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Permuting Tri-Derivations of Semiprime Gamma Rings

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Abstract

We study some properties of permuting tri-derivations on semiprime Γ-rings with a certain assumption. Let M be a 3-torsion free semiprime Γ-ring satisfying a certain assumption and let I be a non-zero ideal of M. Suppose that there exists a permuting tri-derivation D: $M\times M\times M \to M$ such that d is an automorphism commuting on I and also d is a trace of D. Then we prove that I is a nonzero commutative ideal. Various characterizations of M are obtained by means of tri-derivations.

Keywords: Tri-derivation; Semiprime Γ-ring; Commutative ideal; Commuting map; Permuting map.

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1. Preliminaries

Gamma rings were first introduced by Nabusawa [1] and then Barnes [2] generalized the definition of Γ-rings. In this paper we work on Γ-rings due to Barnes [2]. Throughout this paper, M will represent a Γ-ring and $Z(M)$ will be its center. A Γ-ring M is prime if $x\Gamma M\Gamma y = 0$ implies that $x = 0$ or $y = 0$, and is semiprime if $x\Gamma M\Gamma x = 0$ implies $x = 0$. Let $x, y\in M$, $\alpha\in\Gamma$, the commutator $x\alpha y - y\alpha x$ will be denoted by $[x, y]_\alpha$. We know that $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha\beta y + x[\beta, \alpha]_z y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha\beta z + y[\beta,\alpha]_x z$ for all $x, y, z\in M$, $\alpha, \beta\in\Gamma$. We shall take an assumption (*) $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z\in M$, $\alpha, \beta\in\Gamma$. Using the assumption (*) the above identities reduce to $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha\beta y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha\beta z$, for all $x, y, z\in M$ and for all $\alpha, \beta\in\Gamma$ which are used extensively in our results.

Let I be a nonempty subset of M. Then a map d: $M \to M$ is said to be commuting (resp. centralizing) on I if $[d(x), x]_\alpha = 0$ for all $x\in I$, $\alpha\in\Gamma$ (resp. $[d(x), x]_\alpha\in Z(M)$ for all $x\in I$,

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$\alpha\in\Gamma$), and is called central if $d(x)\in Z(M)$ for all $x\in M$, $\alpha\in\Gamma$. Every central mapping is obviously commuting but not conversely in general, and d is called skew-centralizing on a subset I of M (resp. skew-commuting on a subset I of M) if $d(x)\alpha x + x\alpha d(x)\in Z(M)$ holds for all $x\in I$, $\alpha\in\Gamma$ (resp. $d(x)\alpha x + x\alpha d(x) = 0$ holds for all $x\in I$, $\alpha\in\Gamma$). Recall that M is said to be n-torsion free, where $n \neq 0$ is an integer, if whenever $nx = 0$, with $x\in M$ then $x = 0$. An additive map d: $M \rightarrow M$ is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all x, $y\in M$, $\alpha\in\Gamma$. By a bi-derivation we mean a bi-additive map D: $M \times M \rightarrow M$ (i.e., D is additive in both arguments), which satisfies the relations $D(x\alpha y, z) = D(x, z)\alpha y + x\alpha D(y, z)$ and $D(x, y\alpha z) = D(x, y)\alpha z + y\alpha D(x, z)$ for $x, y\in M$, $\alpha\in\Gamma$. Let D be symmetric, that is $D(x, y) = D(y, x)$ for the $x, y\in M$. The map d: $M \rightarrow M$ defined by $d(x) = D(x, x)$ for all $x\in M$ is called the trace of D. A map D: $M \times M \times M \rightarrow M$ will be said to be permuting if the equation $D(x, y, z) = D(x, z, y) = D(z, x, y) = D(y, z, x) = D(z, y, x)$ for all $x, y, z\in M$. A map d: $M \rightarrow M$ defined by $d(x) = D(x, x, x)$ for all $x\in M$, where D: $M \times M \times M \rightarrow M$ is a permuting map is called the trace of D. It is obvious that, in case when D: $M \times M \times M \rightarrow M$ is a permuting map which is also tri-additive (i.e., additive in each argument), the trace d of D satisfies the relation $d(x + y) = d(x) + d(y) + 3D(x, x, y) + 3D(x, y, y)$ for all $x, y\in M$. Since we have $D(0, y, z) = D(0 + 0, y, z) = D(0, y, z) + D(0, y, z)$ for all $y, z\in M$, we obtain $D(0, y, z) = 0$ for all $y, z\in M$. Hence we get $D(0, y, z) = D(x - x, y, z) = D(x, y, z) + D(-x, y, z) = 0$ and so we see that $D(-x, y, z) = -D(x, y, z)$ for all $x, y, z\in M$. This implies that d is an odd function. A tri-additive map D: $M \times M \times M \rightarrow M$ will be called a tri-derivation if the relations $D(x\alpha w, y, z) = D(x, y, z)\alpha w + x\alpha D(w, y, z)$, $D(x, y\alpha w, z) = D(x, y, z)\alpha w + y\alpha D(x, w, z)$ and $D(x, y, z\alpha w) = D(x, y, z)\alpha w + z\alpha D(x, y, w)$ are fulfilled for all $x, y, z, w\in M$, $\alpha\in\Gamma$. If D is permuting, then the above three relations are equivalent to each other.

