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Dynamics of Boundary Graphs

G. Marimuthu[1*] and M. S. Saraswathy[2]

[1]Department of Mathematics, The Madura College, Madurai-625 011, India

[2]Department of Mathematics, Thiagarajar College of Engineering, Madurai-625 015, India

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Abstract

In a graph G, the distance d(u,v) between a pair of vertices u and v is the length of a shortest path joining them. A vertex v is a boundary vertex of a vertex u if $d(u,w) \le d(u,v)$ for all $w \in N(v)$. The boundary graph $B(G)$ based on a connected graph G is a simple graph which has the vertex set as in G. Two vertices u and v are adjacent in B(G) if either u is a boundary of v or v is a boundary of u. If G is disconnected, then each vertex in a component is adjacent to all other vertices in the other components and is adjacent to all of its boundary vertices within the component. Given a positive integer m, the m^{th} iterated boundary graph of G is defined as $B^m(G) = B(B^{m-1}(G))$. A graph G is periodic if $B^m(G) \cong G$ for some m. A graph G is said to be an eventually periodic graph if there exist positive integers m and $k > 0$ such that $B^{m+i}(G) \cong B^i(G), \forall i \ge k$. We give the necessary and sufficient condition for a graph to be eventually periodic.

Keywords: Boundary graph; Periodic graph.

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1. Introduction and Definitions

The graphs considered here are nontrivial and simple. For other graph theoretic notation and terminology, we follow [1]. In a graph G, the distance d(u,v) between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The radius $r(G)$ of G is defined as $r(G) = \min\{e(u) : u \in V(G)\}$ and the diameter $d(G)$ of G is defined as $d(G) = \max\{e(u) : u \in V(G)\}$. A graph G for which $r(G) = d(G)$ is called a self-centered graph of radius $r(G)$. A vertex v is called an eccentric vertex of a vertex u if d(u,v) = $e(u)$.

* *Corresponding author*: yellowmuthu@yahoo.com

A vertex v of G is called an eccentric vertex of G if it is an eccentric vertex of some vertex of G. The eccentric graph based on G is denoted by G_e, whose vertex set is $V(G)$ and two vertices u and v are adjacent in G_e if and only if $d(u,v) = \min\{e(u),e(v)\}$.

Gimbert *et al.* [3] studied the iterations of eccentric digraphs. The eccentric digraph of a digraph G, denoted by $ED(G)$, is the digraph on the same vertex set as in G but with an arc from a vertex u to a vertex v in $ED(G)$ if and only if v is an eccentric vertex of u in G. Given a positive integer k, the k^{th} iterated eccentric digraph of G is written as $ED^k(G) = ED(ED^{k-1}(G))$ where $ED^0(G) = G$. For every digraph G, there exists smallest integer $p' > 0$ and $t' \geq 0$ such that $ED^{t'}(G) \cong ED^{p'+t'}(G)$, where $\cong$ denotes graph isomorphism. We call p', the iso-period of G and t', the iso-tail of G; these quantities are denoted by $p'(G)$ and $t'(G)$, respectively.

Kathiresan and Marimuthu [4] introduced a new type of graph called radial graph. Two vertices of a graph G are said to be radial to each other if the distance between them is equal to the radius of the graph. Two vertices of graph G are said to be radial to each other if the distance between them is equal to the radius of the graph. The radial graph of a graph G denoted by $R(G)$ has the vertex as in G and two vertices are adjacent in $R(G)$ if and only if they are radial in G. If G is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of G. A graph G is called a radial graph if $R(H) \cong G$ for some graph H. In [5] Kathiresan et al. studied the properties of iteration of radial graphs. Given a positive integer m, the m^{th} iterated radial graph of G is defined as $R^m(G) = R(R^{m-1}(G))$. Note that $R^0(G) \cong G$. A graph G is periodic if $R^m(G) \cong G$ for some m. If p is the least positive integer with this property, then G is called a periodic graph with iso-period p. When $p=1$, G is called as a fixed graph. A graph G is said to be eventually periodic if there exist positive integers m and $k>0$, such that $R^{m+i}(G) \cong R^i(G), \forall i \geq k$. If p and k are the least positive integers with this property, then G is eventually periodic with iso-period p and iso-tail k.

