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Stability of Axially Symmetric Cosmological Model in Lyra Geometry

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Abstract

The present study concerns an examination of a cosmological model that exhibits axially symmetric and incorporates perfect fluid within the context of Lyra geometry. The field equations are solved by utilising the relation between the metric coefficients and the equation of state for a stiff fluid. The physical and kinematic properties, namely the Hubble parameter, the deceleration parameter, the energy density, and the pressure, have been examined, and their graphical behaviour has been analysed. Furthermore, we examined several additional parameters, including the redshift and the Om(z) diagnostic, and verified the stability of the model by utilising the sound speed ratio and perturbation technique.

Keywords: Axially symmetric space-time; Lyra Geometry; Perfect fluid.

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1. Introduction

Since the introduction of Einstein's theory of gravitation, attempts have been made to unify the field theories; such a theory would be required to generalize the usual Riemannian space-time. Weyl [1] made one of the best attempts in this direction. He proposed a more general theory in which electromagnetism is also described geometrically. Later, Lyra [2] suggested a modification of Riemannian geometry, which may also be considered a modification of Weyl's geometry, by introducing a gauge function into the structure less manifold, which removes the non-integrability condition of the length of a vector under parallel transport and a cosmological constant is naturally introduced from the geometry. In the subsequent investigations, Sen [3], Dunn *et al.* [4] formulated a new scalar-tensor theory of gravitation and constructed an analog of Einstein's field equations based on Lyra's geometry.

Halford [5] says that the current theory predicts the same effects within the limits of what can be seen when it comes to the classic tests of the solar system. Soleng [6] has pointed out that the constant displacement field in Lyra's geometry will either include a creation field and be equal to Hoyle's creation field in cosmology [7-9] or contain a

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special vacuum field, which together with the gauge vector term may be considered a cosmological term.

The field equations (in geometrized units for which $c = 1$, $G = 1$), in normal gauge for Lyra's manifold, obtained by Sen [3] as

$$R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\varphi_i\varphi_j - \frac{3}{4}g_{ij}\varphi^k\varphi_k = -8\pi T_{ij} \tag{1}$$

where φ_i is the displacement vector field and defined as $\varphi_i = (\beta(t),0,0,0)$ and Conservation law $T_{ij} = 0$ (2)

Agrawal *et al.* [10] studied vacuum solutions of *FRW* and axially symmetric space-time in *f(R)* theory of gravity, Mete *et al*. [11] investigated an axially symmetric non-static space-time in the presence of bulk stress in the scalar-tensor theory formulated by Saez and Ballester, Sahoo *et al*. [12] studied an axially symmetric cosmological model in *f(R, T)* gravity in the presence of perfect fluid. Vinutha *et al.* [13] studied the Kantowski–Sachs perfect fluid cosmological model in R2-Gravity. Hadole *et al*. [14] studied Bianchi type VI_0 string cosmological model in Lyra's manifold. Pradhan *et al.* [15] examined some exact Bianchi type-V cosmological models in scalar tensor theory : kinematic tests. Reddy *et al*. [16] studied axially symmetric cosmic strings and domain walls in Lyra geometry. Adhau *et al*. [17] investigated an axially symmetric non-static space-time in the presence of thick domain walls with scalar-tensor theories formulated by Brans and Dicke. Sharma *et al.* [18] presented a comparative study of transit FRW and axially symmetric cosmological models in the framework of domain walls in *f(R, T)* modified theory of gravity considering the time-dependent deceleration parameter. Hegazy *et al*. [19] studied Bianchi's type VI_0 model in Lyra geometry. Yadav *et al.* [20] investigated a model of the quintessence universe with the dominance of dark energy in Lyra geometry. Aditya *et al*. [21] studied the Bianchi type-IX model in the presence of anisotropic dark energy and a massive scalar meson field in Lyra geometry. Mollah *et al.* [22] investigated the behavior of viscous fluid in string cosmological models within the framework of Lyra geometry. Reddy [23] static and non-static plane-symmetric string cosmological models are obtained in the Lyra manifold.

Many authors have studied the stability of cosmological models using various techniques, some of them are Nimkar *et al.* [24], Geovanny *et al*. [25], Katore *et al.* [26, 27], Shah *et al.* [28], Ahmed *et al*. [29], Sharif *et al.* [30], Knutsen [31,32], and Wanas *et al.* [33], Vinutha *et al*. [34], Koussour *et al.* [35], Yadav *et al.* [36], Chiang-Mei Chen *et al.* [37], and Saha *et al*. [38]. Also, Santhi *et al.* [39], Thakur [40], Zhai *et al*. [41], and Debnath *et al*. [42] proposed a new parameter called the $Om(z)$ diagnostic.