Let M be commutative Γ-ring. A map D: $M \times M \times M \rightarrow M$ defined by $(x, y, z) \rightarrow d(x)\alpha d(y)\beta d(z)$ for all $x, y, z\in M$, $\alpha, \beta\in\Gamma$, is a tri-derivation where d is a derivation on M.

Ozturk *et al.* [3] studied on symmetric bi-derivations on prime Γ-rings. Some fruitful results of prime Γ-rings were obtained by these authors. Ozturk [4] obtained some properties concerning to the mapping permuting tri-derivations on prime and semiprime Γ-rings. Permuting tri-derivations in prime and semiprime Γ-rings had been studied by Ozden *et al.* [5]. Some remarkable results of these Γ-rings were obtained by them. An example of a permuting tri-derivation has also been given by these authors [5].

In this paper, we study and investigate some results concerning a permuting tri-derivation D on non-commutative 3-torsion free semiprime Γ-rings M. Some characterizations of semiprime Γ-rings are obtained by means of permuting tri-derivations.

First we prove the following lemmas which will be needed in our results.

Lemma 1.1

Let M be a semiprime Γ-ring. Then M contains no nonzero nilpotent ideal.

Proof.

Let I be a nilpotent ideal of M. Then $(I\Gamma)^n I = 0$ for some positive integer n. Let us assume that n is minimum. Now suppose that $n \geq 1$. Since $I\Gamma M \subset I$, we then have $(I\Gamma)^{n-1} I\Gamma M\Gamma(I\Gamma)^{n-1}I \subset (I\Gamma)^{n-1}I(I\Gamma)^{n}I = (I\Gamma)^{n}I(I\Gamma)^{n-2}I = 0$. Hence by the semiprimeness of M we get $(I\Gamma)^{n-1}I = 0$, a contradiction to the minimality of n. Therefore $n = 1$. Thus $I\Gamma I = 0$. Then $I\Gamma M\Gamma I \subset I\Gamma I = 0$. Since M is semiprime, it gives $I = 0$. This completes the proof.

The above lemma gives us the following corollary.

Corollary 1.2

Every prime Γ-ring has no nilpotent ideals.

Lemma 1.3 [**15 Theorem 4.1**]

Let M be a 2, 3-torsion free prime Γ-ring. Let $D(., ., .)$ be permuting tri-derivation of M with the trace d. If

$$a\alpha d(x) = 0,\ x\in M,\ \alpha\in\Gamma \tag{1}$$

where a is a fixed element of M, then either $a = 0$ or $D = 0$.

Lemma 1.4

Let M be a 2-torsion free semiprime Γ-ring. If $x\alpha x = 0$ then $x\in Z(M)$ for all $x\in M$, $\alpha\in\Gamma$.

Proof:

We have $x\alpha x = 0$ for all $x\in M$, $\alpha\in\Gamma$. Replacing x by $x + y$, we get $x\alpha y + y\alpha x = 0$ for all x, $y\in M$, $\alpha\in\Gamma$.

Right-multiplying by βx we obtain $x\alpha y\beta x = 0$ for all $x, y\in M$, $\alpha, \beta\in\Gamma$. Replacing y by $y\gamma z$ and right-multiplying by αy we get $x\alpha y\gamma z\beta x\alpha y = 0$ for all $x, y, z\in M$, $\alpha, \beta, \gamma\in\Gamma$. Since M is semiprime Γ-ring, we obtain $x\alpha y = 0$ for all $x, y\in M$, $\alpha\in\Gamma$. By the same method, we get $y\alpha x = 0$ for all $x, y\in M$, $\alpha\in\Gamma$. By subtracting those, we obtain $[x, y]_\alpha = 0$, for all $x, y\in M$, $\alpha\in\Gamma$, then $x\in Z(M)$ for all $x\in M$.

Lemma 1.5

Let M be a semiprime Γ-ring satisfying the condition (*). If $x\alpha x\in Z(M)$ then $x\in Z(M)$ for all $x\in M$, $\alpha\in\Gamma$.

