# Common Fixed Points of Compatible Mappings

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#### Abstract

In this paper we prove two common fixed point theorems by considering four mappings in complete metric space. In the first theorem we consider two pairs of compatible mappings of type (A) and in the second theorem we consider two pairs of compatible mappings of type (B). Our results modify and extend some earlier results in the literature.

*Keywords:* Fixed point; Complete metric space; Compatible mappings.

## 1. Introduction

The first important result in the theory of fixed point of compatible mappings was obtained by Jungck [1] as a generalization of commuting mappings. In 1993 Jungck *et al*. [2] introduced the concept of compatible mappings of type (A) by generalizing the definition of weakly uniformly contraction maps. The concept of compatible mappings of type (B) was introduced by Pathak and Khan [3] in the year 1995. Recently, Nema and Qureshi [4] proved two common fixed point theorems of compatible mappings of type (P). Koireng *et. al.* [5] also proved another theorem of compatible mappings of type (R).

The aim of this paper is to prove some common fixed point theorems of compatible mappings of type (A) and type (B) in metric spaces by considering four self mappings. Our results extend and modify the results in [4-6].

## 2. Preliminaries

We recall definitions of various types of compatible mappings and other results which will be needed in the sequel.

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**Definition 1.1** [1]: Let $S$ and $T$ be mappings from a complete metric space $X$ into itself. The mappings $S$ and $T$ are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

**Definition 1.2** [2]: Let $S$ and $T$ be mappings from a complete metric space $X$ into itself. The mappings $S$ and $T$ are said to be compatible of type (A) if $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ and

$\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in X.$$

**Definition 1.3** [5]: Let $S$ and $T$ be mappings from a complete metric space $X$ into itself. The mappings $S$ and $T$ are said to be compatible of type (B) if

$$\lim_{n\to\infty} d(STx_n, TTx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, SSx_n)\right]$$

and $\lim_{n\to\infty} d(TSx_n, SSx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, TTx_n)\right]$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

**Proposition 2.4** [2]: Let $S$ and $T$ be mappings from a complete metric space $(X, d)$ into itself. If a pair $\{S, T\}$ is compatible of type (A) on $X$ and $Sz = Tz$ for $z \in X$, then

$STz = TSz = SSz = TTz$.

**Proposition 2.5**[2]: Let $S$ and $T$ be mappings from a complete metric space $(X, d)$ into itself. If a pair $\{S, T\}$ is compatible of type (A) on $X$ and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some

$z \in X$, then we have

(i) $d(TSx_n, Sz) \to 0$ as $n \to \infty$ if $S$ is continuous,

(ii) $STz = TSz$ and $Sz = Tz$ if $S$ and $T$ are continuous at $z$.

**Proposition 2.6**[5]: Let $S$ and $T$ be mappings from a complete metric space $(X, d)$ into itself. If a pair $\{S, T\}$ is compatible of type (B) on $X$ and $Sz = Tz$ for $z \in X$, then

$STz = TSz = SSz = TTz$.

**Proposition 2.7**[5]: Let $S$ and $T$ be mappings from a complete metric space $(X, d)$ into itself. If a pair $\{S, T\}$ is compatible of type (B) on $X$ and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some

$z \in X$, then we have

(i) $d(TTx_n, Sz) \to 0$ as $n \to \infty$ if $S$ is continuous.

(ii) $d(SSx_n, Tz) \to 0$ as $n \to \infty$ if $T$ is continuous.

(iii) $STz=TSz$ and $Sz=Tz$ if $S$ and $T$ are continuous at $z$.

**Proposition 2.8** [3]: If $S$ and $T$ be continuous from a metric space $X$ into itself then

(i) $S$ and $T$ are compatible if and only if they are compatible of type (B).

(ii) $S$ and $T$ are compatible of type (A) if and only if they are compatible of type (B).

**Remark** [3]: Proposition 2.8 is not true if $S$ and $T$ are not continuous.

Following example will illustrate the remark.

**Example**: Let $X = R$ with the metric $d$ and define two mappings $S$ and $T$: $X \to X$ as follows:

$$Sx = \begin{cases} \dfrac{1}{x^4} & \text{if } x \neq 0 \\ \dfrac{1}{4} & \text{if } x = 0 \end{cases} \qquad \text{and} \qquad Tx = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ \dfrac{1}{3} & \text{if } x = 0 \end{cases}$$

Both $S$ and $T$ are discontinuous at $z = 0$. Consider a sequence $\{x_n\}$ in $X$ defined by $x_n = n^2$, $n = 1, 2, 3,....$ Then $\lim\limits_{n \to \infty} Sx_n = \lim\limits_{n \to \infty} Tx_n = 0 = z$. Since $d(STx_n, TSx_n) = 0$ the pair $\{S, T\}$ is compatible on $X$. But $d(STx_n, TTx_n) \to \infty$ as $n \to \infty$ the pair $\{S, T\}$ is not compatible of type (A) on $X$.

**Lemma 2.9** [6] Let $A$, $B$, $S$ and $T$ be mapping from a metric space $(X, d)$ into itself satisfying the following conditions:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2) $[d(Ax, By)]^2 \leq a[d(Ax, Sx)d(By, Ty)+d(By, Sx)d(Ax, Ty)]$
$\qquad\qquad + b[d(Ax, Sx)d(Ax, Ty)+d(By, Ty)d(By, Sx)]$

where $0 \leq a + 2b < 1$; $a, b \geq 0$

(3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for $x_1$ there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in $X$ such that

$y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$

then the sequence $\{y_n\}$ is Cauchy sequence in $X$.

## 3. Main Results

In this section we prove a common fixed point theorem of compatible mappings of type (A). Another theorem of compatible mappings of type (B) is also given without prove.

