JSR Publications

Available Online

J. Sci. Res. **4** (2), 349-356 (2012)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

On Left Centralizers of Semiprime Γ-Rings

K. K. Dey[*] and A. C. Paul

Department of Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh

Received 5 October 2011, accepted in final revised form 23 January 2012

Abstract

Let M be a semiprime Γ-ring satisfying an assumption $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z\in M$, $\alpha, \beta\in\Gamma$. In this paper, we prove that a mapping $T: M \rightarrow M$ is a centralizer if and only if it is a centralizing left centralizer. We also show that if T and S are left centralizers of M such that $T(x)\alpha x + x\alpha S(x)\in Z(M)$ (the center of M) for all $x\in M$, $\alpha\in\Gamma$, then both T and S are centralizers.

Keywords: Semiprime Γ-ring; Left (right) centralizer; Centralizer; Commuting mapping; Centralizing mapping: Extended centroid.

doi: http://dx.doi.org/10.3329/jsr.v4i2.8691 J. Sci. Res. **4** (2), 349-356 (2012)

1. Introduction and Preliminaries

Let M and Γ be additive abelian groups. M is called a Γ-ring if for all $x, y, z\in M$, $\alpha, \beta\in\Gamma$ the following conditions are satisfied :

(i) $x\beta y\in M$,

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$,

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

Every ring is a Γ-ring and many notions on the ring theory are generalized to Γ-rings. Let M be a Γ-ring. A subring I of M is an additive subgroup which is also a Γ-ring. A *right ideal* of M is a subring I such that $I\Gamma M \subset I$. Similarly a *left ideal* can be defined. If I is both a right and a left ideal then we say that I is an ideal.

Let S be a subset of M. If $x\alpha y+ y\alpha x\in S$, for all $x, y\in S$, $\alpha\in\Gamma$, then S is called a Jordan subring of M.

The commutator $x\alpha y - y\alpha x$ will be denoted by $[x, y]_\alpha$. We know that $[x\beta y, z]_\alpha = [x, z]_\alpha\beta y + x\beta[y, z]_\alpha + x[\beta, \alpha]_z y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha\beta z + y[\beta,\alpha]_x z$. We take an assumption (*) $x\beta z\alpha y = x\alpha z\beta y$ for all $x, y, z\in M$ and $\alpha, \beta\in\Gamma$. Using the assumption the

[*] *Corresponding author*: kkdmath@yahoo.com

basic commutator identities reduce to $[x\beta y, z]_\alpha = [x, z]_\alpha\beta y + x\beta[y, z]_\alpha$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha\beta z$.

Throughout, M denotes a Γ-ring with center $Z(M)$. M is said to be semiprime if $x\Gamma M\Gamma x = 0$ implies $x = 0$, it is prime if $x\Gamma M\Gamma y = 0$ implies $x = 0$ or $y = 0$. An additive mapping T: $M\rightarrow M$ is called a left (right) centralizer if $T(x\alpha y) = T(x)\alpha y$ ($T(x\alpha y) = x\alpha T(y)$) for all $x, y\in M$, $\alpha\in\Gamma$. If $a\in M$, then $L_a(x) = a\alpha x$ and $R_a(x) = x\alpha a$, $(x\in M, \alpha\in\Gamma)$ define a left centralizer and a right centralizer of M, respectively. An additive mapping T: $M \rightarrow M$ is called a centralizer if $T(x\alpha y) = T(x)\alpha y = x\alpha T(y)$ for all $x, y\in M$, $\alpha\in\Gamma$. A mapping $f : M \rightarrow M$ is called centralizing (skew centralizing) if $[f(x), x]_\alpha\in Z(M)$ ($f(x)\alpha x + x\alpha f(x)\in Z(M)$) for all $x\in M$, $\alpha\in\Gamma$, in particular, if $[f(x), x]_\alpha = 0$ ($f(x)\alpha x + x\alpha f(x) = 0$) for all $x\in M$, $\alpha\in\Gamma$, then it is called commuting (skew-commuting). Obviously every commuting (skew-commuting) mapping $f : M \rightarrow M$ is centralizing (skew centralizing). We recall if $f : M \rightarrow M$ is commuting, then $[f(x), y]_\alpha = [x, f(y)]_\alpha$ for all $x, y\in M$, $\alpha\in\Gamma$. A mapping $f : M \rightarrow M$ is called central if $f(x)\in Z(M)$ for all $x\in M$.

