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Edge Pair Sum Labeling

P. Jeyanthi[1*] and T. Saratha Devi[2]

[1]Research Centre, Department of Mathematics, Govindammal Aditanar College for Women, Thiruchendur 628 215. Tamilnadu, India

[2]Department of Mathematics, G. Venkataswamy Naidu College, Kovilpatti, Tamilnadu, India

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Abstract

An injective map $f : E(G) \rightarrow \{\pm 1, \pm 2, \dots, \pm q\}$ is said to be an edge pair sum labeling of a graph $G(p, q)$ if the induced vertex function $f^*: V(G) \rightarrow Z - \{0\}$ defined by $f^*(v) = \sum_{e \in E_v} f(e)$ is one – one, where E_v denotes the set of edges in G that are incident with a vertex v and $f^*(V(G))$ is either of the form $\left\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{p}{2}}\right\}$ or $\left\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{p-1}{2}}\right\} \cup \left\{k_{\frac{p}{2}}\right\}$ according as p is even or odd. A graph with an edge pair sum labeling is called an edge pair sum graph. In this paper we prove that path P_n, cycle C_n, triangular snake, $P_m \cup K_{1,n}$, $C_n \odot K_m^c$ are edge pair sum graphs.

Keywords: Pair sum graph, edge pair sum labeling, edge pair sum graph.

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1. Introduction

We consider only finite, simple, planar and undirected graphs. A graph $G(p,q)$ has the vertex-set $V(G)$ and the edge-set $E(G)$ with $|V(G)|=p$ and $|E(G)| = q$. A vertex labeling f of a graph G is an assigned of labels to the vertices of G that induces a label for each edge xy depending on the vertex labels. An edge labeling f of a graph G is an assigned of labels to the edges of G that induces a label for each vertex v depending on the labels of the edges incident on it. Terms and terminology are used in the sense of Harary [1].

Ponraj and Parthipan [2] introduced the concept of pair sum labeling. An injective map $f: V(G) \rightarrow \{\pm 1, \pm 2, \dots, \pm p\}$ is said to be a pair sum labeling of a graph $G(p,q)$ if the induced edge function $f_e: E(G) \rightarrow Z - \{0\}$ defined by $f_e(uv) = f(u)+f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{q}{2}}\}$ or $\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{q-1}{2}}\} \cup \{k_{\frac{q+1}{2}}\}$

* *Corresponding author:* jeyajeyanthi@rediffmail.com

according as q is even or odd. A graph with a pair sum labeling is called a pair sum graph. The pair sum behavior of graphs like complete graph, path, bistar, cycle, all trees of order ≤8, and ≤9 and some more standard graphs are investigated in refs. [3-6].

Motivated by Ponraj and Parthipan [2], we define a new labeling called an edge pair sum labeling analogous to pair sum labeling. Let $G(p, q)$ be a graph. An injective map f : $E(G) \to \{\pm 1, \pm 2, \dots, \pm q\}$ is said to be an edge pair sum labeling if the induced vertex function $f^*: V(G) \to Z - \{0\}$ *is* defined by $f^*(v) = \sum_{e \in E_v} f(e)$ is one – one where E_v denotes the set of edges in G that are incident with a vertex v and $f^*\big(V(G)\big)$ is either of the form $\left\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{p}{2}}\right\}$ or $\left\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{p-1}{2}}\right\} \cup \left\{k_{\frac{p}{2}}\right\}$ according as p is even or odd. A graph with an edge pair sum labeling is called an edge pair sum graph.

We use the following definitions in the subsequent sequel.

Definition 1.1
The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 1.2
The corona $G \odot H$ is the graph obtained by taking one copy of G and n copies of H and joins the ith vertex of G with an edge to every vertex in the ith copy of H where $|V(G)| = n$.

2. Main results

Theorem 2.1: Every path P_n is an edge pair sum graph for $n \geq 3$.

Proof: Let $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_{n-1}\}$ be the vertex set and edge set of P_n respectively, where $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$. We consider the following four cases:

Case (i) $n = 3$.
Define the labeling $f: E(G) \to \{\pm 1, \pm 2\}$ by $f(e_1) = -2$, $f(e_2) = 1$. The induced vertex labeling are $f^*(v_1) = -2$, $f^*(v_2) = -1$, $f^*(v_3) = 1$. Hence, f is an edge pair sum labeling of P_3.

Case (ii) $n = 4$.
Define the labeling $f: E(G) \to \{\pm 1, \pm 2, \pm 3\}$ by $f(e_1) = -2$, $f(e_2) = -1$, $f(e_3) = 3$. The induced vertex labelings are $f^*(v_1) = -2$, $f^*(v_2) = -3$, $f^*(v_3) = 2$, $f^*(v_4) = 3$. Hence, f is an edge pair sum labeling of P_3.