**Theorem 3.1:** Let $A$, $B$, $S$ and $T$ be self maps of a complete metric space $(X, d)$ satisfying the following conditions:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2) $[d(Ax, By)]^2 \leq a[d(Ax, Sx)d(By, Ty)+d(By, Sx)d(Ax, Ty)]$
$$+ b[d(Ax, Sx)d(Ax, Ty)+d(By, Ty)d(By, Sx)]$$

where $0 \leq a + 2b < 1$; $a, b \geq 0$

(3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for $x_1$ there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in $X$ such that

$y_{2n+1}=Tx_{2n+1}=Ax_{2n}$ and $y_{2n}= Sx_{2n}= Bx_{2n-1}$

then the sequence $\{y_n\}$ is Cauchy sequence in $X$.

(4) One of $A$, $B$, $S$ or $T$ is continuous.

(5) $\{A, S\}$ and $\{B, T\}$ are compatible of type (A) on $X$.

Then $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.

**Proof:** By lemma 2.9, $\{y_n\}$ is Cauchy sequence and since $X$ is complete so there exists a point $z \in X$ such that $\lim y_n = z$ as $n \to \infty$. Consequently subsequences $Ax_{2n}$, $Sx_{2n}$, $Bx_{2n-1}$ and $Tx_{2n+1}$ converges to $z$.

Let $S$ be continuous. Since $A$ and $S$ are compatible of type (A) on $X$, then by proposition 2.5 we have $S^2x_{2n} \to Sz$ and $ASx_{2n} \to Sz$ as $n \to \infty$.
Now by condition (2) of lemma 2.9, we have

$[d(ASx_{2n}, Bx_{2n-1})]^2 \leq a[d(ASx_{2n}, S^2x_{2n})d(Bx_{2n-1}, Tx_{2n-1}) + d(Bx_{2n-1}, S^2x_{2n})d(ASx_{2n-1}, Tx_{2n-1})]$
$+ b[d(ASx_{2n}, S^2x_{2n})d(ASx_{2n}, Tx_{2n-1}) + d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, S^2x_{2n})]$

As $n \to \infty$, we have

$[d(Sz, z)]^2 \leq a[d(Sz, z)]^2,$

which is a contradiction. Hence $Sz = z$,

Now $[d(Az, Bx_{2n-1})]^2 \leq a[d(Az, Sz)d(Bx_{2n-1}, Tx_{2n-1}) + d(Bx_{2n-1}, Sz)d(Az, Tx_{2n-1})]$
$+ b[d(Az, Sz)d(Az, Tx_{2n-1}) + d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, Sz)]$

Letting $n \to \infty$, we have $[d(Az, z)]^2 \leq b[d(Az, z)]^2$. Hence $Az = z$.

Now since $Az = z$, by condition (1) $z \in T(X)$. Also $T$ is self map of $X$ so there exists a point $u \in X$ such that $z = Az = Tu$. More over by condition (2), we obtain,

$[d(z, Bu)]^2 = [d(Az, Bu)]^2 \le a[d(Az, Sz)d(Bu, Tu) + d(Bu, Sz)d(Az, Tu)]$

$+ b[d(Az, Sz)d(Az, Tu) + d(Bu, Tu)d(Bu, Sz)]$

i.e., $[d(z, Bu)]^2 \le b[d(z, Bu)]^2$.

Hence $Bu = z$ i.e., $z = Tu = Bu$.

By proposition 2.4, we have $TBu=BTu$

Hence $Tz = Bz$.

Now,

$[d(z, Tz)]^2 = [d(Az, Bz)]^2 \le a[d(Az, Sz)d(Bz, Tz) + d(Bz, Sz)d(Az, Tz)]$

$+ b[d(Az, Sz)d(Az, Tz) + d(Bz, Tz)d(Bz, Sz)]$

i.e., $[d(z, Tz)]^2 \le a[d(z, Tz)]^2$ which is a contradiction. Hence $z = Tz$ i.e, $z = Tz = Bz$.

Therefore $z$ is common fixed point of $A$, $B$, $S$ and $T$. Similarly we can prove this when any one of $A$, $B$ or $T$ is continuous.

Finally, in order to prove the uniqueness of $z$, suppose $w$ be another common fixed point of $A$, $B$, $S$ and $T$ then we have,

$[d(z, w)]^2 = [d(Az, Bw)]^2 \le a[d(Az, Sz)d(Bw, Tw) + d(Bw, Sz)d(Az, Tw)]$

$+ b[d(Az, Sz)d(Az, Tw) + d(Bw, Tw)d(Bw, Sz)]$

which gives

$[d(z, Tw)]^2 \le a[d(z, Tw)]^2$. Hence $z = w$.

This completes the proof.

**Theorem 3.2:** Let $A$, $B$, $S$ and $T$ be self maps of a complete metric space $(X, d)$ satisfying the following conditions:

(1)  $A (X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2)  $[d(Ax, By)]^2 \le a[d(Ax, Sx)d(By, Ty)+d(By, Sx)d(Ax, Ty)]$

$+ b[d(Ax, Sx)d(Ax, Ty)+d(By, Ty)d(By, Sx)]$

where   $0 \le a + 2b < 1$; $a, b \ge 0$

(3)  Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for $x_1$ there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in $X$ such that

$y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$

then the sequence $\{y_n\}$ is Cauchy sequence in $X$.

(4) One of $A$, $B$, $S$ or $T$ is continuous.

(5) $\{A, S\}$ and $\{B, T\}$ are compatible of type (B) on $X$.

Then $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.

**Proof:** Proof is in the same line as in theorem 3.1.

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