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Solitary Wave Solutions of Some Coupled Nonlinear Evolution Equations

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Abstract

In this article, the modified simple equation (MSE) method has been executed to find the traveling wave solutions of the coupled (1+1)-dimensional Broer-Kaup (BK) equations and the dispersive long wave (DLW) equations. The efficiency of the method for finding exact solutions has been demonstrated. It has been shown that the method is direct, effective and can be used for many other nonlinear evolution equations (NLEEs) in mathematical physics. Moreover, this procedure reduces the large volume of calculations.

Keywords: MSE method; NLEE; BK equations; DLW equations; Solitary wave solutions.

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1. Introduction

Nowadays NLEEs have been the subject of all-embracing studies in various branches of nonlinear sciences. A special class of analytical solutions named traveling wave solutions for NLEEs has a lot of importance, because most of the phenomena that arise in mathematical physics and engineering fields can be described by NLEEs. NLEEs are frequently used to describe many problems of protein chemistry, chemically reactive materials, in ecology most population models, in physics the heat flow and the wave propagation phenomena, quantum mechanics, fluid mechanics, plasma physics, propagation of shallow water waves, optical fibers, biology, solid state physics, chemical kinematics, geochemistry, meteorology, electricity etc. Therefore investigating traveling wave solutions is becoming more and more attractive in nonlinear sciences day by day. However, not all equations posed of these models are solvable. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as the Hirota's bilinear transformation method [1, 2], the tanh-function method [3, 4], the extended tanh-method [5, 6], the Exp-function method [7-11], the Adomian decomposition method [12], the F-expansion method [13], the auxiliary equation method [14], the Jacobi elliptic function method [15], the modified exp-function method

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[16], the (G'/G)-expansion method [17-26], the Weierstrass elliptic function method [27], the homotopy perturbation method [28-30], the homogeneous balance method [31, 32], the modified simple equation method [33-36], the enhanced (G'/G)-expansion method [37], the exp(-Φ(ξ))-expansion method [38], the ansatz method [39], the functional variable method [40] and so on.

The objective of this article is to implement the infliction of the MSE method to construct the exact traveling wave solutions for NLEEs in mathematical physics via the BK equation and the DLW equations.

The article is prepared as follows: In section 2, the MSE method has been discussed. In section 3, we apply this method to the nonlinear evolution equations pointed out above; in section 4, physical explanation and in section 5 conclusions are given.

2. The MSE method

In this section we describe the MSE method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation in two independent variables x and t is given by

$$\Re(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \cdots) = 0, \tag{1}$$

where $u(x,t)$ is an unknown function, $\Re$ is a polynomial of $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [33-36]:

Step 1. Combining the independent variables x and t into a compound variable

$\xi = x \pm \omega t$, we suppose that

$$u(\xi) = u(x,t), \qquad \xi = x \pm \omega t. \tag{2}$$

The traveling wave transformation Eq. (2) permits us to reduce Eq. (1) into the following ordinary differential equation (ODE):

$$\Re(u, u', u'', u''', \cdots) = 0, \tag{3}$$

where $\Re$ is a polynomial in $u(\xi)$ and its derivatives, while $u'(\xi) = \dfrac{du}{d\xi}$, $u''(\xi) = \dfrac{d^2 u}{d\xi^2}$, and so on.

Step 2. We suppose that the solution of Eq. (3) can be presented in the following form

$$u(\xi) = C_0 + \sum_{k=1}^{n} C_k \left(\frac{\varphi'(\xi)}{\varphi(\xi)} \right), \tag{4}$$

where C_k $(k = 1, 2, 3, \cdots)$ are arbitrary constants to be determined, such that $C_n \neq 0$, and $\varphi(\xi)$ is an unknown function to be determined later.

Step 3. We determine the positive integer n come out in Eq. (4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (3).

Step 4. We substitute Eq. (4) into (3) and then we account the function $\varphi(\xi)$. As a result of this substitution, we get a polynomial of $\left(\varphi'(\xi)/\varphi(\xi)\right)$ and its derivatives. In this polynomial, we equate the coefficients of same power of $\varphi^{-i}(\xi)$ to zero, where $i \geq 0$. This procedure yields a system of equations which can be solved to find C_k, $\varphi(\xi)$ and $\varphi'(\xi)$. Then the substitution of the values of C_k, $\varphi(\xi)$ and $\varphi'(\xi)$ into Eq. (4) completes the determination of exact solutions of Eq. (1).