Drawing inspiration from the previously mentioned works, this research investigates an axially symmetric cosmological model with perfect fluid in the presence of Lyra geometry. The work is organised as follows: after the introduction, Section 2 provides a description of the metric and field equations. Section 3 is devoted to the solution of the field equations. Section 4 is dedicated to the examination of the physical and kinematical properties of the model. Section 5, Stability Analysis, and the last section contain some conclusions.

2. Metric and Field Equations

Consider the axially symmetric space-time given by Bhattacharya and Karade [43] in the form

$$ds^2 = dt^2 - A^2(t)(d\chi^2 + f^2(\chi)d\varphi^2) - B^2(t)dz^2 \quad (3)$$

Where A, B are functions of the proper time t alone while f is a function of co-ordinate χ alone.

The energy-momentum tensor for perfect fluid is given by

$$T^{ik} = (p+\rho)u^i u^k - pg^{ik} \quad (4)$$

Here p is the pressure, ρ is the energy density, and u_i is the four-velocity vectors of the distribution, respectively.

From Eq. (4), we have

$$T_1^1 = T_2^2 = T_3^3 = -p \text{ and } T_4^4 = \rho \quad (5)$$

The trace of energy-momentum tensor is given by

$$T = T_1^1 + T_2^2 + T_3^3 + T_4^4 = -3p + \rho$$

Using the equations (1), (2), (4), and (5), the field equations of metric (3) are

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4}{A}\frac{B_4}{B} + \frac{3}{4}\beta^2 = -8\pi p \quad (6)$$

$$\frac{2A_{44}}{A} + \left(\frac{A_4}{A}\right)^2 - \left(\frac{f_{11}}{f}\right)\frac{1}{A^2} + \frac{3}{4}\beta^2 = -8\pi p \quad (7)$$

$$\frac{2A_4}{A}\frac{B_4}{B} + \left(\frac{A_4}{A}\right)^2 - \left(\frac{f_{11}}{f}\right)\frac{1}{A^2} - \frac{3}{4}\beta^2 = 8\pi\rho \quad (8)$$

$$\rho_4 + (p+\rho)\frac{B_4}{B} = 0 \quad (9)$$

Where the subscript '4' denote ordinary differentiation with respect to t. Together with equations (7) and (8), the functional dependency of the metric provides

$$\frac{f_{11}}{f} = k^2, k^2 = \text{Constant}$$

If $k = 0$, then $f(\chi) = (\text{constant } f)\chi$, $0 < \chi < \alpha$

By selecting the appropriate units for φ, it is possible to set this constant to 1. Thus, equations (7) and (8) will be transformed into the following as $f(\chi) = \chi$ results in the flat model of the universe [16].

$$\frac{2A_{44}}{A} + \left(\frac{A_4}{A}\right)^2 + \frac{3}{4}\beta^2 = -8\pi p \quad (10)$$

$$\frac{2A_4}{A}\frac{B_4}{B} + \left(\frac{A_4}{A}\right)^2 - \frac{3}{4}\beta^2 = 8\pi\rho \quad (11)$$

3. Solutions of Feld Equations

The field equations (6), (10), and (11) are three equations in five unknowns $A, B, \beta, p \& \rho$. Hence to get a determinate solution, one has to assume the relation between metric coefficients, i.e., $A = B^n$ and the condition of stiff fluid $\rho = p$ gives,

$$\frac{B_{44}}{B} + 2n\frac{B_4}{B} = 0 \tag{12}$$

The above equation admits an exact solution given by

$$A = N^n\{k_1 t + k_2\}^{n/2n+1} \tag{13}$$

$$B = N\{k_1 t + k_2\}^{1/2n+1} \tag{14}$$

Also,

$$p = -\frac{(3n+1)}{32\pi}\left[\frac{2nk_3{}^2 + k_4}{\{k_1 t + k_2\}^2}\right] \tag{15}$$

$$\rho = \frac{1}{8\pi}\left[\frac{(n+3)k_4 - n^2 k_3{}^2}{\{k_1 t + k_2\}^2}\right] \tag{16}$$

and $\beta^2 = \frac{4}{3}\left\{\frac{2n(n+1)k_3{}^2 - (n+3)k_4}{\{k_1 t + k_2\}^2}\right\}$ (17)