Case (iii) n is even. Take $n = 2k$, $k \geq 3$.
Define the labeling $f: E(G) \to \{\pm 1, \pm 2, \dots, \pm(n-1)\}$ by

$$f(e_i) = \begin{cases} -2 & \text{if } i = k-1, \\ -1 & \text{if } i = k, \\ 3 & \text{if } i = k+1, \end{cases}$$

$$f(e_i) = \begin{cases} 2k+1-2i & \text{if } 1 \leq i \leq k-2, \\ 2k-1-2i & \text{if } k+2 \leq i \leq 2k-1. \end{cases}$$

The induced vertex labelings are

$f^*(v_1) = f(e_1) = 2k-1$, $f^*(v_n) = f(e_{2k-1}) = -(2k-1)$, for $2 \leq i \leq k-2$ $f^*(v_i) = f(e_{i-1}) + f(e_i) = 4(k+1-i)$, $f^*(v_{k-1}) = 3$, $f^*(v_k) = -3$, $f^*(v_{k+1}) = 2$, $f^*(v_{k+2}) = -2$ and for $k+3 \leq i \leq 2k-1$ $f^*(v_i) = f(e_{i-1}) + f(e_i) = 4(k-i)$.

From the above vertex labeling we get $f^*(V(G)) = \{\pm 2, \pm 3, \pm 12, \pm 16, \dots, \pm 4(k-1, \pm(2k-1)$. Hence, f is an edge pair sum labeling of Pn.

Case (iv) n is odd. Take $n = 2k + 1$, $k \geq 2$.

Define the labeling $f: E(G) \to \{\pm 1, \pm 2, \dots, \pm(n-1)\}$ by

$$f(e_i) = \begin{cases} 1 & \text{if } i = k+1, \\ 2 & \text{if } \quad i = k, \\ -5 & \text{if } i = k-1, \end{cases}$$

$$f(e_i) = \begin{cases} 5 & \text{if } \quad i = k+2, \\ -(2k+3-2i) & \text{if } 1 \leq i \leq k-2, \\ -2k+1+2i & \text{if } k+3 \leq i \leq 2k. \end{cases}$$

The induced vertex labelings are

$f^*(v_1) = f(e_1) = -2k-1$, $f^*(v_n) = f(e_{2k}) = (2k+1)$,
for $2 \leq i \leq k-1$ $f^*(v_i) = f(e_{i-1}) + f(e_i) = 4(-k+i-2)$, $f^*(v_k) = -3$, $f^*(v_{k+1}) = 3$, $f^*(v_{k+2}) = 6$ and for $k+3 \leq i \leq 2k$ $f^*(v_i) = f(e_{i-1}) + f(e_i) = -4(k-i)$.

From the above argument we get $f^*(V(G)) = \{\pm 3, \pm 12, \pm 16, \dots, \pm 4k, \pm(2k+1) \cup 6$. Hence, f is an edge pair sum labeling of Pn. □

Theorem 2.2: Every cycle C_n $(n \geq 3)$ is an edge pair sum graph.

Proof : Let $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_n\}$ be the vertex set and the edge set of C_n, where $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$ and $e_n = v_n v_1$.

Define the edge labeling $f: E(G) \to \{\pm 1, \pm 2, \dots, \pm n\}$ by considering the following three cases.

Case (i) n = 3.

$f(e_1) = -1$, $f(e_2) = 2$, $f(e_3) = -3$. The induced vertex labelings are $f^*(v_1) = -4$, $f^*(v_2) = 1$, $f^*(v_3) = -1$. Clearly, $f^*(V(G)) = \{\pm 1, -4\}$. Hence f is an edge pair sum labeling of C_3.

Case (ii) n = 4.

$f(e_1) = 1$, $f(e_2) = 2$, $f(e_3) = -1$, $f(e_4) = -2$. The induced vertex labelings are $f^*(v_1) = -1$, $f^*(v_2) = 3$, $f^*(v_3) = 1$, $f^*(v_4) = -3$. Clearly, $f^*(V(G)) = \{\pm 1, \pm 3\}$. Hence f is an edge pair sum labeling of C_4.

Case (iii) n is even. Take $n = 2m$.

Subcase (i) m is odd.
For $1 \le i \le m$ $f(e_i) = i$ and $f(e_{m+i}) = -i$. The induced vertex labelings are as follows:

For $1 \le i \le m-1$ $f^*(v_{i+1}) = f(e_i) + f(e_{i+1}) = (2i+1)$, $f^*(v_{m+1}) = f(e_m) + f(e_{m+1}) = (m-1)$,
for $1 \le i \le m-1$ $f^*(v_{m+1+i}) = f(e_{m+i}) + f(e_{m+1+i}) = -(2i+1)$, $f^*(v_1) = f(e_{2m}) + f(e_1)$ = - (m - 1) .

Thus we get $f^*\big(V(G)\big) = \{\pm3, \pm5, \pm7, \ldots, \pm(2m-1), \pm(m-1)\}$. Hence, f is an edge pair sum labeling.

Subcase (ii) m is even and $m > 2$.

For $1 \le i \le m$ $f(e_i) = i$, $f(e_{m+1}) = -2$, $f(e_{m+2}) = -1$,
for $1 \le i \le m-2$ $f(e_{m+2+i}) = f(e_{m+i}) - 2$.