The theory of centralizers in rings is well established. Many mathematicians worked on centralizers of rings and found out some remarkable results. The theories of Banach algebras and C^*-algebra with centralizers are established by many authors.

Bresar [1-3] studied centralizing mappings with derivation in prime rings. Mayne [4] worked on centralizing automorphisms of prime rings. Recently, Vukman [5-7] and Zalar [8] studied on centralizer of semiprime rings and 2-torsion free semiprime rings. Samman and Chaudhry [9] established the necessary and sufficient condition for a mapping to be a centralizer. If two left centralizers T and S of a semiprime ring R satisfying $T(x)x + xS(x)\in Z(R)$ for all $x\in R$, then they also prove that both T and S are centralizers. Haque and Paul [10] worked on Jordan centralizers on a Γ-ring with certain assumption. For the extended centroid we refer to [11, 12]. They proved that every Jordan left centralizer on a 2-torsion free semiprime Γ-ring is a left centralizer. They also proved that every Jordan centralizer on a 2-torsion free semiprime Γ-ring satisfying a certain condition is a centralizer.

In this paper, we develop the results of [9] in Gamma rings. Our results are the generalizations of the results of Samman and Chaudhry [9]. The results in this paper for left centralizers are also true for right centralizers because of left-right symmetry.

2. Left Centralizers on Semiprime Γ-rings

In this section, we prove our main results.

Theorem 2.1 Let S be a set and M be a semiprime Γ-ring. If the functions f and g of S into M satisfy

$$f(s)\alpha x\beta g(t) = g(s)\alpha x\beta f(t) \text{ for all } s, t\in S, x\in M, \alpha, \beta\in\Gamma, \tag{1}$$

then there exist idempotent elements $e_1, e_2, e_3\in C$, the extended centroid on M and an invertible $k\in C$ such that $e_i\alpha e_j = 0$ for $i \neq j$, $e_1 + e_2 + e_3 = 1$ and $e_1\alpha f(s) = k\beta e_1\alpha g(s)$, $e_2\alpha g(s) = 0$, $e_3\alpha f(s) = 0$ hold for all $s\in S$, $\alpha, \beta\in\Gamma$.

Proof. Obviously, the identity holds in case x is an element from $C(M)$, the central closure of M. Thus there is no loss of generality in assuming that M is centrally closed. Let $A = M\Gamma f(s)\Gamma M$ and $B = M\Gamma g(s)\Gamma M$. We have $A^{\perp} = p\Gamma M$ and $B^{\perp} = q\Gamma M$ for some idempotent elements $p, q\in C$. We set $e_1 = (1-p)\alpha(1-q)$, $e_2 = (1-p)\alpha q$ and $e_3 = p$. Clearly e_i's (i =1, 2, 3) are mutually orthogonal idempotent elements with sum 1. Since $q\alpha g(s)\in B^{\perp}$, $s\in S$, $\alpha\in\Gamma$, we have $q\alpha g(s)\beta x\delta q\alpha g(s) = 0$, which implies $q\alpha g(s) = 0$. Hence $e_2\alpha g(s) = 0$, $s\in S$, $\alpha\in\Gamma$. Similarly we see that $e_3\alpha f(s) = 0$, $s\in S$, $\alpha\in\Gamma$.

We note that $(e_1\alpha A)^{\perp} = (e_1\alpha B)^{\perp} = (1-e_1)\alpha M$, that is, $(e_1\alpha A)^{\perp} = (e_1\alpha B)^{\perp} = (1-e_1)\Gamma M$. Hence $E = e_2\Gamma A\oplus(1-e_1)\Gamma M$ is an essential ideal of M. Define ϕ: $E\rightarrow M$ by $\phi(e_1\alpha(\sum_{i=1}^{3} x_i\beta f(s_i)\delta y_i) + (1-e_1)\lambda r) = e_1\alpha(\sum_{i=1}^{3} x_i\beta g(s_i)\delta y_i) + (1-e_1)\lambda r$.

In order to show that ϕ is well defined, we suppose that

$e_1\alpha(\sum_{i=1}^{3} x_i\beta f(s_i)\delta y_i) = 0$. Consequently $e_1\alpha(\sum_{i=1}^{3} x_i\beta f(s_i)\delta y_i)\gamma z\lambda g(t) = 0$ holds for all $z\in M$, $t\in S$, $\alpha, \beta, \delta, \gamma, \lambda\in\Gamma$.