Proof:

We have $x\alpha x\in Z(M)$ for all $x\in M$, $\alpha\in\Gamma$. Then $[x\alpha x, z]_\beta = 0$ for all $z\in M$, $\alpha, \beta\in\Gamma$. Replacing x by $x + y$, we get $[x\alpha y + y\alpha x, z]_\beta = 0$ for all $x, y, z\in M$, $\alpha, \beta\in\Gamma$. Since $y\beta z\alpha x = y\alpha z\beta x$, we have $x\alpha[y, z]_\beta + [x, z]_\beta\alpha y + [y\alpha x, z]_\beta = 0$ for all $x, y, z\in M$, $\alpha, \beta\in\Gamma$.

Similarly, $[y, z]_\beta \alpha x + y\alpha[x, z]_\beta + [x\alpha y, z]_\beta = 0$ for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$.

Using the relation $[x\alpha y + y\alpha x, z]_\beta = 0$ and replacing y by $x\alpha x$ we obtain $[x, z]_\beta \alpha x \alpha x = 0$ for all $x, z \in M$, $\alpha, \beta \in \Gamma$.

Left-multiplying by $x\alpha$ and right-multiplying $\alpha[x, z]_\beta \alpha x$, we get $(x\alpha[x, z]_\beta \alpha x)\alpha(x\alpha[x, z]_\beta \alpha x) = 0$ for all $x, z \in M$, $\alpha, \beta \in \Gamma$. We obtain $x\alpha[x, z]_\beta \alpha x = 0$ for all $x, z \in M$, $\alpha, \beta \in \Gamma$. Left-multiplying by $[x, z]_\beta \alpha$ with using Lemma 1.4, we obtain $[x, z]_\beta \alpha x = 0$ for all $x, z \in M$, $\alpha, \beta \in \Gamma$. Right-multiplying by δz, we get $[x, z]_\beta \alpha x \delta z = 0$ for all $x, z \in M$, $\alpha, \beta, \delta \in \Gamma$. Again using the relation $[x, z]_\beta \alpha x = 0$ and replacing z by $z\delta z$,we obtain $[x, z]_\beta \alpha z \delta x = 0$ for all $x, z \in M$, $\alpha, \beta, \delta \in \Gamma$. Subtracting we obtain $x \in Z(M)$ for all $x \in M$.

Lemma 1.6

Let M be a 3-torsion free prime Γ-ring satisfying the condition (*) and let I be a non zero ideal of M. If there exists a permuting tri-derivation $D: M \times M \times M \to M$ such that d is commuting on I, where d is the trace of D, then we have $D = 0$.

Proof.

Suppose that

$$[d(x), x]_\beta = 0 \text{ for all } x \in I, \beta \in \Gamma \tag{2}$$

Linearizing (2) we get,

$$[d(x), y]_\beta + [d(y), x]_\beta + 3[D(x, x, y), x]_\beta + 3[D(x, y, y), x]_\beta + 3[D(x, x, y), y]_\beta + 3[D(x, y, y), y]_\beta = 0 \text{ for all } x, y \in I, \beta \in \Gamma \tag{3}$$

Putting –x instead of x in (3) and since d is odd, we obtain

$$[D(x, x, y), x]_\beta = 0 \text{ for all } x, y \in I, \beta \in \Gamma \tag{4}$$

Putting $x = x + y$ in (4) and then we obtain

$$[d(y), x]_\beta + 3[D(x, y, y), x]_\beta = 0 \text{ for all } x, y \in I, \beta \in \Gamma \tag{5}$$

Replacing $y\alpha x$ for x in (3) we get

$[d(y), y\alpha x]_\beta + 3[D(y\alpha x, x, y), y]_\beta = y\alpha[d(y), x]_\beta + 3d(y)\alpha[x, y]_\beta + 3y\alpha[D(x, y, y), y]_\beta = 0$ for all $x, y \in I$, $\alpha, \beta \in \Gamma$, which implies that

$$y\alpha([d(y), x]_\beta + 3\,[D(x, y, y), y]_\beta) + 3d(y)\alpha[x, y]_\beta = 0 \tag{6}$$

By using (5) we have $d(y)\alpha[x, y]_\beta = 0$ for all $x, y \in I$, $\alpha, \beta \in \Gamma$ on account of (5). Since I is a nonzero non-commutative prime Γ-ring, it follows from (3) and Lemma 1.3 that, for all $y \in I$ with $y \notin Z(M)$, we have $d(y) = 0$ since for every fixed $y \in I$, a map $x \to [x, y]_\beta$ is a derivation on I.