3. Applications

3.1. *The coupled (1+1)-dimensional Broer–Kaup (BK) equation*

The BK equation describes the bi-directional propagation of long wave in shallow water. Now we will bring to bear the MSE method to find the exact solutions and then the solitary wave solutions to the BK equation in the form,

$$\begin{aligned} u_t &= u\,u_x + v_x - \frac{1}{2}u_{xx} \\ v_t &= (u\,v)_x + \frac{1}{2}v_{xx}. \end{aligned} \tag{5}$$

Suppose that the traveling wave transformation equation be

$$u(\xi) = u(x,t),\ v(\xi) = v(x,t), \qquad \xi = x - \omega t\ . \tag{6}$$

The wave transformation (6) reduces Eqs. in (5) into the following ODEs

$$-\omega u' - u\,u' - v' + \frac{1}{2}u'' = 0\,. \tag{7}$$

$$\omega v' + (u\,v)' + \frac{1}{2}v'' = 0\,. \tag{8}$$

Integrating Eqs. (7) and (8) with respect to ξ, and neglecting the constant of integration, we obtain

$$-\omega u - \frac{1}{2}u^2 - v + \frac{1}{2}u' = 0\,. \tag{9}$$

$$\omega v + (u\,v) + \frac{1}{2}v' = 0\,. \tag{10}$$

Eq. (9) yields

$$v = -\omega u - \frac{1}{2}u^2 + \frac{1}{2}u'\,. \tag{11}$$

Substituting Eq. (11) into Eq. (10), we obtain

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$$\omega^2 u + \frac{3}{2}\omega u^2 + \frac{1}{2} u^3 - \frac{1}{4}u'' = 0 . \tag{12}$$

Balancing the highest order derivative u'' and nonlinear term u^3 from Eq. (12), we obtain $3n = n + 2$, which gives $n = 1$.

Therefore, Eq. (4) can be written as

$$u(\xi) = C_0 + C_1\left(\frac{\varphi'(\xi)}{\varphi(\xi)}\right). \tag{13}$$

where C_0 and C_1 are constants to be determined such that $C_1 \neq 0$, while $\varphi(\xi)$ is an unknown function to be determined. It is trouble-free to find that

$$u' = C_1\left\{\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2\right\}. \tag{14}$$

$$u'' = C_1\left(\frac{\varphi'''}{\varphi}\right) - 3C_1\left(\frac{\varphi''\varphi'}{\varphi^2}\right) + 2C_1\left(\frac{\varphi'}{\varphi}\right)^3 . \tag{15}$$

$$u^2 = C_0^2 + 2C_0C_1\left(\frac{\varphi'}{\varphi}\right) + C_1^2\left(\frac{\varphi'}{\varphi}\right)^2 . \tag{16}$$

$$u^3 = C_0^3 + 3C_0^2 C_1\left(\frac{\varphi'}{\varphi}\right) + 3C_0 C_1^2\left(\frac{\varphi'}{\varphi}\right)^2 + C_1^2\left(\frac{\varphi'}{\varphi}\right)^3 . \tag{17}$$

Now substituting the values of u, u^2, u^3, u'' into Eq. (12) and then equating the coefficients of φ^0, φ^{-1}, φ^{-2}, φ^{-3} to zero, we respectively obtain

$$\frac{1}{2}C_0^3 + \frac{3}{2}\omega C_0^2 + \omega^2 C_0 = 0 . \tag{18}$$

$$\omega^2 C_1\varphi' - \frac{1}{4}C_1\varphi''' + \frac{3}{2}C_0^2C_1\varphi' + 3\omega C_0C_1\varphi' = 0 . \tag{19}$$

$$\frac{1}{2}C_1\varphi'\varphi'' + \frac{3}{2}\omega C_1^2(\varphi')^2 + \frac{3}{2}C_0 C_1^2(\varphi')^2 + \frac{1}{4}C_1\varphi'\varphi'' = 0 . \tag{20}$$

$$\frac{1}{2}C_1^3(\varphi')^3 - \frac{1}{2}C_1(\varphi')^3 = 0 . \tag{21}$$

Solving Eq. (18), we obtain $C_0 = 0, -\omega, -2\omega$.

Solving Eq. (21), we obtain $C_1 = \pm 1$ since $C_1 \neq 0$.