Since by (1) we have $f(s_i)\,\delta y_i\gamma z\lambda\, g(t) = g(s_i)\,\delta y_i\gamma z\,\lambda f(t)$, it follows that $e_1\alpha(\sum_{i=1}^{3} x_i\beta g(s_i)\delta y_i)\gamma z\lambda f(t) = 0$ for all $z\in M$, $t\in S$, $\alpha, \beta, \delta, \gamma, \lambda\in\Gamma$.

Thus the elements $e_1\alpha(\sum_{i=1}^{3} x_i\beta g(s_i)\delta y_i)$ lies in $A^{\perp}$. Since $A^{\perp} = p\Gamma M$ and $e_1 = (1-p)\alpha(1-q)$, it follows that $e_1\alpha(\sum_{i=1}^{3} x_i\beta g(s_i)\delta y_i) = 0$. This proves that ϕ is well defined.

Clearly ϕ is an M_{Γ}-module homomorphism. Then there exist $k\in C$ such that $\phi(u) = k\beta u$ for every $u\in E$, $\beta\in\Gamma$. Hence $e_1\alpha f(s) = k\beta e_1\alpha g(s)$ for all $s\in S$, $\alpha,\beta\in\Gamma$. It remains to prove that k is invertible. Note that $k\Gamma E = e_1\Gamma B\oplus(1-e_1)\Gamma M$. Since $e_1\Gamma B\oplus(1-e_1)\Gamma M$ is an essential ideal (namely $(e_1\Gamma B)^{\perp} = (1-e_1)\Gamma M$), k can not be a divisor of zero. Consequently, C is the extended centroid of M, k is invertible. The proof is complete.

Theorem 2.2. Let M be a 2-torsion free semiprime Γ-ring satisfying the condition (*) and U be a Jordan subring of M. If an additive mapping F of M into itself is centralizing on U, then F is commuting on U.

Proof: A linearization of $[F(x), x]_{\alpha}\in Z$ gives $[F(x), y]_{\alpha} + [F(y), x]_{\alpha}\in Z$ for all $x, y\in U$, $\alpha\in\Gamma$.

Replacing y by $x\beta x$,

$[F(x), x\beta x]_{\alpha} + [F(x\beta x), x]_{\alpha}\in Z$. Since $[F(x), x]_{\alpha}\in Z$, we have $[F(x), x\beta x]_{\alpha}$

$= x\beta[F(x), x]_{\alpha} + [F(x), x]_{\alpha}\beta x$

$= [F(x), x]_{\alpha}\beta x + [F(x), x]_{\alpha}\beta x = 2[F(x), x]_{\alpha}\beta x$. Thus

$2[F(x), x]_{\alpha}\beta x + [F(x\beta x), x]_{\alpha}\in Z$ for all $x\in U$, $\alpha, \beta\in\Gamma$. (2)

By assumption $[F(x\beta x), x\beta x]_{\alpha}\in Z$, for all $x\in U$, $\alpha, \beta\in\Gamma$. That is

$[F(x\beta x), x]_{\alpha}\beta x + x\beta[F(x\beta x), x]_{\alpha}\in Z.$ (3)

Now fix $x\in U$ and let $z = [F(x), x]_\alpha$, $u = [F(x\beta x), x]_\alpha$. We must show that $z = 0$. By (2) we have

$0 = [F(x), 2z\beta x + u]_\alpha$

$= 2z\beta[F(x), x]_\alpha + 2[F(x), z]_\alpha\beta x + [F(x), u]_\alpha = 2z\beta z + [F(x), u]_\alpha$

Thus $[F(x), u]_\alpha = -2z\beta z$ (4)

According to (3) we have $0 = [F(x), u\beta x + x\beta u]_\alpha = [F(x), u]_\alpha\beta x + u\beta[F(x), x]_\alpha + [F(x), x]_\alpha\beta u + x\beta[F(x), u]_\alpha$, applying (4) we then get $-4z\beta z\beta x + 2z\beta u = 0$.

Thus $z\beta u = 2z\beta z\beta x$. Multiplying (4) by $z\beta$ and using the last relation we obtain

$-2z\beta z\beta z = z\beta[F(x), u]_\alpha = [F(x), z\beta u]_\alpha - [F(x), z]_\alpha\beta u = [F(x), z\beta u]_\alpha$

$= [F(x), 2z\beta z\beta x]_\alpha = 2z\beta z\beta[F(x), x]_\alpha + [F(x), 2z\beta z]_\alpha\beta x = 2z\beta z\beta z$. Hence $z\beta z\beta z = 0$.