Now, let $x \in I$ with $x \in Z(M)$ and $y \in I$ with $y \notin Z(M)$. Then $x + y \notin Z(M)$ and $-y \notin Z(M)$. Thus we have

$d(x + y) = d(x) + 3D(x, x, y) + 3D(x, y, y) = 0$ which shows that $d(x - y) = d(x) - 3D(x, x, y) + 3D(x, y, y) = 0$ which shows that

$$d(x) + 3D(x, y, y) = 0 \tag{7}$$

Replacing $y \in I$ ($y \notin Z(M)$) by $2y$ in (7) we obtain that $D(x, y, y) = 0$ and so the relation (7) gives $d(x) = 0$ for all $x \in I$ with $x \in Z$. Therefore we obtain $d(x) = 0$ for all $x \in I$.

On the other hand, since the relation $D(x, x, y) + D(x, y, y) = 0$ fulfilled for all $x, y \in I$, it follows that

$$D(x, x, y) + D(x, y, y) = 0 \text{ for all } x, y \in I, \tag{8}$$

and substituting $y + z$ for y in (8) we obtain that $2D(x, y, z) = 0 = D(x, y, z)$ for all $x, y \in I$.

Let us substitute $w\alpha x$ ($w \in M$) for x in the above relation $D(x, y, z) = 0$ for all $x, y, z \in I$. Then we have

$D(w, y, z)\alpha x = 0$. Hence $D(w, y, z)\alpha x\beta D(w, y, z) = 0$.Since M is prime, we get $D(w, y, z) = 0$ for all $y, z \in I$, $w \in M$. Also, substituting $y\delta v$ ($v \in M$) for y in this relation, we have $y\delta D(w, v, z) = 0$ and so $D(w, v, z)\beta y\delta D(w, v, z) = 0$. Again, by primeness of M, we obtain that $D(w, v, z) = 0$ for all $z \in I$, $w, v \in M$. Furthermore, replacing z by $u\gamma z$ ($u \in M$, $\gamma \in \Gamma$) in the relation $D(w, v, z) = 0$, we get $D(w, v, u) = 0$. The primeness of M implies that $D(w, v, u) = 0$ for all $u, v, w \in M$.

Lemma 1.7

Let M be a 3-torsion free semiprime Γ-ring satisfying the condition (*) and I be a nonzero ideal of M. If there exists a permuting tri-derivation D: $M \times M \times M \rightarrow M$ such that d is centralizing on I, where d is the trace of D, then d is commuting on I.

Proof:

Assume that

$$[d(x), x]_\beta \in Z(M) \text{ for all } x \in I, \beta \in \Gamma \tag{9}$$

By linearizing (9) we get,

$$[d(x), y]_\beta + [d(y), x]_\beta + 3[D(x, x, y), x]_\beta + 3[D(x, y, y), x]_\beta + 3[D(x, x, y), y]_\beta + 3[D(x, y, y), y]_\beta \in Z(M), \text{ for all } x, y \in I, \beta \in \Gamma. \tag{10}$$

We substitute $-x$ for x in (10) we get

$$[D(x, y, y), x]_\beta + [D(x, x, y), y]_\beta \in Z(M), \text{ for all } x, y \in I, \beta \in \Gamma \tag{11}$$

Replacing x by $x + y$ in (11) we have

$$[d(y), x]_\beta + 3[D(x, y, y), y]_\beta \in Z(M), \text{ for all } x, y \in I, \beta \in \Gamma \tag{12}$$

Taking $x = y\delta y$ in (12) and invoking (9) show that

$$[d(y), y\delta y]_\beta + 3[D(y\delta y, y, y), y]_\beta = 8[d(y), y]_\beta \delta y \in Z(M), \text{ for all } y \in I, \beta, \delta \in \Gamma \quad (13)$$

Commuting with d(y) in (13) gives

$$8[d(y), y]_\beta \delta [d(y), y]_\beta = 0, \text{ for all } y \in I, \beta, \delta \in \Gamma \quad (14)$$

On the other hand, substituting x for $y\gamma x$ in (14)

$$[d(y), y\gamma x]_\beta + 3[D(y\gamma x, x, y), y]_\beta = y\gamma[d(y), x]_\beta + 3d(y)\gamma[x, y]_\beta + 3[D(x, y, y), y]_\beta$$
$$+ 4[d(y), y]_\beta \gamma x \in Z(M) \text{ for all } x, y \in I, \beta, \gamma \in \Gamma \quad (15)$$

Hence we have

$$[y\gamma\{[d(y), x]_\beta + 3[D(x, y, y), y]_\beta \gamma [x, y]_\beta\}, y]_\beta + [3d(y)\gamma[x, y]_\beta + 4[d(y), y]_\beta \gamma x, y]_\beta = 0$$
$$\text{for all } x, y \in I, \beta, \gamma \in \Gamma \quad (16)$$

So we get $[3d(y)\gamma[x, y]_\beta, y]_\beta + 7[d(y), y]_\beta \gamma [x, y]_\beta = 0$, for all $x, y \in I$, $\beta, \gamma \in \Gamma$, according to (14).