Solving Eqs. (19) and (20), we obtain

$$\varphi'(\xi) = M\, A\exp(L M\xi) . \tag{22}$$

Integrating, Eq. (22) with respect to ξ, we obtain

$$\varphi(\xi) = \frac{1}{L}(A\exp(LM\xi) + LB). \tag{23}$$

where $L = 4\omega^2 + 6C_0^2 + 12C_0$, $M = -\dfrac{1}{2C_1(\omega_0 + C_0)}$ and *A*, *B* are constants of integration.

Substituting the values of φ and φ' into Eq. (13), we obtain the following exact solution

$$u(x,t) = C_0 + C_1 \frac{LM\,A\exp(LM(x-\omega t))}{A\exp(LM(x-\omega t)) + LB}. \tag{24}$$

Case-I: When $C_0 = 0, -2\omega$ and $C_1 = -1$, also when $C_0 = -\omega$ and $C_1 = \pm 1$, Eq. (24) yields trivial solutions. So this case is discarded.

Case-II: When $C_0 = 0$ and $C_1 = 1$, putting the values of C_0, C_1, L, M into Eq. (24) and then simplifying, we obtain

$$u(x,t) = -2\omega A\left\{\frac{\cosh(\omega(x-\omega t)) + \sinh(\omega(x-\omega t))}{\left(A + 4B\omega^2\right)\cosh(\omega(x-\omega t)) - \left(A - 4B\omega^2\right)\sinh(\omega(x-\omega t))}\right\} \tag{2.5}$$

We can freely choose the constants *A* and *B*. Therefore, substituting the value of *L*, *M* and then setting $A = 4\omega^2$ and $B = 1$, Eq. (25) reduces to

$$u_1(x,t) = \omega\left(\tanh(\omega(x-\omega t)) - 1\right). \tag{26}$$

Substituting Eq. (26) into Eq. (11), we obtain

$$v_1(x,t) = \omega^2 \operatorname{sech}^2(\omega(x-\omega t)). \tag{27}$$

Again, if we set $A = -4\omega^2$ and $B = 1$, Eq. (25) reduces to

$$u_2(x,t) = \omega\left(\coth(\omega(x-\omega t)) - 1\right). \tag{28}$$

Substituting Eq. (28) into Eq. (11), we obtain

$$v_2(x,t) = -\omega^2 \operatorname{csch}^2(\omega(x-\omega t)). \tag{29}$$

Case III: When $C_0 = -2\omega$ and $C_1 = 1$, we obtain same results included in Eq.(26)-(29).

3.2. *The coupled (1+1)-dimensional dispersive long wave (DLW) equation*

In this section, we will apply the modified simple equation method to find the exact solutions and then the solitary wave solutions of DLW equation,

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$$u_t + u u_x + v_x = 0, \\ v_t + (u v)_x + \frac{1}{3} u_{xxx} = 0. \tag{30}$$

The traveling wave transformation is

$$u(\xi) = u(x,t),\ v(\xi) = v(x,t), \qquad \xi = x - \omega t. \tag{31}$$

Using traveling wave Eq. (31), Eq. (30) reduces into the following ODEs:

$$-\omega u' + u u' + v' = 0. \tag{32}$$

$$-\omega v' + (u v)' + \frac{1}{3} u''' = 0. \tag{33}$$

Integrating Eq. (32) and Eq. (33) with respect to ξ, choosing constant of integration as zero, we obtain the following ODEs:

$$-\omega u + \frac{1}{2} u^2 + v = 0. \tag{34}$$

$$-\omega v + (u v) + \frac{1}{3} u'' = 0. \tag{35}$$

From Eq. (34), we obtain

$$v = \omega u - \frac{1}{2} u^2. \tag{36}$$

Substituting Eq. (36) into Eq. (35), yields

$$-\omega^2 u + \frac{3}{2}\omega u^2 - \frac{1}{2} u^3 + \frac{1}{3} u'' = 0. \tag{37}$$

Now balancing the highest order derivative u'' and nonlinear term u^3, we obtain $n = 1$.

Now for $n = 1$, solution $u(\xi) = C_0 + \sum_{k=1}^{n} C_k \left(\frac{\varphi'(\xi)}{\varphi(\xi)} \right)^k$ becomes

$$u(\xi) = C_0 + C_1 \left(\frac{\varphi'(\xi)}{\varphi(\xi)} \right). \tag{38}$$

where C_0 and C_1 are constants to be determined such that $C_1 \neq 0$, while $\varphi(\xi)$ is an unknown function to be determined.