Since the center of a semiprime Γ-ring contains no nonzero nilpotent elements, we conclude that $z = 0$. This proves the theorem.

Theorem 2.3 Let T be a centralizing left centralizer of a semiprime Γ-ring M satisfying the condition (*). Then T is commuting.

Proof. If M is 2-torsion free, then T is commuting follows from Theorem 2.3 by taking $U = M$ in it. If M is not a 2-torsion free semiprime Γ-ring, then

$2[T(x), x]_\alpha = 0$ for all $x\in M$, $\alpha\in\Gamma$ (5) and

$2([T(x), y]_\alpha + [T(y), x]_\alpha) = 0$ for all $x, y\in M$, $\alpha\in\Gamma$. (6)

By assumption $[T(x), x]_\alpha\in Z(M)$. Linearizing this, we get

$[T(x), y]_\alpha + [T(y), x]_\alpha\in Z(M)$ for all $x, y\in M$, $\alpha\in\Gamma$. (7)

Using (5) – (7) and the hypothesis that $[T(x), x]_\alpha\in Z(M)$, the following identity follows easily

$[T(x), x\beta y + y\beta x]_\alpha + [T(y), x\beta x]_\alpha = 0$ for all $x, y\in M$, $\alpha, \beta\in\Gamma$. (8)

Replacing y by $y\delta x$ in (8), we get $[T(x), x\beta y\delta x + y\delta x\beta x]_\alpha + [T(y\delta x), x\beta x]_\alpha = 0$, which gives $(x\beta y + y\beta x)\delta[T(x), x]_\alpha + [T(x), x\beta y + y\beta x]_\alpha\delta x + T(y)\delta[x, x\beta x]_\alpha + [T(y), x\beta x]_\alpha\delta x = 0$ for all $x, y\in M$, $\alpha, \beta, \delta\in\Gamma$. Combining this with (8), we get $(x\beta y + y\beta x)\delta[T(x), x]_\alpha = 0$ for all $x, y\in M$, $\alpha, \beta, \delta\in\Gamma$, which gives $(x\beta y - y\beta x + 2y\beta x)\delta[T(x), x]_\alpha = 0$, for all $x, y\in M$, $\alpha, \beta, \delta\in\Gamma$. Thus $(x\beta y - y\beta x)\delta[T(x), x]_\alpha = 0$ for all $x, y\in M$, $\alpha, \beta, \delta\in\Gamma$. In particular, (replacing y by $T(x)$ and δ by α) $(x\beta T(x) - T(x)\beta x)\delta[T(x), x]_\alpha = -[T(x), x]_\alpha\beta[T(x), x]_\alpha = 0$ for all $x\in M$, $\alpha, \beta\in\Gamma$. Since a semiprime Γ-ring has no nontrivial central nilpotents, therefore $[T(x), x]_\alpha = 0$ for all $x\in M$, $\alpha\in\Gamma$.

Theorem 2.4 Let T be a centralizing left centralizer of a semiprime Γ-ring M satisfying the condition (*), then T is a centralizer of M.

Proof. We have $T(x\alpha y) = T(x)\alpha y$ for all $x, y\in M$ and $\alpha\in\Gamma$. We now show that $T(x\alpha y) = x\alpha T(y)$ for all $x, y\in M$, $\alpha\in\Gamma$. Since T is a centralizing left centralizer of M, therefore by Theorem 2.3, it is commuting. Thus $[T(x), x]_\alpha = 0$ for all $x\in M$, $\alpha\in\Gamma$. That is, $T(x)\alpha x - x\alpha T(x) = 0$ for all $x\in M$, $\alpha\in\Gamma$. Linearizing this, we get $T(x)\alpha y + T(y)\alpha x - y\alpha T(x) - x\alpha T(y) = 0$ for all $x, y\in M$, $\alpha\in\Gamma$.

