2}.$$

(b). $\aleph(\overline{S(K_p)}) = \left(\binom{p}{2} + p - 2\right)\binom{\binom{p+1}{2}+1}{2} + (p-2)\binom{p+1}{2}.$

Proof: The vertices of $\overline{S(K_p)}$ is partitioned into, $V_1 = \{v_1, v_2, \ldots v_p\}$, $V_2 = \left\{v_{p+1}, v_{p+2}, \ldots v_{p+\binom{p}{2}}\right\}$ with vertices of sets V_1 and V_2 have degree $p-1$ and 2 respectively. $\Rightarrow m = \binom{p}{2} + p - 2, n = \binom{p}{2}, p_m = \binom{p}{2}, p_n = p, p = \binom{p+1}{2}$. Substituting these values in the result **(VII)**,

$$\aleph(\overline{S(K_p)}) = \left(\binom{p}{2} + p - 2\right)\binom{\binom{p+1}{2}+1}{2} + (p-2)\binom{p+1}{2}$$

Theorem 3: Let $S(G(p,r))$ is a graph obtained by subdivision operation on *r*-regular graph *G*, then

(a). $\aleph(G(p,r)) = r\binom{\binom{p+1}{2}+1}{2} + (r-2)\binom{\binom{p}{2}+1}{2}.$

(b). $\aleph(\overline{S(G(p,r))}) = \left(pr/2 + p - 3\right)\binom{\binom{p+1}{2}+1}{2} + (p+r-5)\binom{p+1}{2}.$

Proof: (a). The vertices of $S(G(p,r))$ is partitioned into $V_1 = \{v_1, v_2, \ldots v_p\}$,

$V_2=\left\{v_{p+1},v_{p+2},\ldots v_{p+pr/2}\right\}$ with vertices of sets V_1 and V_2 have degree r and 2 respectively. $\Rightarrow m=r, n=2, p_m=p, p_n=pr/2, p=\binom{p+1}{2}$. By substituting these values in result

(VII), $\aleph(S(G(p,r)))=r\begin{pmatrix}\binom{p+1}{2}+1\\2\end{pmatrix}+(r-2)\begin{pmatrix}\binom{p}{2}+1\\2\end{pmatrix}$.

Proof: (b). The vertices of $S(G(p,r))$ is partitioned into, $V_1=\{v_1,v_2,\ldots v_p\}$, $V_2=\left\{v_{p+1},v_{p+2},\ldots v_{p+pr/2}\right\}$ with vertices of sets V_1 and V_2 have degree $\frac{(p-2)(r+2)}{2}+1$ and $\frac{p(r+2)}{2}-3$, respectively.

$\Rightarrow m=\frac{p(r+2)}{2}-3, n=\frac{(p-2)(r+2)}{2}+1, p_m=pr/2, p_n=p, p=\binom{p+1}{2}$.

By substituting these values in the result (**VII)**,

$$\aleph\left(\overline{S(G(p,r))}\right)=\left(pr/2+p-3\right)\begin{pmatrix}\binom{p+1}{2}+1\\2\end{pmatrix}+(p+r-5)\binom{p+1}{2}.$$

□

Theorem 4: Let $S(G(p,q))$ is a graph obtained by subdivision operation on graph G, then

(a). $\aleph(S(G(p,q)))=\aleph(G(p,q))+q(3q+1)$

(b). $\aleph\left(\overline{S(G(p,q))}\right)=(p+q-1)\binom{p+1}{2}+(p+q-3)\left(pq+\binom{q+1}{2}\right)-\aleph(G(p,q))$.

or

$$\aleph\left(\overline{S(G(p,q))}\right)+\aleph(S(G(p,q)))=(p+q-1)\binom{p+1}{2}+(p+q-3)\left(pq+\binom{q+1}{2}\right)-q(3q+1).$$