Now substituting (38) into Eq. (37) and then equating the coefficients of φ^0, φ^{-1}, φ^{-2}, φ^{-3} to zero, we respectively obtain

$$-\frac{1}{2} C_0^3 + \frac{3}{2}\omega C_0^2 - \omega^2 C_0 = 0. \tag{39}$$

$$-\omega^2 C_1 \varphi' + \frac{1}{3} C_1 \varphi''' - \frac{3}{2} C_0^2 C_1 \varphi' + 3\omega C_0 C_1 \varphi' = 0. \tag{40}$$

$$-\frac{3}{2} C_1 \varphi' \varphi'' + \frac{3}{2}\omega C_1^2 (\varphi')^2 - \frac{3}{2} C_0 C_1^2 (\varphi')^2 - \frac{1}{3} C_1 \varphi' \varphi'' = 0. \tag{41}$$

$$-\frac{1}{2}C_1^3(\varphi')^3+\frac{3}{2}C_1(\varphi')^3=0\,. \tag{42}$$

Solving Eq. (39) yields $C_0=0,\ \omega,\ 2\omega.$ And Eq. (42) yields $C_1=\pm\frac{2}{\sqrt{3}}$ since $C_1\neq 0$. From Eq. (40) and Eq. (41), we obtain

$$\varphi'(\xi)=M\,E\exp(L\,M\xi)\,. \tag{43}$$

Integrating Eq. (43), we obtain

$$\varphi(\xi)=\frac{E\exp(L\,M\xi)+L\,F}{L}\,. \tag{44}$$

where $L=3(\omega^2+\frac{3}{2}C_0^2-3\omega C_0)$, $M=\frac{2}{3(\omega-C_0)C_1}$ and E, F are constants of integration.

Substituting ξ, $\varphi(\xi)$ and $\varphi'(\xi)$ from Eq. (43) and Eq. (44) into Eq. (38), we obtain

$$u(x,t)=C_0+C_1\frac{L\,M\,E\exp(L\,M\,(x-\omega t))}{E\exp(L\,M\ (x-\omega t))+L\,F}\,. \tag{45}$$

Case I: When $C_0=\omega$ Eq. (45) yields trivial solution. Therefore, this case is rejected.

Case II: When $C_0=0$ and $C_1=\pm\frac{2}{\sqrt{3}}$, executing the parallel course of action described in Case-II of subsection 3.1, putting the values of L and M Eq.(45) yields

$$u_{1,2}(x,t)=\omega\left(1\pm\tanh(\frac{\omega\sqrt{3}}{2}(x-\omega t))\right), \tag{46}$$

and

$$u_{3,4}(x,t)=\omega\left(1\pm\coth(\frac{\omega\sqrt{3}}{2}(x-\omega t))\right). \tag{47}$$

Substituting Eq. (46) and Eq. (47) into Eq. (36), we obtain

$$v_1(x,t)=\frac{\omega^2}{2}\mathrm{sec}h^2(\frac{\omega\sqrt{3}}{2}(x-\omega t)), \tag{48}$$

and

$$v_2(x,t)=-\frac{\omega^2}{2}\mathrm{csc}h^2(\frac{\omega\sqrt{3}}{2}(x-\omega t)), \tag{49}$$

respectively.

Case II: When $C_0=2\omega$ and $C_1\pm\frac{2}{\sqrt{3}}$, we obtain same results like Eq.(36) -Eq.(49).

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4. Physical Explanation

In this section, we will put forth the physical explanation of determined exact solutions and solitary wave solutions of nonlinear evolution equations named (1+1)-dimensional Broer-Kaup equation and the (1+1)-dimensional dispersive long wave equation.

The Eq. (26) is the kink solution of BK equation. Kink soliton rise or descent from one asymptotical state at $\xi \to -\infty$ to another asymptotical state at $\xi \to +\infty$. This soliton referred to as topological solitons. The Fig.1 shows the shape of the solitary kink-type solution of the BK equation (only shows the shape of Eq. (26) with wave speed $\omega = 1$ and $-3 \le x, t \le 3$).