Replacing y by $x\beta y$ in the last identity, we get

$$0 = T(x)\alpha x\beta y + T(x\beta y)\alpha x - x\beta y\alpha T(x) - x\alpha T(x\beta y) = T(x)\alpha x\beta y + T(x)\beta y\alpha x - x\beta y\alpha T(x) - x\alpha T(x)\beta y = (T(x)\alpha x - x\alpha T(x))\beta y + T(x)\beta y\alpha x - x\alpha y\beta T(x) = T(x)\beta y\alpha x - x\alpha y\beta T(x).$$

That is,

$$T(x)\beta y\alpha x - x\beta y\alpha T(x) = 0 \text{ for all } x, y\in M,\ \alpha, \beta\in\Gamma. \quad (9)$$

Taking $S = M$, $s = t = x$, $f(x) = T(x)$ and $g(x) = x$ in (9) and applying Theorem 2.1 to (9) we conclude that there exist idempotent elements $e_1, e_2, e_3\in C$ and an invertible $p\in C$ such that $e_i\alpha e_j = 0$ for $i \neq j$, $e_1 + e_2 + e_3 = 1$ and $e_1\alpha T(x) = p\beta e_1\alpha x$, $e_2\alpha x = 0$ and $e_3\alpha T(x) = 0$ for all $x\in M$, $\alpha, \beta\in\Gamma$. Now $e_2\alpha x = 0$ implies $x\alpha e_2 = 0$. Thus $T(x\alpha e_2) = T(0)$, which gives $T(x)\alpha e_2 = T(0)\alpha 0 = 0$. That is, $T(x)\alpha e_2 = 0$ or $e_2\alpha T(x) = 0$. Thus

$$T(x) = e_1 + e_2 + e_3)\alpha T(x) = e_1\alpha T(x) = p\beta e_1\alpha x.$$

That is, $T(x) = p\beta e_1\alpha x$ for all $x\in M$, $\alpha\in\Gamma$. Thus $T(x)\alpha y - x\alpha T(y) = p\beta e_1 x\delta y - x\alpha p\beta e_1\delta y = p\beta e_1\alpha x\delta y - p\beta e_1\alpha x\delta y = 0$. That is,

$$T(x)\alpha y = x\alpha T(y) \text{ for all } x, y\in M,\ \alpha\in\Gamma. \quad (10)$$

$$(T(x\alpha y) - T(x)\alpha y)\beta z\gamma(T(x\alpha y) - T(x)\alpha y) = 0.$$

By the semiprimeness of M, we have, $T(x\alpha y) - T(x)\alpha y = 0$. This implies $T(x\alpha y) = T(x)\alpha y$. Thus $T(x\alpha y) = T(x)\alpha y = x\alpha T(y)$. This shows that T is a centralizer.

Remark 2.5 Obviously every centralizer is commuting because $T(x\alpha x) = T(x)\alpha x = x\alpha T(x)$ for all $x\in M$, $\alpha\in\Gamma$, and hence is a centralizing left centralizer. Thus we have the following corollary.

Corollary 2.6 A mapping T of a semiprime Γ-ring M satisfying the condition (*) is a centralizer if and only if it is a centralizing left centralizer. Let T be a commuting left centralizer of a semiprime Γ-ring, then $T(x)\beta[x, y]_\alpha = x\beta[T(x), y]_\alpha$ holds for all $x, y\in M$, $\alpha, \beta\in\Gamma$.

Proof. Since T is commuting, therefore $[T(x), x]_\alpha = 0$ for all $x\in M$, $\alpha\in\Gamma$. (11)

Linearizing (11), we get

$$[T(x), y]_\alpha + [T(y), x]_\alpha = 0 \text{ for all } x, y\in M,\ \alpha\in\Gamma. \quad (12)$$

Replacing y by $x\beta y$ in (12) and using (12), we get $0 = [T(x), x\beta y]_\alpha + [T(x\beta y), x]_\alpha = [T(x), x\beta y]_\alpha + [T(x)\beta y, x]_\alpha = [T(x), x]_\alpha\beta y + x\beta[T(x), y]_\alpha + [T(x), x]_\alpha\beta y + T(x)\beta[y, x]_\alpha = x\beta[T(x), y]_\alpha - T(x)\beta[x, y]_\alpha$.

That is, $x\beta[T(x), y]_\alpha - T(x)\beta[x, y]_\alpha = 0$ for all $x, y\in M$, $\alpha, \beta\in\Gamma$.

Thus $T(x)\beta[x, y]_\alpha = x\beta[T(x), y]_\alpha$ for all $x, y\in M$, $\alpha, \beta\in\Gamma$.

Remark 2.7 If T is a central left centralizer of a prime Γ-ring M, then either $T = 0$ or M is commutative. This is because $T(x)\beta[x, y]_\alpha = x\beta[T(x), y]_\alpha$ gives $T(x)\beta[x, y]_\alpha = 0$. Replacing y by $y\delta z$ in the last identity and using it, we get $T(x)\beta y\delta[x, z]_\alpha = 0$ for all $x, y, z\in M$. Since M is prime, therefore $T(x) = 0$ or $[x, z]_\alpha = 0$ for all $x, z\in M$, $\alpha\in\Gamma$. That is, $T = 0$ or M is commutative.