Proof: Let $\{v_1,v_2,\ldots v_p\}$ be the vertices of G such that $d_1\ge d_2\ge\ldots.>=d_p$. Vertices of $S(G(p,q))$ is partitioned into $V_1=\{v_1,v_2,\ldots v_p\}$ and $V_2=\{u_1,u_2,\ldots u_q\}$ with their degrees $\{d_{i_1},\ldots,d_{i_p}\}$ and $\{2,2,2,\ldots.\}$ upto q, respectively. Label the vertices of $V_2=\{u_1,u_2,\ldots u_q\}$ as $\{1,2,3,\ldots,q\}$ and $V_1=\{v_1,v_2,\ldots v_p\}$ as

$\{q+p, q+p-1, \ldots\ldots, q+1\}$. Then, the contribution of labels of the vertices of $V_2 = \{u_1, u_2, \ldots u_q\}$ and $V_1 = \{v_1, v_2, \ldots v_p\}$ to $\aleph(S(G(p,q)))$ is $\sum_{i=1}^{q} i*2 = q(q+1)$ and $\sum_{i=1}^{p} (q+i)d_{i_{p+1-i}} = 2q^2 + \aleph(G(p,q))$ respectively. By adding these two, we obtain,

$\aleph(S(G(p,q))) = 2q^2 + \aleph(G(p,q)) + q^2 + q$.

Therefore, $\aleph(S(G(p,q))) = \aleph(G(p,q)) + q(3q+1)$.

Note: In general, $\aleph(S_n(G(p,q))) = \aleph(G(p,q)) + (2^n - 1)q(3(2^n - 1)q + 1)$

Proof: (b). Let $\{v_1, v_2, \ldots v_p\}$ be the vertices of G such that $d_1 \geq d_2 \geq \ldots >= d_p$. Vertices of $\overline{S(G(p,q))}$ is partitioned into $V_1 = \{v_1, v_2, \ldots v_p\}$ and $V_2 = \{u_1, u_2, \ldots u_q\}$ with their degrees $\{p+q-1-d_{i_1}, \ldots, p+q-1-d_{i_p}\}$ and $\{p+q-3, p+q-3, \ldots\ldots p+q-3\}$ up to q terms respectively. Note that degrees of the every vertex of $V_2 = \{u_1, u_2, \ldots u_q\}$ are greater than that of degrees of vertices of $V_1 = \{v_1, v_2, \ldots v_p\}$. Label the vertices of $V_2 = \{u_1, u_2, \ldots u_q\}$ as $\{p+1, p+2, \ldots\ldots, p+q+1\}$ and $V_1 = \{v_{i_1}, v_{i_2}, \ldots v_{i_p}\}$ as $\{p, p-1, \ldots 1\}$. Then, contribution of labels of the vertices of $V_2 = \{u_1, u_2, \ldots u_q\}$ and $V_1 = \{v_1, v_2, \ldots v_p\}$ to $\aleph(\overline{S(G(p,q))})$ is

$$\sum_{i=1}^{q} (p+i)*(p+q-3) = (p+q-3)\left(pq + \binom{q+1}{2}\right) \quad \text{and}$$

$$\sum_{i=1}^{p} i*\left(p+q-1-d_{i_{p+1-i}}\right) = (p+q-1)\binom{p+1}{2} - S(G(p,q)), \text{ respectively.}$$

By adding these two,

$$\aleph(\overline{S(G(p,q))}) = (p+q-1)\binom{p+1}{2} + (p+q-3)\left(pq + \binom{q+1}{2}\right) - \aleph(G(p,q)) \quad \text{or}$$

$$\aleph\left(\overline{S(G(p,q))}\right) + \aleph\left(S(G(p,q))\right) = (p+q-1)\binom{p+1}{2} + (p+q-3)\left(pq + \binom{q+1}{2}\right) - q(3q+1).$$

Theorem 5: Let $G_1(p,q)$ is a graph with p points and q edges.

(a). Let $G = G_1(p,q) \oplus G_1(p,q)$, then $\aleph(G) = 2(2\aleph(G_1) - q)$.

(b). Let $G = G_1(p,q) \otimes G_1(p,q)$, then $\aleph(G) = 4\aleph(G_1) + 2p^3 + p^2 - 2q$.

