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Semi-Image Neighborhood Block Graphs with Crossing Numbers

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Abstract

The advent of graph theory has played a prominent role in wide variety of engineering applications. Minimization of crossing numbers in graphs optimizes its use in many applications. In this paper, we establish the necessary and sufficient condition for Semi-Image neighborhood block graph to have crossing number 3. We also prove that the Semi-Image neighborhood block graph $NB_I(G)$ of a graph never has crossing numbers k, where $1 \le k \le 6$.

Keywords: Semi-Image; Neighborhood block; Crossing numbers.

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1. Introduction

Crossing number minimization is one of the fundamental optimization problems because it is related to various other widely used notions. Besides its mathematical interest, there are numerous applications, most notably those in VLSI design and in combinatorial geometry [1]. Researchers in computer science are focusing their attention to this area of graph theory as the study of crossing numbers of graphs finds applications in network design, higraph system model and circuit layout. Minimizing the number of wire crossings in a circuit greatly reduces the chance of cross-talk in long crossing wires carrying the same signal and also allows for faster operation and less power dissipation. It is also an important measure of non-planarity of a graph.

Let $G = (V, E)$ be a simple finite, undirected and without loops or multiple lines graph with vertex set V and edge set E. A drawing D of a graph G is a representation of G in the

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Euclidean plane R^2 where vertices are represented as distinct points and edges by simple polygonal arcs joining points that correspond to their end vertices. A drawing D is good or clean, if no edge crosses itself, no pair of adjacent edges cross, two edges cross at most once and no more than two edges cross at one point. The number of crossings of D is denoted by $Cr(D)$ and is called the crossing number of the drawing D. The crossing number $Cr(G)$ of a graph G is the minimum $Cr(D)$ taken from over all good or clean drawings D of G. If a graph G admits a drawing D with $Cr(D)$ =0 then G is said to be planar; otherwise non-planar. It is well known that K_5, the complete graph on five vertices and $K_{3,3}$ the complete bipartite graph with three vertices in its classes are non-planar. According to Kuratowski's famous theorem, a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ [2,3]. The study of crossing numbers began during the Second World War with Paul Turan. The bounds for $Cr(K_n)$ and $Cr(K_{m,n})$ are obtained by Guy [4]. The Semi-Image neighborhood block graph $NB_I(G)$ of a graph G is defined as the graph that has the point set $V(G)$ U $V^{'}(G)$ U $b(G)$. Two points of $V(G)$ and $V^{'}(G)$ are adjacent, if they are adjacent in G. Two points V_i and $V^{'}_j$ are adjacent, if $i=j$ as well as two points of $V(G)$ are incident on $b(G)$. This concept is introduced in ref. [5]. The crossing Number problem for generalized Peteresen graph has been investigated in ref. [6].

The following theorems are used to prove our main results.

Theorem A [5]: The Semi-Image neighborhood block graph $NB_I(G)$ of a graph G *i*s planar if and only if G is outer planar.

Theorem B [5]: The Semi-Image neighborhood block graph $NB_I(G)$ of a graph G is outer planar if and only if every component of G is a path.

Lemma 1: If G is K_4 or $K_{2,3}$, then $Cr[NB_I(G)]$ =3.

Proof: Suppose G is K_4 or $K_{2,3}$. then, we can easily say that the graphs $NB_I(K_4)$ and $NB_I(K_{2,3})$ are non-planar and drawings in Fig. 1 and Fig. 2 show that the crossing number is at most three. One may also observe that $NB_I(K_4)$ and $NB_I(K_{2,3})$ can not be drawn without crossing K_4 or $K_{2,3}$ line.