Substituting $d(y)\lambda x$ for x in (15), it follows that

$$d(y)\gamma\{3d(y)\lambda[[x, y]_\beta, y]_\beta + 7[d(y), y]_\beta \gamma [x, y]_\beta\} + 6d(y)\gamma[d(y), y]_\beta \gamma [x, y]_\beta$$
$$+ 7[d(y), y]_\beta \gamma [d(y), y]_\beta \gamma x, \text{ for all } x, y \in I, \beta, \gamma \in \Gamma, \quad (17)$$

which by (16) implies

$$6d(y)\gamma[d(y), y]_\beta \gamma [x, y]_\beta + 7[d(y), y]_\beta \gamma [d(y), y]_\beta \gamma x = 0 \text{ for all } x, y \in I, \beta, \gamma \in \Gamma \quad (18)$$

Letting $x = [d(y), y]_\beta$ in (18) we arrive at $[d(y), y]_\beta \gamma [d(y), y]_\beta \gamma [d(y), y]_\beta = 0$ and so we get

$$7[d(y), y]_\beta \gamma [d(y), y]_\beta \gamma 7[d(y), y]_\beta \gamma [d(y), y]_\beta = 0 \quad (19)$$

Since M is a semiprime Γ-ring $7[d(y), y]_\beta \gamma [d(y), y]_\beta = 0$ for all $x, y \in I$, $\beta, \gamma \in \Gamma$. Hence, the relations (16) and (19) yield $[d(y), y]_\beta \gamma [d(y), y]_\beta = 0$ for all $y \in I$, $\beta, \gamma \in \Gamma$. Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that $[d(y), y]_\beta = 0$ for all $y \in I$, $\beta \in \Gamma$. This completes the proof.

Lemma 1.8

Let M be a 3- torsion free prime Γ-ring and let I be a nonzero ideal of M. If there exists a nonzero permuting tri-derivation D: $M \times M \times M \to M$ such that d is centralizing on I, where d is the trace of D, then M is commutative.

Proof:

Suppose that M is non-commutative. Then it follows from Lemma 1.3 that is commuting on I. Hence Lemma 1.6 gives $D = 0$ which proves the Lemma.

Lemma 1.9

Let M be a semiprime Γ-ring satisfying the condition (*). If there exists $a \in M$ such that $a\alpha[x, y]_\beta = 0$ holds for all pairs $x, y \in M$, $\alpha, \beta \in \Gamma$. In this case, $a \in Z(M)$.

Proof:

We have $[z, a]_\beta \alpha x \delta [z, a]_\beta = z\beta a\alpha x\delta[z, a]_\beta - a\beta z\alpha x\delta[z, a]_\beta = z\beta a\alpha[z, x\delta a]_\beta - z\beta a\alpha[z, x]_\beta\delta a - a\beta[z, z\alpha x\delta a]_\beta + a\beta[z, z\alpha x]_\beta\delta a = 0$.

Hence $a \in Z(M)$. Since $z\alpha a\delta w\gamma[x, y]_\beta = 0$ for all $z, w, x, y \in M$, $\alpha, \beta, \delta, \gamma \in \Gamma$, we can repeat the above argument with $z\alpha a\gamma w$ instead of a to obtain $M\Gamma a\Gamma M \in Z(M)$ and now it is obvious that the ideal generated by a is central.

2. Permuting Tri-Derivations

We prove some results on permuting tri-derivations.

Theorem 2.1

Let M be a 3-torsion free semiprime Γ-ring satisfying the condition (*) and let I be a nonzero ideal of M. If there exists a permuting tri-derivation D: $M \times M \times M \to M$ such that d is an automorphism commuting on I, where d is the trace of D, then I is a nonzero commutative ideal.

Proof:

Suppose that

$$[d(x), x]_\beta = 0 \text{ for all } x \in I, \beta \in \Gamma. \quad (20)$$

Substituting x by $x + y$ leads to

$$[d(x), y]_\beta + [d(y), x]_\beta + 3[D(x, x, y), x]_\beta + 3[D(x, y, y), x]_\beta + 3[D(x, x, y), y]_\beta + 3[D(x, y, y), y]_\beta = 0 \text{ for all } x, y \in I, \beta \in \Gamma \quad (21)$$

Putting $-x$ instead of x in (21) we get

$$[D(x, y, y), x]_\beta + [D(x, x, y), y]_\beta = 0 \text{ for all } x, y \in I, \beta \in \Gamma. \quad (22)$$

Since d is odd, we set $x = x + y$ in (22) and then use (20) and (22) to obtain

$$[d(y), x]_\beta + 3[D(x, y, y), y]_\beta = 0 \text{ for all } x, y \in I, \beta \in \Gamma. \quad (23)$$