Theorem 2.8. Let M be a semiprime Γ-ring satisfying the condition (*) and T and S be left centralizers of M such that
$$T(x)\alpha x + x\alpha S(x)\in Z(M) \text{ for all } x\in M, \alpha\in\Gamma. \quad (13)$$
Then T and S are both centralizers.

Proof. Linearizing (13), we get

$$T(x)\alpha y + T(y)\alpha x + x\alpha S(y) + y\alpha S(x)\in Z(M) \text{ for all } x, y\in M, \alpha\in\Gamma. \quad (14)$$

Thus $[T(x)\alpha y + T(y)\alpha x + x\alpha S(y) + y\alpha S(x), x]_\beta = 0$, which gives

$$[T(x)\alpha y + T(y)\alpha x + x\alpha S(y), x]_\beta = -[y\alpha S(x), x]_\beta \text{ for all } x, y\in M, \alpha, \beta\in\Gamma. \quad (15)$$

Replacing y by $y\beta x$ in (14), we get $T(x)\alpha y\beta x + T(y\beta x)\alpha x + x\alpha S(y\beta x) + y\beta x\alpha S(x) = T(x)\alpha y\beta x + T(y)\beta x\alpha x + x\alpha S(y)\beta x + y\beta x\alpha S(x) = (T(x)\alpha y + T(y)\alpha x + x\alpha S(y))\beta x + y\alpha x\beta S(x)\in Z(M)$.

Thus $[(T(x)\alpha y + T(y)\alpha x + x\alpha S(y))\beta x + y\alpha x\beta S(x), x]_\beta = 0$ for all $x\in M$, $\alpha, \beta\in\Gamma$. This implies that

$$[T(x)\alpha y + T(y)\alpha x + x\alpha S(y), x]_\beta\beta x + [y\beta x\alpha S(x), x]_\beta = 0 \text{ for all } x, y\in M, \alpha, \beta\in\Gamma. \quad (16)$$

Using (15), from (16) we get

$$-[y\alpha S(x), x]_\beta\beta x + [y\alpha x\beta S(x), x]_\beta = 0 \text{ for all } x, y\in M, \alpha, \beta\in\Gamma. \quad (17)$$

Since $[y\alpha S(x)\beta x, x]_\beta = [y\alpha S(x), x]_\beta\beta x$, therefore (17) gives $0 = -[y\alpha S(x)\beta x, x]_\beta + [y\alpha x\beta S(x), x]_\beta = [y\alpha(x\beta S(x) - S(x)\beta x), x]_\beta = [y\alpha[x, S(x)]_\beta, x]_\beta = y\alpha[[x, S(x)]_\beta, x]_\beta + [y, x]_\beta\alpha[x, S(x)]_\beta$.

Thus
$$y\alpha[[x, S(x)]_\beta, x]_\beta + [y, x]_\beta\alpha[x, S(x)]_\beta = 0 \text{ for all } x, y\in M, \alpha, \beta\in\Gamma. \quad (18)$$

Replacing y by $z\lambda y$ in (18) and using (18), we get $0 = z\lambda y\alpha[[x, S(x)]_\beta, x]_\beta + [z\lambda y, x]_\beta\alpha[x, S(x)]_\beta = z\lambda y\alpha[[x, S(x)]_\beta, x]_\beta + z\lambda[y, x]_\beta\alpha[x, S(x)]_\beta + [z, x]_\beta\lambda y\alpha[x, S(x)]_\beta = [z, x]_\beta\lambda y\alpha[x, S(x)]_\beta$.

That is, $$[z, x]_\beta\lambda y\alpha[x, S(x)]_\beta = 0 \text{ for all } x, y, z\in M, \alpha, \beta, \lambda\in\Gamma. \quad (19)$$

In particular, $[S(x), x]_\beta\lambda y\alpha[x, S(x)]_\beta = 0$ which, by semiprimeness of M, implies $[S(x), x]_\beta = 0$. Thus S is a commuting left centralizer and, by Theorem 2.2, is a centralizer.