Proof: (a). Graph *G* has *2p* points and *2q* edges. WLOG, the vertices $\{v_1, v_2, \ldots. v_p\}$ of the graph G_1 are labeled with $\{1,2,\ldots., p\}$ respectively and this set is $MVALS_\aleph$. Then the vertices of G $\{v_1, v_2, \ldots. v_p, v_{p+1,} v_{p+2}, \ldots. v_{2p}\}$ are labeled with $\{1,3,5,\ldots 2p-1,2,4,\ldots.2p\}$, respectively and the vertices v_i and v_{i+p} are having same degree d_i . That is the vertices of G_1 which is to the left and right of the operator $\oplus$ are assigned the odd and even labels respectively. This label set of *G* is $MVALS_\aleph$ for *G*.

$$\text{Thus, } \aleph(G) = \sum_{i=1}^{p}(2i-1)d_i + \sum_{i=p+1}^{2p}(2i)d_i = 2\sum_{i=1}^{p} id_i - \sum_{i=1}^{p} d_i + 2\sum_{i=1}^{p} id_i$$

$$= 4\sum_{i=1}^{p} id_i - 2q$$

$$\text{Therefore, } \aleph(G) = 4\aleph(G_1) - 2q = 2(2\aleph(G_1) - q).$$

Proof: (b). Using the (a), and the degree of the vertex v_i of G is $d_i + p$,

$$\aleph(G) = \sum_{i=1}^{p}(2i-1)(d_i + p) + \sum_{i=p+1}^{2p}(2i)(d_i + p)$$

$$= \sum_{i=1}^{p}(2i-1)(d_i + p) + \sum_{i=p+1}^{2p}(2i)(d_i + p)$$

$$= \sum_{i=1}^{p} 2id_i + \sum_{i=1}^{p} 2ip - \sum_{i=1}^{p} d_i - \sum_{i=1}^{p} p + \sum_{i=1}^{p}(2i)d_i + \sum_{i=1}^{p}(2i)p$$

Therefore, $\aleph(G) = 4\aleph(G_1) + 2p^3 + p^2 - 2q$. □

Theorem 6: If $G_1(p_1, r_1)$ and $G_2(p_2, r_2)$ are r_1 and r_2 regular graphs respectively with $r_1 \geq r_2$.

(a). Let $G = G_1(p_1, r_1) + G_2(p_2, r_2)$, then

$$\aleph(G) = \aleph(G_1) + \aleph(G_2) + p_1 p_2 r_1 \text{ or } r_1 \frac{p_1(p_1+1)}{2} + r_2 \frac{p_2(p_2+1)}{2} + p_1 p_2 r_1 .$$

Proof: (a). The vertices and degree sequences of $G_1(p_1, r_1)$ and $G_2(p_2, r_2)$ be $\{v_1, v_2, \ldots v_{p_1}\}$, $\{u_1, u_2, \ldots u_{p_2}\}$ and $\{d_{i_1}, d_{i_2}, \ldots, d_{i_{p_1}}\}$, $\{d_{j_1}, d_{j_2}, \ldots, d_{j_{p_2}}\}$ respectively and they are labeled with $\{1,2,3,\ldots, p_1\}$ and $\{1,2,3,\ldots, p_2\}$, respectively. Note that[1], any label set of a regular graph is $MVALS_{\aleph}$. Since $r_1 \geq r_2$, In *G*, the labels of the vertices of G_2 is unaltered, where as the labels of vertices of G_1 $\{v_1, v_2, \ldots v_{p_1}\}$ are changed to $\{p_2+1, p_2+2, p_2+3, \ldots, p_2+p_1\}$, respectively. Then

$$\aleph(G) = \sum_{k=1}^{p_1} r_1(p_2+k) + \aleph(G_2)$$

$$= \sum_{k=1}^{p_1} r_1 k + \sum_{k=1}^{p_1} r_1 p_2 + \aleph(G_2)$$

$$\aleph(G) = \aleph(G_1) + \aleph(G_2) + p_1 p_2 r_1 \quad \text{or}$$

By substituting, $\aleph(G_1) = r_1 \dfrac{p_1(p_1+1)}{2}, \aleph(G_2) = r_2 \dfrac{p_2(p_2+1)}{2}$.