Based on the concept of radial graphs, Marimuthu and Sivanandha Saraswathy [6] introduced the concept of boundary graphs. A vertex v is a boundary vertex of a vertex u if $d(u,w) \leq d(u,v)$ for all $w \in N(v)$. The boundary graph $B(G)$ based on a connected graph G is a simple graph which has the vertex set as in G. Two vertices u and v are adjacent in $B(G)$ if either u is a boundary of v or v is a boundary of u. If G is disconnected, then each vertex in a component is adjacent to all the vertices in the other components and is adjacent to all of its boundary vertices within the component. A graph G is called a boundary graph if there exists a graph H such that $B(H) = G$. we defined the neighborhood $N_k(u) = \{w \in N(v) / d(u,w) = k\}$.

Motivated by the work of J. Gimbert et al., [2,3] and KM. Kathiresan et al., [5],We study here an iterated version of a distance dependent mapping. Given a positive integer m, the m^{th} iterated boundary graph of G is defined as $B^m(G) = B(B^{m-1}(G))$. Note that $B^0(G) \cong G$.

Definition 1.1: A graph G is periodic if $B^m(G) \cong G$ for some m. If p is the least positive integer with this property, then G is called a periodic graph with iso-period p. When $p = 1$, G is called as a fixed graph.

Definition 1.2: A graph G is said to be eventually periodic if there exist positive integers m and $k>0$, such that $B^{m+i}(G) \cong B^i(G), \forall i \geq k$. If p and k are the least positive integers with this property, then G is called an eventually periodic graph with iso-period p and iso-tail k .

Figs. 1, 2 and 3 illustrate these definitions showing boundary graph of G and its iterated boundary graphs.

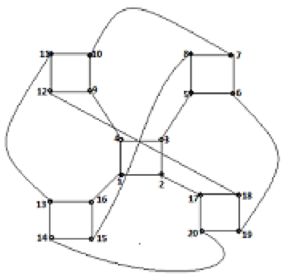

Fig. 1. The graph G.

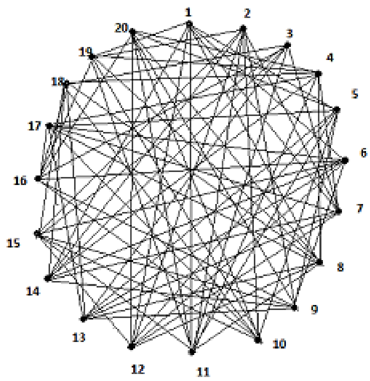

Fig. 2. The graph $B(G)$.

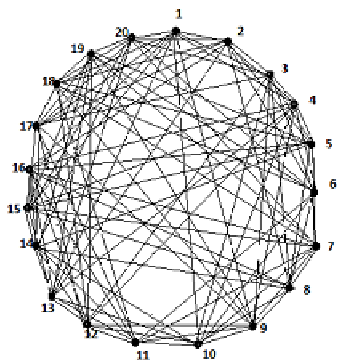

Fig. 3. The graph $B^2(G)$.

In the above example $B^3(G) \cong B(G)$. Here $k(G) = 1$ and $p(G) = 2$ where k denotes the iso-tail and p denotes the iso-period of G.

Let F_{11}, F_{12}, F_{22}, F_{23}, F_{24} and F_3 denote the set of all graphs G such that $r(G) = 1$ and $d(G) = 1$; $r(G) = 1$ and $d(G) = 2$; $r(G) = 2$ and $d(G) = 2$; $r(G) = 2$ and $d(G) = 3$; $r(G) = 2$

and $d(G) = 4$ and $r(G) \geq 3$ respectively and F_4 denote the set of all disconnected graphs. It is well known that $d(G) \geq 4$ implies that $d(\overline{G}) \leq 2$.

2. Previous results

The following theorems are appeared in [6].

Theorem 2.1 [6]: $B(G) = G$ if and only if G is complete. □

Theorem 2.2 [6]: For a graph $G \in F_{12}, B(G) = K_n$ if and only if either $N(u)-\{v\} \subseteq N(v)-\{u\}$ or $N(v)-\{u\} \subseteq N(u)-\{v\}$ for any two adjacent vertices u and v of G. □

Theorem 2.3 [6]: Let G be a graph. Then $B(G) = \overline{G}$ if and only if the following conditions hold.

(i) G has no complete vertex .

(ii) neither $N(u)-\{v\} \subseteq N(v)-\{u\}$ nor $N(v)-\{u\} \subseteq N(u)-\{v\}$ for any two adjacent vertices u and v of G.

(iii) either $N_k(u) = \phi$ or $N_k(v) = \phi$ for any two non-adjacent vertices u and v of G, where $k = d(u,v)+1$. □

Theorem 2.4 [6]: If G has at least one isolated vertex, then G is not a boundary graph.