Eq. (27) is the bell-shaped soliton solution of BK equation. It has infinite wings or infinite tails. This soliton referred to as non-topological solitons. This solution does not depend on the amplitude and high frequency soliton. Fig. 2 shows the shape of the exact bell-shaped soliton solution i.e., non-topological soliton solution of the BK equation. (only shows the shape of Eq. (27) with wave speed $\omega = 1$ within the interval $-3 \le x, t \le 3$).

Eq. (28) and Eq. (29) are singular soliton solutions of BK equation. Fig. 3 and Fig. 4 show the shape of singular solitons of Eq. (28) and Eq. (29) respectively for wave speed $\omega = 1$ within the interval $-3 \le x, t \le 3$.

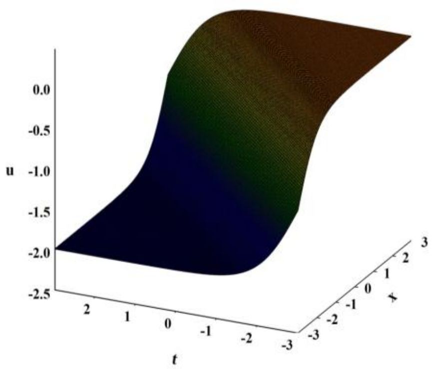

Fig. 1. Shape of Eq.(26) for $\omega = 1$ in the interval $-3 \le x, t \le 3$.

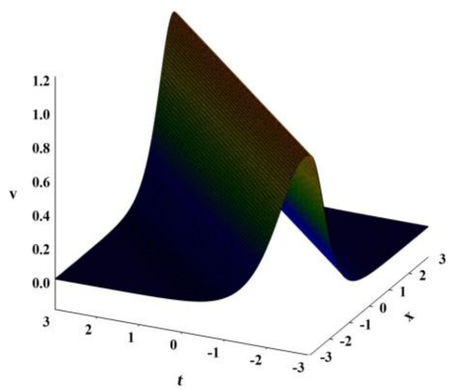

Fig. 2. Shape of Eq.(27) for $\omega = 1$ in the interval $-3 \le x, t \le 3$.

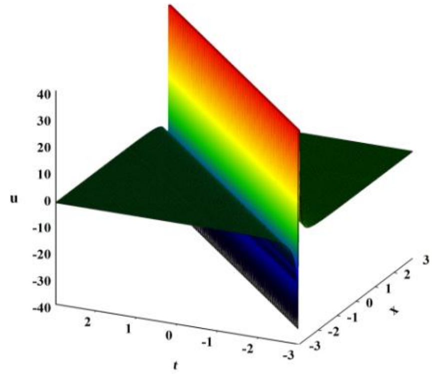

Fig. 3. Shape of Eq.(28) for $\omega = 1$ in the interval $-3 \le x, t \le 3$.

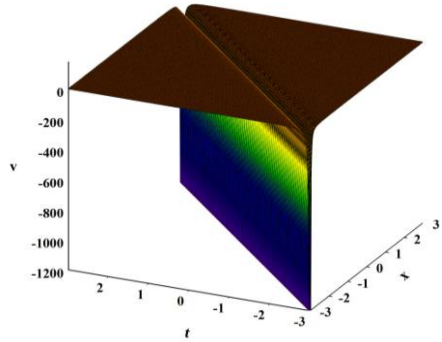

Fig. 4. Shape of Eq.(29) for $\omega = 1$ in the interval $-3 \le x, t \le 3$.

Eq. (46) is the kink solution of DLW equation. This soliton referred to as topological solitons. Fig.5 shows the shape of the solitary kink-type solution of the DLW equation (only shows the shape of Eq. (46) with wave speed $\omega = 2$ and $-3 \le x, t \le 3$).

Eq. (47) is the singular soliton solutions of DLW equation. Fig. 6 shows the shape of singular solitons of Eq. (47) for wave speed $\omega = 2$ within the interval $-3 \le x, t \le 3$.

Eq. (48) is the bell-shaped i.e., non-topological soliton solution of DLW equation. Fig. 7 shows the shape of the exact bell-shaped soliton solution of the DLW equation. (only shows the shape of Eq. (48) with wave speed $\omega = 2$ within the interval $-3 \le x, t \le 3$).

Eq. (49) is the singular soliton solutions of DLW equation. Fig. 8 shows the shape of singular solitons of Eq. (49) for wave speed $\omega = 2$ within the interval $-3 \le x, t \le 3$.