The axially symmetric cosmological model in equation (3) takes the form

$$ds^2 = dt^2 - N^{2n}(k_1 t + k_2)^{2n/2n+1}[d\chi^2 + f^2(\chi)d\varphi^2] - N^2(k_1 t + k_2)^{2/2n+1}dz^2$$

4. Physical and Kinematical Properties of the Model

In this section, some of the important physical parameters are given. Expansion Scalar, Hubble parameter, spatial Volume, Shear Scalar, and deceleration parameter are given by

Expansion Scalar: $\theta = \frac{k_1}{3(k_1 t + k_2)}$ (18)

Hubble Parameter: $H = \frac{k_1}{(k_1 t + k_2)}$ (19)

Spatial Volume: $V = \sqrt{-g} = N^{2n+1}(k_1 t + k_2)\chi$ (20)

Shear Scalar : $\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{k_1{}^2}{54(k_1 t + k_2)^2}$ (21)

Deceleration Parameter: $q = \frac{-3}{\theta^2}\left[\theta_{;\alpha}u^\alpha + \frac{1}{3}\theta^2\right] = 2$ (22)

Average Scale factor $a(t) = V^{1/3} = (N^{2n+1}(k_1 t + k_2)\chi)^{\frac{1}{3}}$ (23)

Graphs are plotted for particular values of the physical parameters and other integration constants.

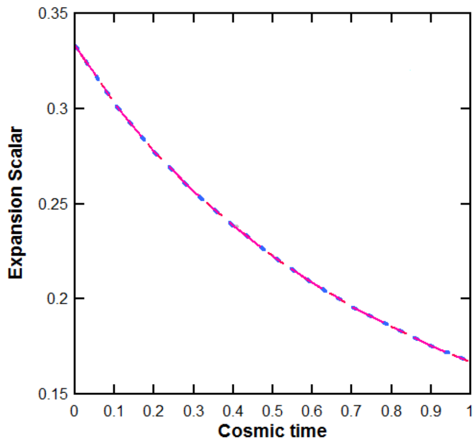

Fig. 1. Plot of Expansion Scalar Vs. Cosmic time for $k_1 = k_2 = 1,2,3$.

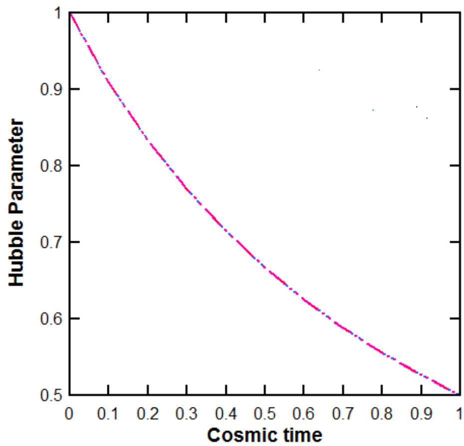

Fig. 2. Plot of Hubble Parameter Vs. Cosmic time for $k_1 = k_2 = 1,2,3$

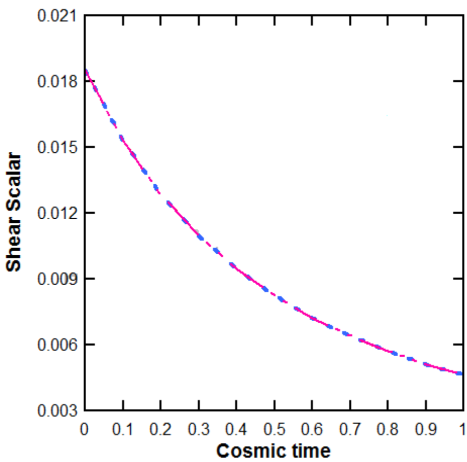

Fig. 3. Plot of Shear Scalar Vs. Cosmic time for $k_1 = k_2 = 1,2,3$.

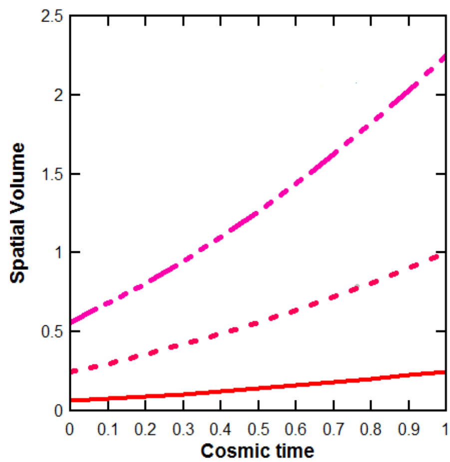

Fig. 4. Plot of Spatial Volume Cosmic time for $n = 0.5$ and $k_1 = k_2 = 1,2,3$.