$$\aleph(G) = r_1 \frac{p_1(p_1+1)}{2} + r_2 \frac{p_2(p_2+1)}{2} + p_1 p_2 r_1 .$$

(b). Let $G = G_1(p_1, r_1) \otimes G_2(p_2, r_2)$, then

$$\aleph(G) = (r_1+r_2)(p_1+p_2)(p_1+p_2+1)/2 ,$$

$$\overline{\aleph(G)} = (p_1+p_2-(1+r_1+r_2))(p_1+p_2)(p_1+p_2+1)/2 .$$

Proof: (b). From the data, *G* is a regular graph with degree $r_1 + r_2$. By using the result (iii),

$$\aleph(G) = (r_1+r_2)(p_1+p_2)(p_1+p_2+1)/2 \text{ and}$$

$$\overline{\aleph(G)} = (p_1+p_2-(1+r_1+r_2))(p_1+p_2)(p_1+p_2+1)/2 . \qquad \square$$

Theorem 7. If $G_1(p_1, q_1)$ is a graph and $G_2(p_2, r_2)$ is r_2-regular graph, then

(a). $$\aleph(G) = \begin{cases} \aleph(G_1) + \aleph(G_2) + p_1 p_2 r, & if \quad G = G_1(p_1, q_1) \oplus G_2(p_2, r_2) \\ \aleph(G_1) + \aleph(G_2) + \dfrac{p_1 p_2}{2}(p_1 + 3p_2 + 2r + 2) & if \quad G = G_1(p_1, q_1) \otimes G_2(p_2, r_2) \end{cases}$$

with $r_2 \geq \delta(G_1)$.

(b). $$\aleph(G)=\begin{cases} \aleph(G_1)+\aleph(G_2)+2p_2q_1, & if \quad G=G_1(p_1,q_1)\oplus G_2(p_2,r_2) \\ \aleph(G_1)+\aleph(G_2)+p_1p_2\left(\frac{p_1+p_2+2}{2}\right) & if \quad G=G_1(p_1,q_1)\otimes G_2(p_2,r_2) \end{cases}$$

with $r_2 \leq \delta(G_1)$.

Proof: (a).
case (i): $r_2 \geq \delta(G_1)$.

Let the vertices $\{v_1, v_2, \ldots v_{p_1}\}$ of G_1 and vertices of G_2 $\{u_1, u_2, \ldots u_{p_2}\}$ are labeled with $\{1,2,3,\ldots,p_1\}$ and $\{1,2,3,\ldots,p_2\}$ respectively, which gives $\aleph(G_1)$ and $\aleph(G_2)$ for G_1 and G_2 respectively. To obtain $\aleph(G)$, In G, the vertices of the $G_1(p_1,q_1)$ and $G_2(p_2,r_2)$ are labeled with $\{1,2,3,\ldots,p_1\}$ and $\{p_1+1, p_1+2, p_1+3,\ldots,p_1+p_2\}$, respectively.

Let $G = G_1 \oplus G_2$.

Thus to $\aleph(G)$, the contribution of G_1 and G_2 is $\aleph(G_1)$ and $\aleph(G_2)+\sum_{i=1}^{p_2} d(u_i)\, p_1$ respectively.

$$\therefore\ \aleph(G)=\aleph(G_1)+\aleph(G_2)+p_1\sum_{i=1}^{p_2} d(u_i)$$
$$\therefore\ \aleph(G)=\aleph(G_1)+\aleph(G_2)+p_1p_2r\,.$$

Proof: (b). If $G = G_1 \otimes G_2$. Then degrees of the vertices of G_1 and G_2 are increased by p_2 and p_1, respectively. Thus in $\aleph(G)$, the contribution of the vertex v, u of G_1 and G_2 is
$l(v)(d(v)+p_2), (l(u)+p_1)(r+p_1)$, respectively.