We now show that T is commuting. By hypothesis and by the assumption, we have

$0 = [T(x)\alpha x + x\alpha S(x), x]_\beta = [T(x), x]_\beta\alpha x + x\alpha[S(x), x]_\beta = [T(x), x]_\beta\alpha x$. That is,

$[T(x), x]_\beta \alpha x = 0$ for all $x, y \in M$, $\alpha, \beta \in \Gamma$. (20)

From (15), we get $[T(x)\alpha y + T(y)\alpha x, x]_\beta = [-x\alpha S(y) - y\alpha S(x), x]_\beta$. Thus $T(x)\alpha[y, x]_\beta + [T(x), x]_\beta \alpha y + [T(y), x]_\beta \alpha x = -x\alpha[S(y), x]_\beta - y\alpha[S(x), x]_\beta - [y, x]_\beta \alpha S(x) = -x\alpha[y, S(x)]_\beta - [y, x]_\beta \alpha S(x) = x\alpha[S(x), y]_\beta + [x, y]_\beta \alpha S(x)$.

That is, for all $x, y \in M$, $\alpha, \beta \in \Gamma$

$$T(x)\alpha[y, x]_\beta + [T(x), x]_\beta \alpha y + [T(y), x]_\beta \alpha x = x\alpha[S(x), y]_\beta + [x, y]_\beta \alpha S(x). \quad (21)$$

Again by hypothesis, we get

$$0 = [T(x)\alpha x + x\alpha S(x), y]_\beta = T(x)\alpha[x, y]_\beta + [T(x), y]_\beta \alpha x + [x, y]_\beta \alpha S(x) + x\alpha[S(x), y]_\beta.$$

That is, for all $x, y \in M$, $\alpha, \beta \in \Gamma$

$$-T(x)\alpha[y, x]_\beta + [T(x), y]_\beta \alpha x = -[x, y]_\beta \alpha S(x) - x\alpha[S(x), y]_\beta. \quad (22)$$

Adding (21) and (22), we get

$$[T(x), x]_\beta \alpha y + [T(y), x]_\beta \alpha x + [T(x), y]_\beta \alpha x = 0 \text{ for all } x, y \in M,\ \alpha, \beta \in \Gamma. \quad (23)$$

Replacing y by $y\beta T(x)$ in (23) and using (20), we get

$$\begin{aligned}
0 &= [T(x), x]_\beta \alpha y\beta T(x) + [T(y\beta T(x)), x]_\beta \alpha x + [T(x), y\beta T(x)]_\beta \alpha x \\
&= [T(x), x]_\beta \alpha y\beta T(x) + [T(y)\beta T(x), x]_\beta \alpha x + [T(x), y\beta T(x)]_\beta \alpha x \\
&= [T(x), x]_\beta \alpha y\beta T(x) + T(y)\beta[T(x), x]_\beta \alpha x + [T(y), x]_\beta \beta T(x)\alpha x + [T(x), y]_\beta \beta T(x)\alpha x \\
&= -[T(y), x]_\beta \alpha x\beta T(x) - [T(x), y]_\beta \alpha x\beta T(x) + [T(y), x]_\beta \beta T(x)\alpha x + [T(x), y]_\beta \beta T(x)\alpha x \\
&= [T(y), x]_\beta \alpha[T(x), x]_\beta + [T(x), y]_\beta \alpha[T(x), x]_\beta.
\end{aligned}$$

That is

$$[T(y), x]_\beta \alpha[T(x), x]_\beta + [T(x), y]_\beta \alpha[T(x), x]_\beta = 0 \text{ for all } x, y \in M,\ \alpha, \beta \in \Gamma. \quad (24)$$

Replacing y by $y\lambda x$ in (24) and using (23), we get

$$\begin{aligned}
0 &= [T(y\lambda x), x]_\beta \alpha[T(x), x]_\beta + [T(x), y\lambda x]_\beta \alpha[T(x), x]_\beta \\
&= [T(y)\lambda x, x]_\beta \alpha[T(x), x]_\beta + [T(x), y\lambda x]_\beta \alpha[T(x), x]_\beta \\
&= [T(y), x]_\beta \lambda x\alpha[T(x), x]_\beta + y\lambda[T(x), x]_\beta \alpha[T(x), x]_\beta + [T(x), y]_\beta \lambda x\alpha[T(x), x]_\beta \\
&= ([T(y), x]_\beta \lambda x + [T(x), y]_\beta \lambda x)\alpha[T(x), x]_\beta + y\lambda[T(x), x]_\beta \alpha[T(x), x]_\beta \\
&= -[T(x), x]_\beta \lambda y\alpha[T(x), x]_\beta + y\lambda[T(x), x]_\beta \alpha[T(x), x]_\beta.
\end{aligned}$$