Theorem 2.5 [6]: Let $G \in F_4$ without isolated vertices. If $\overline{G}$ without complete vertices has the following properties

(i) neither $N(u)-\{v\} \subseteq N(v)-\{u\}$ nor $N(v)-\{u\} \subseteq N(u)-\{v\}$ for any two adjacent vertices u and v of G.

(ii) either $N_k(u) = \phi$ or $N_k(v) = \phi$ for any two non-adjacent vertices u and v of G, where $k = d(u,v)+1$ then, G is a boundary graph. □

3. Main Results

Proposition 3.1: Every graph is either periodic or eventually periodic.

Proof. Consider the set $A = \{B^m(G) : m=0,1,2\ldots\}$ where $B^0(G) = G$. If G has n vertices, then $B^m(G)$ also has n vertices. Moreover, the possible number of graphs in A is atmost $2^{\frac{n(n-1)}{2}}$. Thus, there exist non-negative integer k and positive integer m such that $B^{m+k}(G) \cong B^k(G)$ and hence $B^{m+i}(G) \cong B^i(G), \forall i \geq k.$ If $k = 0$,then G is periodic. If $k > 0$, then G is eventually periodic. □

Proposition 3.2: Let C_n be any cycle. Then C_n is periodic with iso-period 1 if it is odd and eventually periodic with iso-period 1 if it is even.

Proof. Case (i) If n is odd, $B(C_n) \cong C_n$. Hence C_n is periodic with iso-period 1.
Case(ii) If n is even, $B(C_n) \cong \frac{n}{2} K_2$, a disconnected graph with each component K_2. By the definition of $B(G)$, $B^2(C_n)$ is a complete graph. Hence by Theorem 2.1, C_n is eventually periodic with iso-period 1.

Let us find some graphs of order n which is either periodic or eventually periodic.

Observation 3.3: C_n+C_n is a periodic graph for odd values of n, $\forall n \geq 3$ whose $k(G)=0$ and $p(G)=2$ where + denotes the usual addition of graphs. □

Observation 3.4: We also observed that $p(C_m+C_m) = p(C_m) + p(C_m)$ where $m = 2n+1$, $\forall n \geq 2$. □

Observation 3.5: $C_{2m+1} \times C_{2m+1}$ is a fixed graph whose $k(G)=0$ and $p(G)=1$, $\forall m \geq 1$ where $\times$ denotes the Cartesian product of graphs. □

Observation 3.6: $C_{2m} \times C_{2m}$ is eventually periodic with $k(G)=2$, $p(G)=1$. □

Let us say that a class is periodic if every graph in the class is periodic. As we observed earlier C_n+C_n, complete graph, $C_{2m+1} \times C_{2m+1}$ are periodic graphs.

Observation 3.7: Every complete n-partite graph with $|V_i| \geq 2$ for each i^{th} partition is eventually periodic with iso-period 1.

Proof. Let G be a complete n-partite graph with $|V_i| \geq 2$ for each i^{th} partition. Any two vertices v_i and v_j in G are adjacent in $B(G)$ if and only if they are in the same partition. Therefore $B(G)$ is a disconnected graph with each component complete. By the definition of boundary graph, $B^2(G)$ is complete. By Theorem 2.1, G is eventually periodic with iso-period 1.

Proposition 3.8: Every path P_n, $n \geq 3$ is eventually periodic with iso-period 1. □

Proof. Let $v_1, v_2 \ldots v_n$ be a path on n vertices. Since the end vertices are complete, $B(P_n) \in F_{12}$. Further $v_2, v_3 \ldots v_{n-1}$ are non-adjacent vertices of eccentricity 2 in $B(P_n)$ and $B^2(P_n) = K_n$. Hence by Theorem2.1, P_n is eventually periodic with iso-period 1. □

Lemma 3.9: A graph $G \in F_{12}$ is eventually periodic with iso-period 1 if and only if either $N(u) - \{v\} \subseteq N(v) - \{u\}$ or $N(v) - \{u\} \subseteq N(u) - \{v\}$ for any two adjacent vertices u and v of G.

Proof. Let $G \in F_{12}$. Assume for any two adjacent vertices u and v of G, either $N(u)-\{v\} \subseteq N(v)-\{u\}$ or $N(v)-\{u\} \subseteq N(u)-\{v\}$. Then by Theorem 2.2, $B(G) = K_n$. Therefore $B^2(G) = B(B(G)) = B(K_n) \cong K_n$ implies G is eventually periodic with iso-period 1.