The induced vertex labelings are as follows:

For $1 \le i \le m-1$ $f^*(v_{i+1}) = f(e_i) + f(e_{i+1}) = 2i+1$, $f^*(v_{m+1}) = f(e_m) + f(e_{m+1}) = m-2$, for $1 \le i \le m-1$ $f^*(v_{m+1+i}) = f(e_{m+i}) + f(e_{m+1+i}) =$ - (2i + 1), $f^*(v_1) = f(e_{2m}) + f(e_1) = -(m-2)$.

From the above labeling, we get $f^*\big(V(G)\big) = \{\pm3, \pm5, \pm7, \ldots, \pm(2m-1), \pm(m-2)\}$. Hence, f is an edge pair sum labeling.

Case (iv) n is odd. Take $n = 2m + 1$, $m \ge 2$.

Subcase (i) $m \equiv 0{,}2\ (mod 3)$

$f(e_1) = 1$, $f(e_2) = -2$, for $1 \le i \le m-1$ $f(e_{2+i}) = 2 + i$, $f(e_{m+2}) = -1$,
for $1 \le i \le m-1$ $f(e_{m+2+i}) = -(2+i)$.

The induced vertex labelings are

$f^*(v_2) = -1$, $f^*(v_3) = 1$, for $3 \le i \le m$ $f^*(v_{i+1}) = f(e_i) + f(e_{i+1}) = 2i+1$, $f^*(v_{m+2}) = m$, $f^*(v_{m+3}) = -4$, for $3 \le i \le m$ $f^*(v_{m+1+i}) = f(e_{m+i}) + f(e_{m+1+i}) = -(2i+1)$, $f^*(v_1) = -m$.

From the above arguments, we get $f^*\big(V(G)\big) = \{\pm1, \pm7, \pm9, \ldots, \pm(2m+$ *1,±mU−4*. Hence, f is an edge pair sum labeling.

Subcase (ii) $m \equiv 1$ (mod3)

$f(e_1) = 1$, $(e_2) = 2$, $f(e_3) = -3$, for $1 \le i \le m-2$ $f(e_{3+i}) = (3+i)$, $f(e_{m+2}) = -1$, $f(e_{m+3}) = -2$, for $1 \le i \le m-2$ $f(e_{m+3+i}) = -(3+i)$.

The induced vertex labeling are

$f^*(v_2) = 3$, $f^*(v_3) = -1$, $f^*(v_4) = 1$, for $4 \le i \le m$ $f^*(v_{i+1}) = f(e_i) + f(e_{i+1})$ $= (2i+1)$, $f^*(v_{m+2}) = m$, $f^*(v_{m+3}) = -3$, $f^*(v_{m+4}) = -6$, for $4 \le i \le m$ $f^*(v_{m+1+i}) = f(e_{m+i}) + f(e_{m+1+i}) = -(2i+1)$, $f^*(v_1) = -m$.

Clearly, $f^*\big(V(G)\big) = \{\pm1, \pm3, \pm9, \pm11, \ldots, \pm(2m+1), \pm m\} \cup \{-6\}$. Hence, f is an edge pair sum labeling. □

Theorem 2.3: The star graph $K_{1,n}$ is an edge pair sum graph if and only if n is even.

Proof : Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ and $E(K_{1,n}) = \{e_1, e_2, e_3, \ldots, e_n\}$ where $e_i = vv_i$, $1 \le i \le n$.

Define $f(e_1) = -1$, for $2 \le i \le \frac{n+2}{2}$ $f(e_i) = i$, for $3 \le i \le \frac{n+2}{2}$ $f\left(e_{\frac{n-2}{2}+i}\right) = -i$.

The induced vertex labelings are $f^*(v_1) = -1$, for $2 \le i \le \frac{n+2}{2}$ $f^*(v_i) = i$, for $3 \le i \le \frac{n+2}{2}$ $f^*\left(v_{\frac{n-2}{2}+i}\right) = -i$ and $f^*(v) = 1$. From the above labelings we get $f^*(V(G)) = \left\{\pm 1, \pm 3, \pm 4, \ldots, \pm\left(\frac{n}{2}+1\right)\right\} \cup \{2\}$. Hence, *f* is an edge pair sum labeling of $K_{1,n}$ if n is even.

Conversely assume that n is odd. Let f be an edge pair sum labeling of G with labeling $f(e_i) = x_i$, for $1 \le i \le n$. Then $f^*(v_i) = x_i$, $for\ 1 \le i \le n$ and $f^*(v) = \sum_{i=1}^{n} x_i$. Since f is an edge pair sum labeling we have

$$f^*(V(G)) = \left\{\pm k_1, \pm k_2, \pm k_3, \ldots, \pm k_{\frac{n+1}{2}}\right\}.$$

If $f^*(v) = k_1$, then there must be a vertex say v_1 with $f^*(v_1) = -k_1$. Therefore $\sum x_i = -x_1 \implies 2x_1 + \sum_{i=2}^{n} x_i = 0$. Since f is an edge pair sum labeling $\sum_{i=2}^{n} x_i = 0$. Hence $x_1 = 0$ which is a contradiction. Therefore *f* is not an edge pair sum labeling of $K_{1,n}$ if n is odd. □

Theorem 2.4: The complete graph K_4 is not an edge pair sum graph.