$$\therefore\ \aleph(G)=\aleph(G_1)+\sum_{i=1}^{p_1} l(v_i)\, p_2+\aleph(G_2)+\sum_{i=1}^{p_2} l(u_i)\, p_1+p_1\sum_{i=1}^{p_2}(r+p_2)\cdot$$
$$\therefore\ \aleph(G)=\aleph(G_1)+\aleph(G_2)+p_2\left(p_1(p_1+1)/2\right)+p_1\left(p_2(p_2+1)/2\right)+p_1p_2(r+p_2)\cdot$$
$$\therefore\ \aleph(G)=\aleph(G_1)+\aleph(G_2)+\frac{p_1p_2}{2}(p_1+3p_2+2r+2)\cdot$$

Case ii: $r_2 \leq \delta(G_1)$. Let the vertices $\{v_1, v_2, \ldots v_{p_1}\}$ of G_1 and vertices $\{u_1, u_2, \ldots u_{p_2}\}$ of G_2 are labeled with $\{1,2,3,\ldots,p_1\}$ and $\{1,2,3,\ldots,p_2\}$ respectively, which gives $\aleph(G_1)$ and $\aleph(G_2)$ for G_1 and G_2, respectively. To obtain $\aleph(G)$, in G, the vertices of the $G_2(p_2,r_2)$

and $G_1(p_1, q_1)$ are labeled with $\{1,2,3,..., p_2\}$ and $\{p_2+1, p_2+2, p_2+3,..., p_2+p_1\}$, respectively.

Let $G = G_1 \oplus G_2$.

Thus in $\aleph(G)$, the contribution of G_1 and G_2 is $\aleph(G_1) + \sum_{i=1}^{p_1} p_2 d(v_i)$, $\aleph(G_2)$ and respectively.

$\therefore \aleph(G) = \aleph(G1) + \aleph(G2) + 2p_2q_1$.

b) Let $G = G_1 \otimes G_2$. Then degrees of the vertices of G1 and G_2 are increased by p_2 and p_1 respectively. Thus in $\aleph(G)$, the contribution of the vertex *v*, *u* of G_1 and G_2 is $l(v)(d(v)+p_2), l(u)(r+p_1)$, respectively.

$$\therefore \aleph(G) = \aleph(G_1) + \sum_{i=1}^{p_1} l(v_i)\, p_2 + \aleph(G_2) + \sum_{i=1}^{p_1} rp_2$$

$$\therefore \aleph(G) = \aleph(G_1) + \aleph(G_2) + p_2\left(p_1(p_1+1)/2\right) + p_1\left(p_2(p_2+1)/2\right)$$

$$\therefore \aleph(G) = \aleph(G_1) + \aleph(G_2) + p_1p_2\left(\frac{p_1+p_2+2}{2}\right). \qquad \square$$

4. Conclusion

In this paper, $\aleph$ for the various graph operations are derived. The graph labeling finds applications in resource allocation, channel allocation in computer networks and communication. Exploring the application of graph labeling on graph operation is under way. As part of continuity to this work, a new label adjacency matrix representation of the graph and study of its spectra properties are under progress.

References

1. A. Rosa, *in:* Theory of Graphs *(*Internat. Symposium, Rome, 1966) (Gordon and Breach, New York and. Dunod Paris, 1967) pp 349-355.
2. L. W. Beineke and S. M. Hegde, Discuss.Math. Graph Theory **21**, 63 (2001).
3. A. F. Beardon, Austral. J. Combin. **30**, 113 (2004).
4. J. A. Gallian, The electronic journal of combinatorics **17** (2010), #DS6. http://www.combinatorics.org/Surveys/ds6.pdf.
5. M. A. Rajan, V. Lokesha, and P.S. Ranjini, Int. e-Journal Math. Eng. **2** **(**2) 992 (2011) http://internationalejournals.com/vol2_iss2-a109.html
6. F. Harary, Graph Theory (Addison Wesley, Reading, Mass, 1969).
7. M. A. Rajan, V. Lokesha, and P. S. Ranjini, Proc. 23rd joint Iran-South Korea Jang Math. Soc., Iran, 23 (2010).
8. P. S. Ranjini, V. Lokesha, and M. A. Rajan, Int. J. Math. Sc. Eng. Appl. **4**, 221 (2010).