Conversely, assume $G \in F_{12}$ is eventually periodic. Suppose for any two adjacent vertices neither $N(u)-\{v\} \subseteq N(v)-\{u\}$ nor $N(v)-\{u\} \subseteq N(u)-\{v\}$. This implies $uv \notin B(G)$. Therefore non-adjacent vertices in G are adjacent in $B(G)$ together with the full degree vertices in G continue to have the same degree in $B(G)$. Hence $B(G) \in F_{12}$. With the assumption of the condition mentioned for adjacent vertices, $B^2(G) \cong G$, implies G is periodic which is a contradiction.

□

Lemma 3.10: If G is not a boundary graph, then G is eventually periodic.

Proof. Since G is not a boundary graph, there is no graph H such that $B(H) \cong G$. Therefore for any m, $B^m(H) \neq G, m \geq 1$ and thus G is not a periodic graph. Hence by proposition 3.1, G is eventually periodic.

□

Lemma 3.11: Let G be a disconnected graph. If G has at least one isolated vertex, then G is eventually periodic.

Proof. Suppose that G has at least one isolated vertex, then by Theorem 2.4, G is not a boundary graph. Hence by Lemma 3.10, G is eventually periodic. □

Lemma 3.12: Let $G \in F_4$ without isolated vertices. If $\overline{G}$ without complete vertices has the following properties

(i) neither $N(u)-\{v\} \subseteq N(v)-\{u\}$ nor $N(v)-\{u\} \subseteq N(u)-\{v\}$ for any two adjacent vertices u and v of G.

(ii) either $N_k(u) = \phi$ or $N_k(v) = \phi$ for any two non-adjacent vertices u and v of G, where $k = d(u,v)+1$, then G is periodic.

Proof. With the above assumption by Theorem 2.5, G is a boundary graph. Then there exists a graph H such that $B(H) \cong G$. Since G has n vertices, we can find a graph in the set of all graphs with n vertices such that $B^m(G) \cong G$ for some least positive integer m. Therefore G is periodic.

□

Lemma 3.13: Let G be a disconnected graph with at least one complete component. If for any two adjacent vertices u and v in $B(G)$, either $N(u)-\{v\} \subseteq N(v)-\{u\}$ or $N(v)-\{u\} \subseteq N(u)-\{v\}$ then, G is eventually periodic with iso-period 1.

Proof. Let G be a disconnected graph with at least one complete component. Then $B(G) \in F_{12}$. Since for any two adjacent vertices u and v in $B(G)$,either

$N(u)-\{v\} \subseteq N(v)-\{u\}$ or $N(v)-\{u\} \subseteq N(u)-\{v\}$, $B^2(G)$ is complete. Now, consider $B^3(G) = B(B^2(G)) = K_n \cong B^2(G)$. Hence G is eventually periodic with iso-period 1.

□

Open Problem 3.14: Characterize all disconnected periodic graphs in which each component is non-complete.

Theorem 3.15: Let G be a connected graph. If the following conditions hold in two successive iterations in $B^k(G)$, $K \geq 1$

(i) No complete vertex

(ii) neither $N(u)-\{v\} \subseteq N(v)-\{u\}$ nor $N(v)-\{u\} \subseteq N(u)-\{v\}$ for any two adjacent vertices u and v.

(iii) either $N_k(u) = \phi$ or $N_k(v) = \phi$ for any two non-adjacent vertices u and v where $k = d(u,v)+1$, then G is eventually periodic with iso-period 2.

Proof. Suppose two successive iterations in $B^k(G)$, $K \geq 1$ satisfies (i), (ii) and (iii), then by Theorem 2.3, $B^k(G) \cong (\overline{B^{k-1}(G)})$ and $B^{k+1}(G) \cong (\overline{B^k(G)})$ .Consider,
$B^{k+1}(G) \cong (\overline{B^k(G)}) \cong (\overline{\overline{B^{k-1}(G)}}) = B^{k-1}(G)$. This proves that G is eventually periodic with iso-period 2.

From the above theorem it is clear that, If G and $B(G)$ holds the conditions in Theorem 3.15 then G is periodic with iso-period 2.

Remark 3.16: There are some graphs in F_{22} which does not satisfies the condition mentioned in Theorem 3.15, but they are eventually periodic with iso-period 2. □
The following example illustrates the above remark.

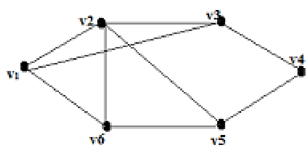

Fig. 4. The graph G.

The graph mentioned in Fig. 4 does not satisfy the condition in Theorem 3.15 but it is eventually periodic with iso-period 2.

Conjecture 3.17: We have observed, but not proven that a self centered graph of radius two is eventually periodic with iso-period 2.