Proof: Let f be an edge pair sum labeling of K_4 with $f(e_1) = x_1$, $f(e_2) = x_2$, $f(e_3) = x_3$, $f(e_4) = x_4$, $f(e_5) = x_5$, $f(e_6) = x_6$. Hence the induced vertex labeling are $f^*(v_1) = x_1 + x_2 + x_6$, $f^*(v_2) = x_1 + x_4 + x_5$, $f^*(v_3) = x_3 + x_4 + x_6$ and $f^*(v_4) = x_2 + x_3 + x_5$.

Suppose that $f^*(v_1) = k_1$ then the other vertex labels must be $f^*(v_2) = -k_1$, $f^*(v_3) = k_2$, $f^*(v_4) = -k_2$. Hence $2x_1 + x_2 + x_4 + x_5 + x_6 = 0$ and $x_2 + 2x_3 + x_4 + x_5 + x_6 = 0$ which implies that $2x_1 - 2x_3 = 0$ and hence $x_1 = x_3$, is a contradiction. Therefore, K_4 is not an edge pair sum graph. □

Remark:
By Theorem 2.1, K_1 and K_2 are not edge pair sum graphs. By theorem 2.2, K_3 is an edge pair sum graph.

Theorem 2.5: The graph $P_m \cup K_{1,n}$ is an edge pair sum graph if m is odd.

Proof : Let $V(P_m) = \{u_1, u_2, \ldots, u_m\}$, $E(P_m) = \{e_i = u_i u_{i+1}\ ; 1 \le i \le m-1\}$, $V(K_{1,n}) = \{v, v_i; 1 \le i \le n\}$ and $E(K_{1,n}) = \{e_i' = vv_i; 1 \le i \le n\}$.

Define the edge labeling $f : E(P_m \cup K_{1,n}) \to \{\pm 1, \pm 2, \pm 3, \ldots, \pm(m+n-1)\}$ by considering the following six cases.

Case (i) m = 3 and n is even.

$f(e_1) = 1, f(e_2) = 2$, for $1 \le i \le 2$ $f(e_i') = -i$, for $1 \le i \le \frac{n-2}{2}$ $f(e_{2+i}') = -(3+i)$ and $f\left(e'_{\frac{n+2}{2}+i}\right) = (3+i)$.

The induced vertex labelings are

$f^*(u_1) = 1, f^*(u_2) = 3, f^*(u_3) = 2, f^*(v_1) = -1,\ f^*(v_2) = -2,$ for $1 \le i \le \frac{n-2}{2}$ $f^*(v_{2+i}) = -(3+i)$ and $f^*\left(v_{\frac{n+2}{2}+i}\right) = (3+i)$, for $1 \le i \le n$ $f^*(v) = \sum f^*(e_i) = -3$. Hence we get $f^*(V(G)) = \left\{\pm 1, \pm 2, \pm 3, \ldots, \pm\left(\frac{n+4}{2}\right)\right\}$. Thus, f is an edge pair sum labeling.

Case (ii) m = 3 and n is odd.

$f(e_1) = 1, f(e_2) = 2,$ for $1 \le i \le 3$ $f(e_i') = -i,$for $1 \le i \le \frac{n-3}{2}$ $f(e_{3+i}') = -(6+i)$ and $f\left(e'_{\frac{n+3}{2}+i}\right) = 6+i$.

The induced vertex labelings are $f^*(u_1) = 1,\ f^*(u_2) = 3,\ f^*(u_3) = 2,$ for $1 \le i \le 3$ $f^*(v_i) = -i,$ for $1 \le i \le \frac{n-3}{2}$ $f^*(v_{3+i}) = -(6+i)$ and $f^*\left(v_{\frac{n+3}{2}+i}\right) = (6+i),$ for $1 \le i \le n$ $f^*(v) = \sum f^*(e_i) = -6$ .The vertex labeling becomes $f^*(V(G)) = \left\{\pm 1, \pm 2, \pm 3, \pm 7, \pm 8, \pm 9, \ldots, \pm\left(\frac{n+9}{2}\right)\right\} \cup \{-6\}$. Hence, f is an edge pair sum labeling.

Case (iii) m = 5, 7, 9 and n is even.