Let us write $y\alpha x$ instead of x in (23), we obtain

$[d(y), y\alpha x]_\beta + 3[D(y\alpha x, y, y), y]_\beta = y\alpha[d(y), x]_\beta + 3d(y)\alpha[x, y]_\beta + 3y\alpha[D(x, y, y), y]_\beta = y\alpha([d(y), x]_\beta + 3[D(x, y, y), y]_\beta) + 3d(y)\alpha[x, y]_\beta = 0$ for all $x, y \in I$, $\alpha, \beta \in \Gamma$. Then $d(y)\alpha[x, y]_\beta = 0$ for all $x, y \in I$, $\alpha, \beta \in \Gamma$, since d is an automorphism, we obtain $y\alpha[x, y]_\beta = 0$ for all $x, y \in I$, $\alpha, \beta \in \Gamma$. Replacing x by $y\alpha x$, we get

$y\alpha x\gamma[x, y]_\beta = 0$ for all $x, y\in I$, $\alpha, \beta, \gamma\in\Gamma$. (24)

Again left-multiplying by x implies that

$x\alpha y\gamma[x, y]_\beta = 0$ for all $x, y\in I$, $\alpha, \beta,\gamma\in\Gamma$. (25)

Subtracting (24) and (25) with using M is semiprime Γ-ring, we completes our proof. By same method in Theorem 2.1, it is easy to proof the following results.

Corollary 2.2

Let M be a 3-torsion free semiprime Γ-ring satisfying the condition (*) and I be an ideal of M. If there exists a permuting tri-derivation D: $M \times M \times M \to M$ such that d is commutating on I, where d is the trace of D, then I is a central ideal.

Theorem 2.3

Let M be a 3-torsion free semiprime Γ-ring satisfying the condition (*). If there exist a permuting tri-derivation D: $M \times M \times M \to M$ such that d is an automorphism commuting on M, where d is the trace of D, then M is commutative.

Theorem 2.4

Let M be a 6-torsion free semiprime Γ-ring satisfying the condition (*). If there exists a permuting tri-derivation D: $M \times M \times M \to M$ such that d is an automorphism centralizing on M, where d is the trace of D, then M is commutative.

Proof:

Assume that

$[d(x), x]_\beta\in Z(M)$ for all $x\in M$ and $\beta\in\Gamma$. (26)

Replacing x by $x + y$ and again using (26), we obtain

$[d(x), y]_\beta + [d(y), x]_\beta + 3[D(x, x, y), x]_\beta + 3[D(x, y, y), x]_\beta + 3[D(x, x, y), y]_\beta$
$+ 3[D(x, y, y), y]_\beta\in Z(M)$ for all $x, y\in M$, $\beta\in\Gamma$. (27)

Replacing x by $-x$ in (27) we get

$[D(x, y, y), x]_\beta + [D(x, x, y), y]_\beta\in Z(M)$ for all $x, y\in M$, $\beta\in\Gamma$. (28)

Replacing x by $x + y$ in (28), we obtain

$[d(y), x]_\beta + 3[D(x, y, y), y]_\beta\in Z(M)$ for all $x, y\in M$, $\beta\in\Gamma$. (29)

Taking $x = y\alpha y$ in (29) and invoking (26), we get

$[d(y), y\alpha y]_\beta + 3[D(y\alpha y, y, y), y]_\beta = 8[d(y), y]_\beta\alpha y\in Z(M)$ for all $y\in M$, $\alpha, \beta\in\Gamma$. (30)

Now commuting (30) with $d(y)$ yields

$8[d(y), y]_\beta\alpha[d(y), y]_\beta = 0$ for all $y\in M$, $\alpha, \beta\in\Gamma$.

Again substituting x by $y\alpha x$ in (29) gives

$[d(y), y\alpha x]_\beta + 3[D(y\alpha x, y, y), y]_\beta = y\alpha([d(y), x]_\beta + 3[D(x, y, y), y]_\beta) + 3d(y)\alpha[x, y]_\beta + 4[d(y), y]_\beta\alpha x\in Z(M)$ for all $x, y\in M$, $\alpha, \beta\in\Gamma$. Then $[y\alpha([d(y), x]_\beta + 3[D(x, y, y), y]_\beta), y]_\beta + [3d(y)\alpha[x, y]_\beta + 4[d(y), x]_\beta\alpha x, y]_\beta = 0$ for all $x, y\in M$, $\alpha, \beta\in\Gamma$. And so we get

$$3d(y)\alpha[[x, y]_\beta, y]_\beta + 7[d(y), y]_\beta\alpha[x, y]_\beta = 0 \text{ for all } x, y\in M, \alpha, \beta\in\Gamma. \quad (31)$$

Since d acts as an automorphism with M is 6-torsion free the relation (31) reduces to $y\alpha[[x, y]_\beta, y]_\beta = 0$ for all $x, y\in M$, $\alpha,\beta\in\Gamma$.