□

Lemma 3.18: Let G be a connected graph .If $\overline{G}$ has the following properties

(i) $\overline{G}$ has no complete vertex .
(ii) neither $N(u)-\{v\}\subseteq N(v)-\{u\}$ nor $N(v)-\{u\}\subseteq N(u)-\{v\}$ for any two adjacent vertices *u* and v of $\overline{G}$.
(iii) either $N_k(u)=\phi$ or $N_k(v)=\phi$ for any two non-adjacent vertices *u* and *v* of $\overline{G}$, where *k*= *d* (*u*,*v*)+1, with $B(G)\cong\overline{G}$, then *G* is periodic with iso-period 2.

Proof. Since $\overline{G}$ has the properties (i), (ii) and (iii) by Theorem 2.3, $B(\overline{G})\cong G$. $B^2(G)=B(B(G))\cong B(\overline{G})\cong G$ implies that *G* is periodic with iso-period 2. □

Lemma 3.19: If *G* is a periodic graph with iso-period *m*>1 and if $\overline{G}$ has the following properties
(i) $\overline{G}$ has no complete vertex .
(ii) neither $N(u)-\{v\}\subseteq N(v)-\{u\}$ nor $N(v)-\{u\}\subseteq N(u)-\{v\}$ for any two adjacent vertices *u* and *v* of $\overline{G}$.
(iii) either $N_k(u)=\phi$ or $N_k(v)=\phi$ for any two non-adjacent vertices *u* and *v* of $\overline{G}$, where *k*= *d (u,v)*+1, then $\overline{G}$ is eventually periodic with iso-period *m*.

Proof. By hypothesis $B^m(G)\cong G$. Then there exists a graph $H=B^{m-1}(G)$ such that $B(H)\cong B(B^{m-1}(G))\cong B^m(G)\cong G$. This implies *G* is a boundary graph. By Theorem 2.3 $B(\overline{G})\cong G$. Consider $B^m(G)\cong G$ implies $B^m(B(\overline{G}))\cong B(\overline{G})$.Therefore $B^{m+1}(\overline{G})\cong B(\overline{G})$. Hence $\overline{G}$ is eventually periodic with iso-period *m*. □

Theorem 3.20: A graph *G* is eventually periodic if and only if one of the following holds
(i) *G* is a complete n-partite graph with $|V_i|\geq 2$ for each i^{th} partition.
(ii) $G\in F_{12}$ and for any two adjacent vertices *u* and *v* in *G* either $N(u)-\{v\}\subseteq N(v)-\{u\}$ or $N(v)-\{u\}\subseteq N(u)-\{v\}$.
(iii) Any two successive iterations in $B^k(G), k\geq 1$ holds the following conditions
(a) No complete vertex
(b) neither $N(u)-\{v\}\subseteq N(v)-\{u\}$ nor $N(v)-\{u\}\subseteq N(u)-\{v\}$ for any two adjacent vertices *u* and *v*.
(c) either $N_k(u)=\phi$ or $N\ (v)\ =\phi$ for any two non-adjacent vertices *u* and *v* where *k* = *d* (*u*,*v*)+1.
(iv) *G* be a disconnected graph with at least one isolated vertex.
(v) Let G be a disconnected graph with at least one component complete. If for any two adjacent vertices *u* and *v* in *B*(*G*), either $N(u)-\{v\}\subseteq N(v)-\{u\}$ or $N(v)-\{u\}\subseteq N(u)-\{v\}$.

Proof. If (i) holds, then by Observation 3.7, *G* is eventually periodic. If (ii) holds, then by Lemma 3.9, *G* is eventually periodic. If (iii) holds true, by Theorem 3.15 *G* is eventually periodic .If (iv) holds, by Lemma 3.11 G is eventually periodic. If (v) holds, then by Lemma 3.13 G is eventually periodic.

Conversely, Suppose G is eventually periodic. Assume that (i), (iii), (iv) and (v) do not hold. Now we have to prove that (ii) definitely holds. Suppose this is not. Let $G \in F_{12}$ and for any two adjacent vertices u and v in G neither $N(u)-\{v\} \subseteq N(v)-\{u\}$ nor $N(v)-\{u\} \subseteq N(u)-\{v\}$. This implies $uv \notin B(G)$. Therefore non-adjacent vertices in G are adjacent in B(G) together with the full degree vertices in G continue to have the same degree in $B(G)$. Hence $B(G) \in F_{12}$. With the assumption of the condition mentioned for adjacent vertices, $B^2(G) \cong G$. implies G is periodic which is a contradiction.

□

Acknowledgments

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