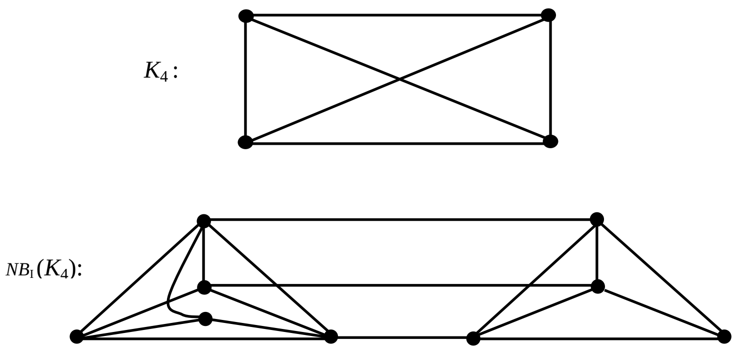

Fig. 1. Semi-Image neighborhood block graph K_4 with crossing number at most 3.

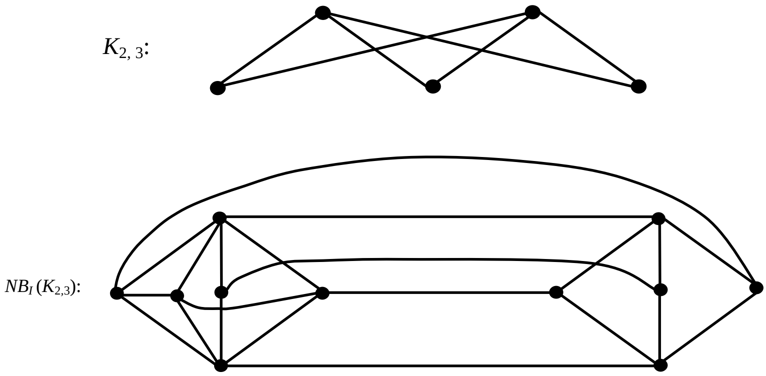

Fig. 2. Semi-Image neighborhood block graph $K_{2,3}$ with crossing number at most three.

However, Fig. 3 shows that the deletion of such a point results a non-planar graph (it contains a homeomorphism of $K_{3,3}$). Therefore, the crossing number must be equal to three. □

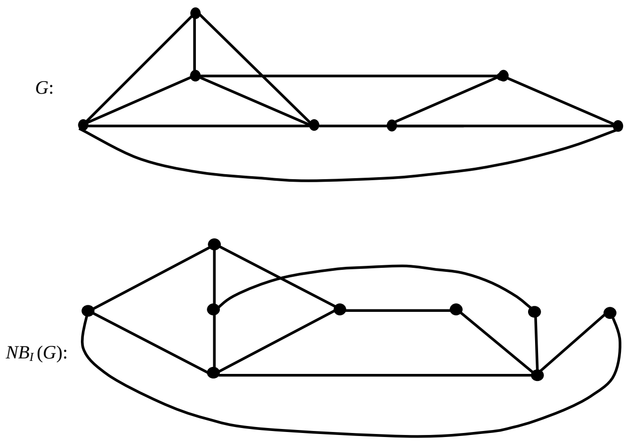

Fig. 3. Semi-Image neighborhood block graph *G* with crossing number three which is a homeomorphism of $K_{3,3}$.

In fact, we can also prove that for any graph *G*, the semi-image neighborhood block graph $NB_I(G)$ of a graph *G*, never has crossing number k; where $1 \leq k \leq 6$ (i.e. k=1,2,4,5 or 6).

Theorem 1: The semi-image neighborhood block graph $NB_I(G)$ of a graph *G* never has crossing number k where $1 \leq k \leq 6$ (i.e. k=1, 2, 4, 5 or 6) except for K_4 *or* $K_{2,3}$.

Proof: Consider the following two cases.

Case 1. Suppose Δ $(G) \leq 2$, then, G is either a path or a cycle. Consider the following sub cases of this case.

Sub case 1.1. Assume G is a path. Then by Theorem A and Theorem B, NB_I (G) is planar and has crossing number zero.

Sub case 1.2. Assume G *i*s a cycle. Then by Theorem A, NB_I (G) is planar. Hence Cr $[NB_I\,(G)] =0$.

Case 2. Suppose Δ $(G) \geq 3$, we consider the following sub cases of this case.

Sub case 2.1**.** Assume G is a tree. Then by Theorem A, Cr $[NB_I\,(G)]$ =0.