The decomposition of a time-like tidal tensor is

$$u_{a;b} = N^{2n}\left(\frac{n}{2n+1}\right)k_1(k_1t+k_2)^{\frac{-1}{2n+1}} + N^{2n}f^2\left(\frac{n}{2n+1}\right)(k_1t+k_2)^{\frac{-1}{2n+1}} - N^2\left(\frac{1}{2n+1}\right)k_1(k_1t+k_2)^{\frac{1-2n}{2n+1}} \tag{24}$$

and, Vorticity $\omega_{11} = \omega_{22} = \omega_{33} = \omega_{44} = 0$ (25)

The vorticity of the model along *x*, *y*, *z*, and *t*-axes is zero. So, the obtained model is non-rotating. Whereas Vorticity is nonzero, the model is rotating.

4.1. *Redshift*

The scale factor a and redshift z are connected through the relation

$$1 + z = \frac{a_0(t)}{a(t)} \tag{26}$$

Where $a_0(t)$ is the present value of the scale factor, take $a_0(t) = 1$.
Using equation (23) and the relation $a(t) = \frac{1}{1+z}$, with z being the redshift, gives us the following relation

$$t = \frac{(1+z)^{-3} - k_2 N^{2n+1}\chi}{k_1 N^{2n+1}\chi} \tag{27}$$

From equation (19) and (27), gives

$$H(z) = k_1 N^{2n+1}\chi(1+z)^3 \tag{28}$$

4.2. *Om (z)* diagnostics

The starting point for Om(z) diagnostics is the Hubble parameter, and it is defined as

$$Om(z) = \frac{\left(\frac{H_z}{H_0}\right)^2 - 1}{(1+z)^3 - 1}$$

Thus, *Om(z)* involves only the first derivative of the scale factor through the Hubble parameter and is easier to reconstruct from observational data, gives

$$Om(z) = \frac{\left[k_1 N^{2n+1}\chi(1+z)^3\right]^2 - {H_0}^2}{{H_0}^2[(1+z)^3 - 1]} \tag{29}$$

From equation (29), it is observed that $Om(z)$ increases as z decreases, so $Om(z)$ increases due to the evolution of the universe.

5. Stability Analysis

In this section, check the stability of obtained model using the ratio of sound speed and perturbation technique.
Firstly, discuss the stability of the model by observing the ratio of sound speed given by $\frac{dp}{d\rho} = {c_s}^2$, A positive value of this ratio i.e., ${c_s}^2 > 0$, indicates a stable model, while a negative value i.e., ${c_s}^2 < 0$ indicates an unstable model.
In the present model,

$${c_s}^2 = \frac{dp}{d\rho} = \frac{(3n+1)(2nk_3{}^2 + k_4)}{4\left(n^2 k_3{}^2 - (n+3)k_4\right)} \tag{30}$$

From Fig. 5, it is observed that for $k_3 = k_4 = 1$and $n \geq 2.5$, the ratio of sound speed ${c_s}^2$ is positive, it gives the present cosmological model is a stable. Whereas for $n < 2.5$, the ratio of sound speed ${c_s}^2$ is negative, so the present cosmological model is unstable. The stability condition of the model also depends on values of constant.

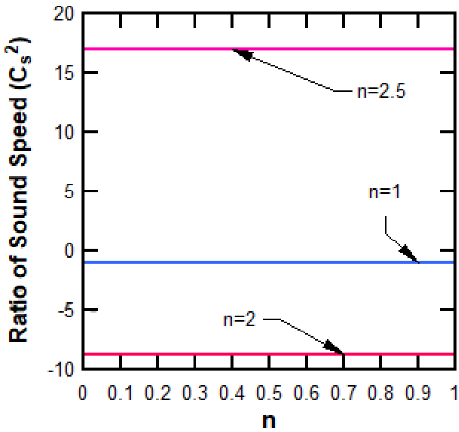

Fig. 5. Plot of ratio of sound speed Vs. constant *n* for $k_3 = k_4 = 1$.