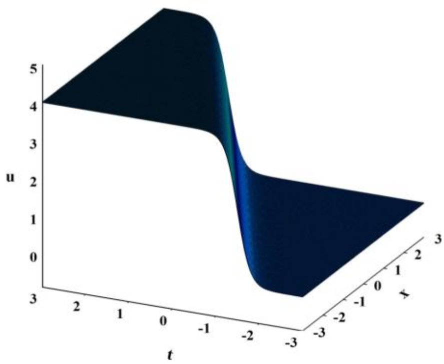

Fig. 5. Shape of Eq.(46) with $\omega = 2$ in the interval $-3 \le x, t \le 3$.

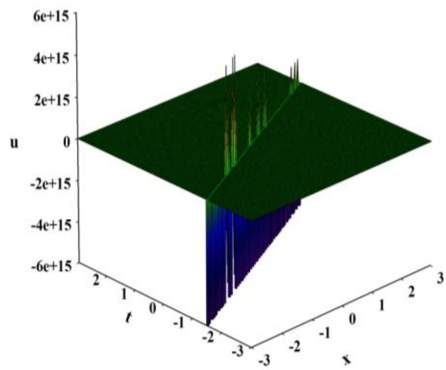

Fig. 6. Shape of Eq.(47) with $\omega = 2$ in the interval $-3 \le x, t \le 3$.

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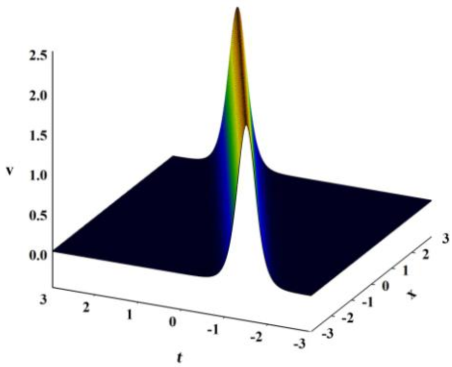

Fig. 7. Shape of Eq.(48) with $\omega = 2$ in the interval $-3 \le x, t \le 3$.

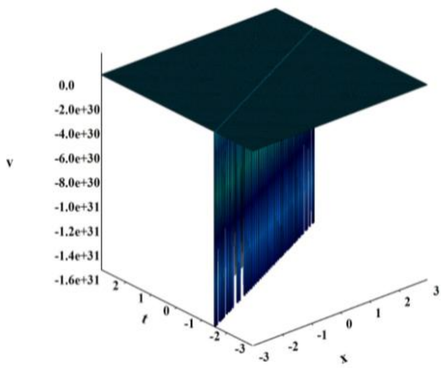

Fig. 8. Shape of Eq.(49) with $\omega = 2$ in the interval $-3 \le x, t \le 3$.

From the obtained solutions we observe that $u(x,t) = u(x - \omega t)$ and $v(x,t) = v(x - \omega t)$ means that for positive value of wave speed, the disturbance moving in the positive *x*-direction. If we take $\omega < 0$ the propagation will be in the negative *x*-direction.

5. Conclusions

The MSE method is powerful and effective mathematical tool in solving nonlinear evolution equation in mathematical physics, applied mathematics and engineering. In this article, the MSE method has been employed for analytic treatment of two nonlinear coupled partial differential equations. The MSE method requires wave transformation formulae. Via MSE method traveling wave solutions, kinks solutions, bell-shaped solutions of BK equations and DLW equations have been derived. Comparing the currently applied method with other methods, such as, the (*G′/G*)-expansion method, the Exp-function method and the projective Riccati equation method, we might conclude that the exact solutions to BK equations and DLW equations can be investigated by using these methods with the help of the symbolic computation software's, such as, Mathematica, Maple etc. to facilitate the complicated algebraic computations. But, by means of the MSE method the exact solutions to these equations have been gained in this article without using the symbolic computation software's since the computations are very simple. Moreover, to explore the exact solutions of the above studied NLEEs through the methods like the (*G′/G*)-expansion method, the F-expansion method, the Jacobi elliptic function method etc. an auxiliary equation is necessary. This study shows that the MSE method is quite efficient and practically well suited to be used in finding exact solutions of NLEEs. Also, we observe that the MSE method is straightforward and can be applied to many other nonlinear evolution equations.

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