Replacing x by $r\delta x$, we get

$$y\alpha x\delta\, [[x, y]_\beta, y]_\beta + 2y\alpha[x, y]_\beta = 0 \text{ for all } x, y\in M, \alpha, \beta, \delta\in\Gamma. \quad (32)$$

Replacing y by $-y$ in (32) and subtracting with (32), gives

$$4y\delta[x, y]_\beta = 0 \text{ for all } x, y\in M, \beta, \delta\in\Gamma. \quad (33)$$

Replacing x by $x\gamma r$ and left-multiplying by s, we obtain

$$4y\delta x\alpha[r, y]_\beta = 0 \text{ for all } x, y, r, s\in M, \alpha, \beta, \delta\in\Gamma. \quad (34)$$

Again in (33) replacing x by xλm and x by sδx, we get

$$4y\gamma s\delta x\alpha[m, y]_\beta = 0 \text{ for all } x, y, m, s\in M, \alpha, \beta, \delta, \gamma\in\Gamma. \quad (35)$$

Subtracting (34) and (35) with using M is 6-torsion free semiprime, we obtain $[s, y]_\beta = 0$ for all $s, y\in M$. Thus, we get M is commutative.

Theorem 2.5

Let M be a 3-torsion free semiprime Γ-ring satisfying the condition (*). If there exists a permuting tri-derivation D: $M \times M \times M \to M$ such that d is commuting on M, where d is the trace of D, then d is a central mapping.

Proof:

We have

$$[d(x), x]_\beta = 0 \text{ for all } x\in M, \beta\in\Gamma. \quad (36).$$

The substitution of x in (36) by $x + y$ leads to

$$[d(x), y]_\beta + [d(y), x]_\beta + 3[D(x, x, y), x]_\beta + 3[D(x, y, y), x]_\beta + 3[D(x, x, y), y]_\beta$$
$$+ 3[D(x, y, y), y]_\beta = 0 \text{ for all } x, y\in M, \beta\in\Gamma. \quad (37)$$

Putting $-x$ instead of x in (37) we obtain,

$$[D(x, y, y), x]_\beta + [D(x, x, y), y]_\beta = 0 \text{ for all } x, y\in M, \beta\in\Gamma. \quad (38)$$

Since d is odd, we set $x = x + y$ in (38) with using (36) and (37), we get

$$[d(y), x]_\beta + 3[D(x, y, y), y]_\beta = 0 \text{ for all } x, y \in M, \beta \in \Gamma. \quad (39)$$

Let us write in (39) $y\alpha x$ instead of x, we obtain according to (39) and since M is 3-torsion semiprime $d(y)\alpha[x, y]_\beta = 0$ for all $x, y \in M$, $\beta \in \Gamma$.

Applying Lemma 1.9, the above relation gives $d(y) \in Z(M)$ for all $y \in M$, thus we completes the proof of the theorem.

Theorem 2.6

Let M be a 3-torsion free semiprime Γ-ring. If there exists a permuting tri-derivation D: $M \times M \times M \rightarrow M$ such that d is commuting on M, where d is the trace of D, then D is commuting (resp. centralizing).

Proof:

We can restrict our attention to the relation

$$[d(x), x]_\beta = 0 \text{ for all } x \in M, \beta \in \Gamma. \quad (40)$$

The substitution of $x + y$ for x in above relation gives

$$[d(x), y]_\beta + [d(y), x]_\beta + 3[D(x, x, y), x]_\beta + 3[D(x, y, y), x]_\beta + 3[D(x, x, y), y]_\beta$$
$$+ 3[D(x, y, y), y]_\beta = 0 \text{ for all } x, y \in M, \beta \in \Gamma \quad (41)$$

Now, by the same method in Theorem 2.5, we arrive at

$$y\delta[d(y), x]_\beta + 3d(y)\delta[x, y]_\beta + 3y\delta[D(x, y, y), y]_\beta = 0 \text{ for all } x, y \in M, \beta, \delta \in \Gamma. \quad (42)$$

which implies that

$$d(y)\delta[x, y]_\beta = 0 \text{ for all } x, y \in M, \beta, \delta \in \Gamma. \quad (43)$$

Applying Lemma 1.5, the above relation gives $d(y) \in Z(M)$ for all $x \in M$. By substitution the relation $d(y) \in Z(M)$ in (41) with using replacing x by y and M is 3-torsion free semiprime, we obtain

$$[D(y, y, y), y]_\beta = 0 \text{ for all } x, y \in M, \beta \in \Gamma \quad (44)$$

Then D is commuting (resp. centralizing) of M.