Now, check the stability of the solution with respect to the perturbation of the metric. By using a perturbative technique, a through investigation of the stability of the relevant solutions may be performed. To ensure the stability of the precise or approximative background solution, perturbations of a gravitational system's fields should be compared to the background evolutionary solution [33-37]. Then investigate the stability of the background solution concerning metric perturbations. Here, the solutions' metric perturbation stability is as follows:

$$a_i \rightarrow a_{\boldsymbol{B}i} + \delta a_i = a_{\boldsymbol{B}i}\left(1 + \frac{\delta a_i}{a_{\boldsymbol{B}i}}\right) = a_{\boldsymbol{B}i}(1 + \delta b_i)$$

where $\delta b_i = \frac{\delta a_i}{a_{\boldsymbol{B}i}}$

Similarly, represent the perturbation in the spatial Volume as $V = \prod_{i=1}^{3} a_i$, directional Hubble parameter $H_i = \frac{\dot{a}_i}{a_i}$ and mean Hubble parameter $H_i = \frac{1}{3}\sum_{i=1}^{3} H_i$as follows

$$V \rightarrow V_{\boldsymbol{B}} + V_B \sum_i \delta b_i,$$

$$H_i \rightarrow H_{\boldsymbol{B}i} + \sum_i \delta b_i,$$

$$H \rightarrow H_{\boldsymbol{B}} + \frac{1}{3}\sum_i \delta b_i$$

The following equations can be used to demonstrate how the metric perturbations δb_i, to the linear order in δb_i, obey them.

$$\sum_i \delta \ddot{b}_i + 2\sum_i H_{Bi}\delta \dot{b}_i = 0 \qquad (31)$$

$$\delta \ddot{b}_i + \frac{\dot{V}_B}{V_B}\delta \dot{b}_i + \sum_j H_{Bi}\delta \dot{b}_j = 0 \tag{32}$$

$$\sum_i \delta \dot{b}_i = 0 \tag{33}$$

Solving equations (31)-(33), it gives

$$\delta \ddot{b}_i + \frac{\dot{V}_B}{V_B}\delta \dot{b}_i = 0 \tag{34}$$

For our model, background spatial volumes V_B is given by

$$V_{\boldsymbol{B}} = N^{2n+1}(k_1 t + k_2)\chi \tag{35}$$

From above equations (31)-(35), obtain

$$\delta b_i = \frac{C}{N^{2n+1}\chi k_1} log(k_1 t + k_2) + C_1$$

Where C and C_1 are integrating constants. Therefore, the "actual" fluctuations for each expansion factor $\delta a_i = \delta b_i a_{\boldsymbol{Bi}}$ is given by

$$\delta a_i = \left[\frac{C}{N^{2n+1}\chi k_1} log(k_1 t + k_2) + C_1\right] N^{-(2n+1)}\chi^{-1}(k_1 t + k_2)^{-1} \tag{36}$$

The behavior of actual fluctuations δa_i as a function of time can be seen in equation (36), which also demonstrates that δa_i begins with a slightly positive value and rapidly decreases to zero as the universe expands. So, even when the gravitational field is perturbed, the background solution remains stable.

Conclusion

The present study examines a cosmological model that is axially symmetric and features Lyra geometry within a perfect fluid. In order to solve the field equations, we use the relation $A = B^n$ and the condition of the stiff fluid. In the present study, negative pressure and positive energy density were found. The concept of repulsive gravity is associated with the negative pressure that is present on a cosmological scale or within spherical entities like planets. Graphically, it can be observed that the spatial volume of the universe exhibits an increase as time progresses. This suggests that the universe's expansion originated from a finite volume and continues to expand as time elapses. Also, it is interesting to note that as t gradually increases, the scalar expansion θ, Hubble parameter H, and shear scalar. σ^2 decreases, which means all the physical parameters are well-behaved. In the present model $Om(z)$ increases as z decreases, so $Om(z)$ increases due to the evolution of the universe. For the values of $k_3 = k_4 = 1$ and $n \geq 2.5$, the ratio of sound speed $c_s{}^2$ is positive, This positive ratio leads to the stability of the current cosmological model. While for $n < 2.5$, the ratio of sound speed $c_s{}^2$ is negative, the present cosmological model is unstable. The values of the constants also have a role in determining the stability condition of the model. As the universe continues to expand, the value of the actual fluctuations δa_i starts out with a little positive sign and quickly approaches zero. As a result, the background solution is stable against the perturbation of the metric.

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