For $1 \le i \le \frac{m-3}{2}$ $f(e_i) = (2i-1),\ f(e_{\frac{m-1}{2}}) = 2,\ f\left(e_{\frac{m+1}{2}}\right) = f\left(e_{\frac{m-3}{2}}\right) + 2,$ for $1 \le i \le \frac{m-3}{2}$ $f\left(e_{\frac{m+1}{2}+i}\right) = -f\left(e_{\frac{m-1}{2}-i}\right), f(e_1') = -2, f(e_2') = -m+2,$ for $1 \le i \le \frac{n-2}{2}$ $f(e_{2+i}') = (m+i)$ and $f\left(e'_{\frac{n+2}{2}+i}\right) = -(m+i).$

The induced vertex labelings are

$f^*(u_1) = 1, f^*(u_m) = -1,$ for $1 \le i \le \frac{m-5}{2}$ $f^*(u_{1+i}) = 4i, f^*\left(u_{\frac{m-1}{2}}\right) = (m-2),$ $f^*\left(u_{\frac{m+1}{2}}\right) = m, f^*\left(u_{\frac{m+3}{2}}\right) = 2,$ for $1 \le i \le \frac{m-5}{2}$ $f^*\left(u_{\frac{m+3}{2}+i}\right) = -f^*\left(u_{\frac{m-1}{2}-i}\right),$ $f^*(v_1) = -2,\ f^*(v_2) = -(m-2),$ for $1 \le i \le \frac{n-2}{2}$ $f^*(v_{2+i}) = (m+i)$ and $f^*\left(v_{\frac{n+2}{2}+i}\right) = -(m+i),$ for $1 \le i \le n$ $f^*(v) = \sum f^*(e_i) = -m.$

From the above arguments we get

$f^*(V(G)) = \left\{\pm 1, \pm 2, \pm 4, \pm 8, \ldots, \pm 4\left(\frac{m-5}{2}\right), \pm(m-2), \pm m, \pm(m+1), \pm(m+2, \ldots, \pm(m+n-22)\right.$. Hence, f is an edge pair sum labeling.

Case (iv) m = 5, 7, 9 and n is odd.

Subcase (i) Let n ≤ 2(m-1).

For $1 \le i \le \frac{m-3}{2}$ $f(e_i) = (2i-1), f(e_{\frac{m-1}{2}}) = 2,\ f\left(e_{\frac{m+1}{2}}\right) = f\left(e_{\frac{m-3}{2}}\right) + 2,$ for $1 \le i \le \frac{m-3}{2}$ $f\left(e_{\frac{m+1}{2}+i}\right) = -f\left(e_{\frac{m-1}{2}-i}\right), f(e_1') = -2, f(e_2') = -m+2, f(e_3') = -m,$ for $1 \le i \le \frac{n-3}{2}$ $f(e_{3+i}') = (m+i)$ and $f\left(e'_{\frac{n+3}{2}+i}\right) = -(m+i)$.

The induced vertex labelings are

$f^*(u_1) = 1, f^*(u_m) = -1,$ for $1 \le i \le \frac{m-5}{2}$ $f^*(u_{1+i}) = 4i, f^*\left(u_{\frac{m-1}{2}}\right) = (m-2),$ $f^*\left(u_{\frac{m+1}{2}}\right) = m, f^*\left(u_{\frac{m+3}{2}}\right) = 2,$ for $1 \le i \le \frac{m-5}{2}$ $f^*\left(u_{\frac{m+3}{2}+i}\right) = -f^*\left(u_{\frac{m-1}{2}-i}\right),$ $f^*(v_1) = -2, f^*(v_2) = -(m-2), f^*(v_3) = -m,$ for $1 \le i \le \frac{n-3}{2}$ $f^*(v_{3+i}) = (m+i)$ and $f^*\left(v_{\frac{n+3}{2}+i}\right) = -(m+i),$ for $1 \le i \le n$ $f^*(v) = \sum f^*(e_i) = -2m$.

The vertex labeling becomes $f^*(V(G)) = \left\{\pm 1, \pm 2, \pm 4, \pm 8, \dots, \pm 4\left(\frac{m-5}{2}\right), \pm(m-2, \pm m, \pm m+1, \dots, \pm(2m+n-32) \cup -2m\right.$. Hence, *f* is an edge pair sum labeling.

Subcase (ii) Let $n > 2(m-1)$.

For $1 \le i \le \frac{m-3}{2}$ $f(e_i) = (2i-1),$ $f(e_{\frac{m-1}{2}}) = 2,$ $f\left(e_{\frac{m+1}{2}}\right) = f\left(e_{\frac{m-3}{2}}\right) + 2,$ for $1 \le i \le \frac{m-3}{2}$ $f\left(e_{\frac{m+1}{2}+i}\right) = -f\left(e_{\frac{m-1}{2}-i}\right),$ $f(e_1') = -2,$ $f(e_2') = -m+2,$ $f(e_3') = -m,$ for $1 \le i \le \frac{n-3}{2}$ $f\left(e_{3+i}'\right) = (2m+i)$ and $f\left(e'_{\frac{n+3}{2}+i}\right) = -(2m+i)$.