Theorem 2.7

Let M be a non-commutative 3-torsion free semiprime Γ-ring satisfying the condition (*). If there exists a permuting tri-derivation D: $M \times M \times M \rightarrow M$ such that d is skew-commuting on M, where d is the trace of D, then d is commuting.

Proof:

We have $d(x)\alpha x + x\alpha d(x) = 0$ for all $x \in M$. Replacing x by $x + y$, we obtain

$$d(y)\alpha x + 3D(x, x, y)\alpha x + 3D(x, y, y)\alpha x + d(x)\alpha y + 3D(x, x, y)\alpha y + 3D(x, y, y)\alpha y + x\alpha d(y) + 3x\alpha D(x, x, y) + 3x\alpha D(x, y, y) + y\alpha d(y) + 3y\alpha D(x, x, y) + 3y\alpha D(x, y, y) = 0 \text{ for all } x, y\in M, \alpha\in \Gamma \quad (45)$$

We substitute $-x$ for x in (45) we get $3D(x, y, y)\alpha x + 3D(x, x, y)\alpha y + 3x\alpha D(x, y, y) + 3y\alpha D(x, x, y) = 0$ for all $x, y\in M$, $\alpha\in \Gamma$.

Since M is 3-torsion free, we obtain

$$D(x, x, y)\alpha x + D(x, x, y)\alpha y + x\alpha D(x, y, y) + y\alpha D(x, x, y) = 0 \text{ for all } x, y\in M, \alpha\in \Gamma \quad (46)$$

Again we substituting $x\beta y$ for x in (46) then we get

$$x\alpha D(y, y, y)\beta y + D(x, y, y)\alpha x\beta y + x\alpha y\beta D(y, y, y) + D(x, y, y)\alpha y = 0 \text{ for all } x, y\in M, \alpha, \beta\in \Gamma \quad (47)$$

We substitute $-x$ for x in (47) and compare (47) with the result to get $D(x, y, y)\alpha x\beta y = 0$ for all $x, y\in M$. Replacing x by y and since d is the trace of D, we obtain $d(y)\alpha y\beta y = 0$ for all $y\in M$. Left- multiplying by y and right-multiplying by $d(y)\delta y$ with using Lemma 1.4, we obtain

$$y\delta d(y)\beta y = 0 \text{ for all } y\in M, \beta, \delta\in \Gamma. \quad (48)$$

Left- multiplying (48) by $d(y)$ with using Lemmas (1.1and 1.3) gives

$$d(y)\beta y = 0 \text{ for all } y\in M, \beta\in \Gamma. \quad (49)$$

Right- multiplying (48) by $d(y)$ with using Lemmas (1.1 and 1.3) and subtracting the result with (49), we obtain $[d(y), y]_\beta = 0$ for all $y\in M$, $\beta\in \Gamma$.

By Theorem 2.3, we complete our proof.

By the same method in Theorem 2.8, with using Lemmas (1.4 and 1.5), it is easy to proof the following corollary.

Theorem 2.8

Let M be a non-commutative 3-torsion free semiprime Γ-ring satisfying the condition (*) and I be a non-zero ideal of M. If there exists a permuting tri-derivation D: $M \times M \times M \rightarrow M$ such that d is skew-commuting on I, where d is the trace of D, then d is commuting on I.

Proof:

Using the same method in Theorem 2.7, with Lemma 1.7, we complete the proof of the Theorem.

Theorem 2.9

Let M be a non commutative 3-torsion free semiprime Γ-ring satisfying the condition (*) and I be a nonzero ideal of M. If there exists a permuting tri-derivation D: $M \times M \times M \to M$ such that d is skew- centralizing on I where d is the trace of D, then d is commuting on I.

Proof:

Using same method in Theorem 2.7, we obtain $[d(x)\alpha y\delta y, r]_\beta \in Z(M)$ for all $x \in I$, $r \in M$, α, β, $\delta \in \Gamma$, replacing r by y with using Lemma 1.7, we complete the proof of the theorem.

Corollary 2.10

Let M be a 3-torsion free prime Γ-ring satisfying the condition (*) and I be a nonzero ideal of M. If there exists a nonzero a permuting tri-derivation D: $M \times M \times M \to M$ such that d is skew-centralizing on I where d is the trace of D, then M is commutative.

Proof:

Suppose that M is non-commutative, then by the same method in Theorem 2.9, we get $[d(x), \mathrm{x}]_\beta \in Z(M)$ for all $x \in I$, $\beta \in \Gamma$. Hence by Lemma 1.8, the proof of the corollary is complete.

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