Thus

$$[T(x), x]_\beta \lambda y\alpha[T(x), x]_\beta = y\lambda[T(x), x]_\beta \alpha[T(x), x]_\beta \text{ for all } x, y \in M,\ \alpha, \beta, \lambda \in \Gamma. \quad (25)$$

Replacing y by $x\alpha y$ in (25) and using (20), we get $x\alpha y\lambda[T(x), x]_\beta \alpha[T(x), x]_\beta = [T(x), x]_\beta \alpha x\alpha y\lambda[T(x), x]_\beta = 0$.

That is,

$$x\alpha y\lambda[T(x), x]_\beta\alpha[T(x), x]_\beta = 0 \text{ for all } x, y\in M, \alpha, \beta, \lambda\in\Gamma, \quad (26)$$

which gives $T(x)\beta x\alpha y\lambda[T(x), x]_\beta\alpha[T(x), x]_\beta = 0$.

Further, replacing y by $T(x)\beta y$ in (26), we get $x\alpha T(x)\beta y\lambda[T(x), x]_\beta\alpha[T(x), x]_\beta = 0$. Combining the last two identities, we get $(T(x)\beta x - x\beta T(x))\alpha y\lambda[T(x), x]_\beta\alpha[T(x), x]_\beta = 0$. That is, $[T(x), x]_\beta\gamma y\lambda[T(x), x]_\beta\alpha[T(x), x]_\beta = 0$, which gives $[T(x), x]_\beta\alpha[T(x), x]_\beta\alpha y\lambda[T(x), x]_\beta\alpha[T(x), x]_\beta = 0$. Since M is semiprime, therefore,

$$[T(x), x]_\beta\alpha[T(x), x]_\beta = 0 \text{ for all } x\in M, \alpha, \beta\in\Gamma. \quad (27)$$

Using (23), from (21) we get $[T(x), x]_\beta\alpha y\lambda[T(x), x]_\beta = 0$, which by semiprimeness of M implies $[T(x), x]_\beta = 0$. Thus T is a commuting left centralizer and hence by Theorem 2.2, T is a centralizer.

Taking $S = T$ in Theorem 2.7, we get the following corollary.

Corollary 2.9. Let T be a skew centralizing left centralizer of a semiprime Γ-ring M satisfying the condition (*). Then T is a centralizer.

The following corollary is also obvious.

Corollary 2.10. Let T and S be left centralizers of a semiprime Γ-ring M satisfying the condition (*) such that $T(x)\alpha x + x\alpha S(x) = 0$ for all $x\in M$, $\alpha\in\Gamma$. Then both T and S are centralizers.

References

1. M. Brešar, Proc. Amer. Math. Soc. **114**, 641 (1992).
2. M. Brešsar, J. Algebra **156**, 385 (1993). http://dx.doi.org/10.1006/jabr.1993.1080
3. M. Brešsar, Proc. Amer. Math. Soc. **120**, 709 (1994).
4. J.H. Mayne, Canad. Math. Bull. **19**, 113 (1976). http://dx.doi.org/10.4153/CMB-1976-017-1
5. J. Vukman, Comment. Math. Univ. Carolinae **38**, 231 (1997).
6. J. Vukman, Comment. Math. Univ. Carolinae **40**, 447 (1999).
7. J. Vukman, Comment. Math. Univ. Carolinae **42**, 237 (2001).
8. B. Zalar, Comment. Math. Univ. Carolinae **34**, 609 (1991).
9. M. S. Samman, M. A. Chaudhry, Int. J. Pure Appl. Math. **20** (4) 487 (2005).
10. M. F. Hoque and A. C. Paul, Int. Math. Forum **6** (13), 627 (2011).
11. M. A. Ozturk and Y. B. Jun, Commun. Korean Math. Soc. **15 (**3**)**, 469 (2000).
12. M. A. Ozturk and Y. B. Jun, Turk. J. Math. **25**, 367 (2001).
13. K. K. Dey and A. C. Paul, J. Sci. Res. **3** (2), 331 **(**2011). http://dx.doi.org/10.3329/jsr.v3i2.7278
14. K. K. Dey and A. C. Paul, J. Sci. Res. **4 (**1**)**, 33 (2012). http://dx.doi.org/10.3329/jsr.v4i1.7911