The induced vertex labelings are

$f^*(u_1) = 1, f^*(u_m) = -1,$ for $1 \le i \le \frac{m-5}{2}$ $f^*(u_{1+i}) = 4i, f^*\left(u_{\frac{m-1}{2}}\right) = (m-2),$ $f^*\left(u_{\frac{m+1}{2}}\right) = m, f^*\left(u_{\frac{m+3}{2}}\right) = 2,$ for $1 \le i \le \frac{m-5}{2}$ $f^*\left(u_{\frac{m+3}{2}+i}\right) = -f^*\left(u_{\frac{m-1}{2}-i}\right)$ $, f^*(v_1) = -2,$ $f^*(v_2) = -(m-2), f^*(v_3) = -m,$ for $1 \le i \le \frac{n-3}{2}$ $f^*(v_{3+i}) = (2m+i)$ and $f^*\left(v_{\frac{n+3}{2}+i}\right) = -(2m+i)$, for $1 \le i \le n$ $f^*(v) = \sum f(e_i) = -2m.$
From the above labelings we get

$f^*(V(G)) = \left\{\pm 1, \pm 2, \pm 4, \pm 8, \dots, \pm 4\left(\frac{m-5}{2}\right), \pm(m-2), \pm m, \pm(2m+1), \pm(2m+2, \dots, \pm(4m+n-32) \cup -2m\right.$.

Hence, *f* is an edge pair sum labeling.

Case (v) $m \ge 11$ and n is even.

Subcase (i) Let $m < n$.

For $1 \le i \le \frac{m-3}{2}$ $f(e_i) = (2i-1), f(e_{\frac{m-1}{2}}) = 2,$ $f\left(e_{\frac{m+1}{2}}\right) = f\left(e_{\frac{m-3}{2}}\right) + 2,$ for $1 \le i \le \frac{m-3}{2}$ $f\left(e_{\frac{m+1}{2}+i}\right) = -f\left(e_{\frac{m-1}{2}-i}\right), f(e_1') = -2,$ $f(e_2') = -m+2,$ for $1 \le i \le \frac{n-2}{2}$ $f\left(e_{2+i}'\right) = (2m-10+i)$ and $f\left(e'_{\frac{n+2}{2}+i}\right) = -(2m-10+i).$

The induced vertex labelings are

$f^*(u_1) = 1,$ $f^*(u_m) = -1,$ for $1 \le i \le \frac{m-5}{2}$ $f^*(u_{1+i}) = 4i,$ $f^*\left(u_{\frac{m-1}{2}}\right) = (m-2),$ $f^*\left(u_{\frac{m+1}{2}}\right) = m,$ $f^*\left(u_{\frac{m+3}{2}}\right) = 2,$ for $1 \le i \le \frac{m-5}{2}$ $f^*\left(u_{\frac{m+3}{2}+i}\right) = -f^*\left(u_{\frac{m-1}{2}-i}\right),$

$f^*(v_1) = -2,\ f^*(v_2) = -(m-2),$ for $1 \le i \le \frac{n-2}{2}$ $f^*(v_{2+i}) = (2m-10+i)$ and $f^*\left(v_{\frac{n+2}{2}+i}\right) = -(2m-10+i)$, for $1 \le i \le n$ $f^*(v) = \sum f(e_i) = -m$.

From the above labeling we get

$f^*(V(G)) = \left\{\pm1, \pm2, \pm4, \pm8, \dots, \pm4\left(\frac{m-5}{2}\right), \pm(m-2), \pm m, \pm(2m-9), \pm(2m-8, \dots, \pm(4m+n-222)\right.$. Hence, f is an edge pair sum labeling.

Subcase (ii) Let $m > n$.

For $1 \le i \le \frac{m-3}{2}$ $f(e_i) = (2i-1),\ f(e_{\frac{m-1}{2}}) = 2, f\left(e_{\frac{m+1}{2}}\right) = f\left(e_{\frac{m-3}{2}}\right) + 2,$
for $1 \le i \le \frac{m-3}{2}$ $f\left(e_{\frac{m+1}{2}+i}\right) = -f\left(e_{\frac{m-1}{2}-i}\right), f(e_1') = -2,\ f(e_2') = -m+2,$
for $1 \le i \le \frac{n-2}{2}$ $f(e_{2+i}') = (2+4i)$ and $f\left(e'_{\frac{n+2}{2}+i}\right) = -(2+4i)$.

The induced vertex labeling are

$f^*(u_1) = 1, f^*(u_m) = -1$, for $1 \le i \le \frac{m-5}{2}$ $f^*(u_{1+i}) = 4i,$
$f^*\left(u_{\frac{m-1}{2}}\right) = (m-2), f^*\left(u_{\frac{m+1}{2}}\right) = m, f^*\left(u_{\frac{m+3}{2}}\right) = 2$, for $1 \le i \le \frac{m-5}{2}$
$f^*\left(u_{\frac{m+3}{2}+i}\right) = -f^*\left(u_{\frac{m-1}{2}-i}\right),\ f^*(v_1) = -2,\ f^*(v_2) = -(m-2)$, for $1 \le i \le \frac{n-2}{2}$
$f^*(v_{2+i}) = (2+4i)$ and $f^*\left(v_{\frac{n+2}{2}+i}\right) = -(2+4i)$, for $1 \le i \le n$ $f^*(v) =$
$\sum f^*(e_i) = -m$. Thus we get
$f^*(V(G)) = \left\{\pm1, \pm2, \pm4, \pm8, \dots, \pm4\left(\frac{m-5}{2}\right), \pm(m-2), \pm m, \pm6, \pm10, \dots, \pm(2n-2)\right.$. Hence, f is an edge pair sum labeling.