Sub case 2.2. Assume G is not a tree. Again, we consider the sub cases of this sub case.

Sub case 2.2.1. Suppose every block of G is a cycle or G has a block which contains at least two cycles such that $i(G)=0$, then by Theorem A, $NB_I(G)$ has crossing number zero.

Sub case 2.2.2. Suppose every block of G has a cycle or G has a block which contains at least two cycles and assume that there exists a block B with $i(B) \geq 1$. Then by Lemma 1, NB_I (B) has at least three crossings. Hence $Cr[NB_I(G)] \geq 7$. □

After trying all possibilities, in each case, we can find that NB_I (G) of a graph G never has crossing number k; where k=1 *or* 2.

Now, the characterization of semi-image neighborhood block graphs with crossing number three is illustrated as Theorem 2 below.

Theorem 2: The semi-image neighborhood block graph NB_I (G) of a graph G has crossing number three if and only if K_4 or $K_{2,3}$.

Proof: Suppose G is a graph satisfying K_4 or $K_{2,\,3}$, then by Theorem A and Theorem 1, NB_I (G) has crossing number exceeding two. Hence, we can show that its crossing number cannot exceed three.

First, consider K_4 of Figure 1. Let *'v'* be the point of degree three as in Figure 1. By Theorem A, NB_I $(G\text{-}v)$ is planar and NB_I (G) can be drawn in the plane with three crossings similar to the illustration in Figure 1.

Now, consider $K_{2,3}$ of Figure 2. Let *'v'* be the point of degree two as in Figure 2. By Theorem A, NB_I $(G\text{-}v)$ is planar and NB_I (G) can be drawn in the plane with three crossings similar to the illustration in Figure 2.

Conversely, suppose NB_I (G) has crossing number three and assume G is a graph with at most three points. Then by Theorem A, NB_I (G) is planar, a contradiction!

Suppose G is a graph with p=4 points and assume Δ (G) =3. Then, the following cases are considered.

*Case*1. Assume G is a graph of order 4. We consider the following sub cases of this case.

Sub case 1.1. Assume G is a tree. Then by Theorem A, NB_I (G) is planar, a contradiction!

Sub case 1.2**.** Assume G is not a tree. We consider the following sub cases of this sub case.

*Sub case*1.2.1. Suppose G is C_4, then by Theorem A, NB_I (G) is planar, a contradiction!

Sub case 1.2.2. Suppose G is a path of length one, together with the triangle adjoined to some end point, then by Theorem A, $NB_I(G)$ is planar, a contradiction!

From the above cases, it can be concluded that G is K_4 or $K_{2, 3}$. □

Next, we can also prove that the Semi-Image neighborhood block graph of any non-planar graph has crossing number greater than six.

Theorem3: The Semi-Image neighborhood block graph of any non-planar graph has crossing number at least seven.

Proof: By Kuratowski's theorem on planar graphs, it is sufficient to prove that NB_I (K_5) and NB_I $(K_{3, 3})$ have crossing number at least seven. The graphs NB_I $(K_{3, 3})$ and NB_I (K_5) are isomorphic to K_2 x $K_{3,3}$ and K_2 x K_5. If $NB_I(K_5)$ and $NB_I(K_{3,3})$ have crossing number less than seven, then the deletion of a few pairs of its points must result in a non-planar graph with at least two crossings. But, Theorem 1 and Theorem 2 imply that any graph obtained by removing a point from K_5 or $K_{3, 3}$ has a non-planar semi-image neighborhood block graph with at least three crossings. Hence, the Semi-Image neighborhood block graph of both K_5 and $K_{3, 3}$ has at least seven crossings in each drawing.

3. Conclusion

In this paper, the necessary and sufficient condition is established for Semi-Image neighborhood block graph to have crossing number three. The proof of "Semi-Image neighborhood block graph NB_I (G) of a graph never has crossing numbers k, where $1 \leq k \leq 6$" is also given. Finally, the necessary and sufficient conditions are provided for semi-image neighborhood block graph $NB_I(G)$ of a graph to have forbidden sub graphs for crossing numbers k (k = 1, 2 or 3).

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