Case (vi) $m \ge 11$ and n is odd.

Subcase (i) Let $m \ge n$.

For $1 \le i \le \frac{m-3}{2}$ $f(e_i) = (2i-1),\ f(e_{\frac{m-1}{2}}) = 2,\ f\left(e_{\frac{m+1}{2}}\right) = f\left(e_{\frac{m-3}{2}}\right) + 2,$
for $1 \le i \le \frac{m-3}{2}$ $f\left(e_{\frac{m+1}{2}+i}\right) = -f\left(e_{\frac{m-1}{2}-i}\right),\ f(e_1') = -2,\ f(e_2') = -m+2,$
$f(e_3') = -m$, for $1 \le i \le \frac{n-3}{2}$ $f(e_{3+i}') = (2+4i)$ *and* $f\left(e'_{\frac{n+3}{2}+i}\right) = -(2+4i)$.

The induced vertex labelings are

$f^*(u_1) = 1, f^*(u_m) = -1$, for $1 \le i \le \frac{m-5}{2}$ $f^*(u_{1+i}) = 4i,$
$f^*\left(u_{\frac{m-1}{2}}\right) = (m-2), f^*\left(u_{\frac{m+1}{2}}\right) = m, f^*\left(u_{\frac{m+3}{2}}\right) = 2$,for $1 \le i \le \frac{m-5}{2}$
$f^*\left(u_{\frac{m+3}{2}+i}\right) = -f^*\left(u_{\frac{m-1}{2}-i}\right), f^*(v_1) = -2,\ f^*(v_2) = -(m-2),\ f^*(v_3) = -m,$

for $1 \le i \le \frac{n-3}{2}$ $f^*(v_{3+i}) = (2+4i)$ and $f^*\left(v_{\frac{n+3}{2}+i}\right) = -(2+4i)$, for $1 \le i \le n$ $f^*(v) = \sum f\,(e_i) = -2m$.

From the above arguments we get
$f^*(V(G)) = \{\pm 1, \pm 2 \pm 4, \pm 8, , \ldots, \pm 4\left(\frac{m-5}{2}\right), \pm(m-2), \pm m, \pm 6, \pm 10, \ldots, \pm(2n-4)U-2m$. Hence, f is an edge pair sum labeling.

Subcase (ii) Let $m < n$.

For $1 \le i \le \frac{m-3}{2}$ $f(e_i) = (2i-1)$, $f(e_{\frac{m-1}{2}}) = 2$, $f\left(e_{\frac{m+1}{2}}\right) = f\left(e_{\frac{m-3}{2}}\right) + 2$,
for $1 \le i \le \frac{m-3}{2}$ $f\left(e_{\frac{m+1}{2}+i}\right) = -f\left(e_{\frac{m-1}{2}-i}\right)$, $f(e_1') = -2$, $f(e_2') = -m+2$, $f(e_3') = -m$, for $1 \le i \le \frac{m-3}{2}$ $f\left(e_{3+i}'\right) = (2m-10+i)$ and $f\left(e'_{\frac{m+3}{2}+i}\right) = -(2m-10+i)$,
for $1 \le i \le \frac{n-m}{2}$ $f\left(e'_{m+i}\right) = (2m+i)$ and $f\left(e'_{\frac{m+n}{2}+i}\right) = -(2m+i)$.

The induced vertex labelings are

$f^*(u_1) = 1$, $f^*(u_m) = -1$, for $1 \le i \le \frac{m-5}{2}$ $f^*(u_{1+i}) = 4i$, $f^*\left(u_{\frac{m-1}{2}}\right) = (m-2)$, $f^*\left(u_{\frac{m+1}{2}}\right) = m$, $f^*\left(u_{\frac{m+3}{2}}\right) = 2$, for $1 \le i \le \frac{m-5}{2}$ $f^*\left(u_{\frac{m+3}{2}+i}\right) = -f^*\left(u_{\frac{m-1}{2}-i}\right)$,
$f^*(v_1) = -2$, $f^*(v_2) = -(m-2)$, $f^*(v_3) = -m$, for $1 \le i \le \frac{m-3}{2}$ $f^*(v_{3+i}) = (2m-10+i)$ and $f^*\left(v_{\frac{m+3}{2}+i}\right) = -(2m-10+i)$, for $1 \le i \le \frac{n-m}{2}$ $f^*(v_{m+i}) = (2m+i)$ and $f^*\left(v_{\frac{m+n}{2}+i}\right) = -(2m+i)$, for $1 \le i \le n$ $f^*(v) = \sum f(e_i) = -2m$.

From the above arguments we get

$f^*(V(G)) = \{\pm 1, \pm 2, \pm 4, \pm 8, \ldots, \pm 4\left(\frac{m-5}{2}\right), \pm(m-2), \pm m, \pm(2m+1), \pm(2m+2, \ldots, \pm(3m+n2) \pm 2m-9, \pm(2m-8, \ldots, \pm 5m-232, -2m$

Hence, *f* is an edge pair sum label □

Theorem 2.6: Let *G(p, q)* is an edge pair sum graph . Then $G \odot K_n^c$ is also an edge pair sum graph if n is even.

Proof: Let f be an edge pair sum labeling of G. Then $f^*(V(G)) = \{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{p}{2}}\}$ if p is even.

$f^*(V(G)) = \{\pm k_1, \pm k_2, \ldots \ldots., \pm k_{\frac{p-1}{2}}\} \cup \{k_{\frac{p}{2}}\}$ if p is odd.

Let the edge set of $G \odot K_n^c$ be $E(G \odot K_n^c) = E(G) \cup \{e_{ij}\ ; 1 \le i \le p$ and $1 \le j \le n\}$ and the vertex be $V(G \odot K_n^c) = V(G) \cup \{v_{ij} : 1 \le i \le p$ and $1 \le j \le n\}$.

Here $|E(G \odot K_n^c)| = q + np$. Take $k = \frac{n}{2}$.

Define $h : E(G \cdot K_n^c) \to \{\pm 1, \pm 2, \pm 3, \dots, \pm(q + pn)\}$.
h(e) = f(e) if e ∈ E(G)
$h(e_{ij}) = q + k(i-1) + j \qquad 1 \le i \le p, 1 \le j \le k, \qquad h(e_{i(k+j)}) = -(q + ki-1+j \qquad 1 \le i \le p,\ 1 \le j \le k.$

The induced vertex labeling are

$h^*(v_{ij}) = q + k(i-1) + j \qquad 1 \le i \le p, 1 \le j \le k,$
$h^*(v_{i(k+j)}) = -(q + k(i-1) + j) \qquad 1 \le i \le p,\ 1 \le j \le k.$

Then $h^*(V(G \odot K_n^c)) = \left\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{p}{2}}\right\}$ if p is even.
$h^*(V(G \odot K_n^c)) = \left\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{p-1}{2}}\right\} \cup \left\{k_{\frac{p}{2}}\right\}$ if p is odd. Hence, h is an edge pair sum labeling. □

Corollary 2.7 : The graph $C_n \odot K_m^c$ is an edge pair sum graph.

Proof: By Theorem 2.6, $C_n \odot K_m^c$ is an edge pair sum graph if m is even. Let m is odd and take $= \frac{m-1}{2}$.

The vertex set $V(C_n \odot K_m^c) = V_1 \cup V_2$, where $V_1 = \{v_i : 1 \le i \le n\}$ and $V_2 = \{v_{ij} : 1 \le i \le n \text{ and } 1 \le j \le m\}$ and the edge set $E(C_n \odot K_m^c) = \{\{e_i = v_i v_{i+1} : 1 \le i \le n-1, e_n = v_n v_1 \cup e_{ij} : 1 \le i \le n \text{ and } 1 \le j \le m$.

Define $h : E(C_n \odot K_m^c) = \{\pm 1, \pm 2, \pm 3, \dots\dots\dots, \pm(n + nm)\}$.
$h(e_i) = i \qquad 1 \le i \le n,$
$h(e_{ij}) = -i \qquad j = 1, \quad 1 \le i \le n,$
$h(e_{ij}) = n + k(i-1) + j - 1 \qquad 1 \le i \le n, \quad 2 \le j \le$
$h(e_{ik+j}) = -(n + k(i-1) + j - 1) \quad 1 \le i \le n, \quad 2 \le j \le \frac{m+1}{2}.$

The induced vertex labels are as follows:

$h(v_i) = i \qquad 1 \le i \le n,$
$h(v_{ij}) = -i \qquad j = 1, \quad 1 \le i \le n,$
$h(v_{ij}) = n + k(i-1) + j - 1 \qquad 1 \le i \le n, \quad 2 \le j \le \frac{m+1}{2},$
$h(v_{ik+j}) = -(n + k(i-1) + j - 1) \quad 1 \le i \le n, \quad 2 \le j \le \frac{m+1}{2}.$
$h(V(C_n \odot K_m^c) = \{\pm 1, \pm 2, \dots, \pm p, \pm(n+1), \pm(n+2), \dots, \pm(n + k(n-1) + m-12.$

Hence, $C_n \odot K_m^c$ is an edge pair sum graph. □

Observation 2.7: If G(p, q) is r- regular edge pair sum graph with even number of vertices then $\sum_{v \in V} f^*(v) d(v) = 0$.

Proof: Let f be an edge pair sum labeling of G. Then $f^*(V(G)) = \left\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{p}{2}}\right\}$

$\sum_{v \in V} f^*(v) d(v) = r(\sum_{v \in V} f^*(v)) = 0.$ □

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