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Prime Gamma-Near-Rings with σ-Derivations

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Abstract

Let N be a prime Γ-near-ring and σ be an automorphism on N. In this paper, we prove that if d is a σ-derivation of N such that $\sigma d = d\sigma$ with $d^2 = 0$, then $d = 0$. The composition of two derivations σ and τ are considered and investigated the conditions that the derivation is a στ-derivation.

Keywords: Γ-near-ring; Prime Γ-near-ring; Derivation; σ-derivation; Automorphism.

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1. Introduction

A Γ-near-ring is a triple $(N, +, \Gamma)$, where:

(i) $(N, +)$ is a group (not necessarily abelian),
(ii) Γ is a non-empty set of binary operations on N such that for each $\alpha \in \Gamma$, $(N, +, \alpha)$ is a left near-ring.
(iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in N$ and $\alpha, \beta \in \Gamma$.

Exactly speaking, it is a left Γ-near-ring because it satisfies the left distributive law. We will use the word Γ-near-ring to mean left Γ-near-ring. For a Γ-near-ring N, the set $N_0 = \{x \in N : 0\alpha x = 0, \alpha \in \Gamma\}$ is called the zero-symmetric part of N. A Γ-near-ring N is said to be zero-symmetric if $N = N_0$. Throughout this paper, we consider N to be a zero symmetric left Γ-near-ring with center $Z(N)$. The Γ-near-ring N is called a prime Γ-near-ring if $x\Gamma N\Gamma y = \{0\}$ with $x, y \in N$ implies $x = 0$ or $y = 0$. An additive mapping d: $N \to N$ is called a derivation if $d(x\alpha y) = x\alpha d(y) + d(x)\alpha y$ holds for all $x, y \in N$, $\alpha \in \Gamma$.

In ref. [1], Bell and Mason introduced the notion of derivations in near-ring and obtained basic properties of derivations in near-rings. They also studied some remarkable results [2]. Afterwards Cho [3] worked on derivations in prime near-rings. In ref. [4], Kamal studied the characterizations of σ-derivations on prime near-rings. Kazaz and Alkan [5] introduced the notion of two sided α-derivations of a Γ-near-ring and obtained

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some generalizations of the works of Argac [6, 7]. Samman [8] studied on an α-derivations of prime Γ-near-rings. He also obtained the composition for α-derivation and a Posner-type result with this. In refs. [9-11], Dey *et. al*. worked on derivations and generalized derivations of Γ-rings and Γ-near-rings and obtained some important properties of these rings. Also Dey and Paul [12] extended the results of near-rings to Γ-near-rings.

In this paper, we define a σ-derivation in Γ-near-rings. An example of this type of this derivation is given to ensure its existence. We generalize the results of ref. [13] in Γ-near-rings. One of our results in Γ-rings is an analogous version of a well-known result of Posner for the composition of derivation of rings and near-rings.

2. Prime Γ-Near-Rings with σ-Derivations

Let N be a Γ-near-ring and let σ be an automorphism of N. An additive mapping d: $N \to N$ is called a right σ-derivation (simply we call σ-derivation) if $d(x\alpha y) = \sigma(x)\alpha d(y) + d(x)\alpha y$ for all $x, y\in N$, $\alpha\in\Gamma$. The composition of derivations of Γ-rings is obtained for the case of Γ-near-rings. The existence of such a derivation is ensured by the following example.

Example 2.1. Let J be a Γ-near-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ which is not a Γ-ring such that $(J, +)$ is abelian. Let M be commutative Γ-ring satisfying the condition as in J. Take $N = J \oplus M$. Then it is clear that N is a Γ-near-ring but not a Γ-ring. It is seen that M is an ideal of N and its elements commute with all elements of N. Let σ be a nontrivial automorphism of N and take $m\in M$. Define d^{α}_{m}: $N \to N$ by $d^{\alpha}_{m}(x) = \sigma(x)\alpha m - x\alpha m$ for all $x\in N$, $\alpha\in\Gamma$. Then we shall show that d^{α}_{m} is a σ-derivation. For all x, $y\in N$, $\beta\in\Gamma$,

$$d^{\alpha}_{m}(x\beta y) = \sigma(x\beta y)\alpha m - x\beta y\alpha m = \sigma(x\beta y)\alpha m - \sigma(x)\beta y\alpha m + \sigma(x)\beta y\alpha m - x\beta y\alpha m$$

$$= \sigma(x)\beta\sigma(y)\alpha m - \sigma(x)\beta y\alpha m + y\beta[\sigma(x)\alpha m - m\alpha x] = \sigma(x)\beta[\sigma(y)\alpha m - y\alpha m] + [\sigma(x)\alpha m - m\alpha x]\beta y$$

$$= \sigma(x)\beta d^{\alpha}_{m}(y) + d^{\alpha}_{m}(x)\beta y$$

This shows that

$$d^{\alpha}_{m}(x\beta y) = \sigma(x)\beta d^{\alpha}_{m}(y) + d^{\alpha}_{m}(x)\beta y \text{ for all } x, y\in N, \alpha, \beta\in\Gamma.$$

Hence d^{α}_{m} is a σ-derivation.

We begin with the following Lemmas which are more useful to develop our main results.

Lemma 2.2. Let d be an additive endomorphism of a 2-torsion free Γ-near-ring N. Then d is a σ-derivation if and only if $d(x\alpha y) = d(x)\alpha y + \sigma(x)\alpha d(y)$ for all $x, y\in N$, $\alpha\in\Gamma$.

Proof. By definition, if d is a σ-derivation then for all $x, y\in N$, $\alpha\in\Gamma$,

$d(x\alpha y) = \sigma(x)\alpha d(y) + d(x)\alpha y$. Then $d(x\alpha(y + y)) = \sigma(x)\alpha d(y + y) + d(x)\alpha(y + y) = 2\sigma(x)\alpha d(y) + 2d(x)\alpha y$,

and

$$d(x\alpha y + x\alpha y) = 2d(x\alpha y) = 2(\sigma(x)\alpha d(y) + d(x)\alpha y),$$

so that

$$\sigma(x)\alpha d(y) + d(x)\alpha y = d(x)\alpha y + \sigma(x)\alpha d(y).$$

The proof of the converse statement is similar.

This Lemma indicates an equivalent definition of a σ-derivation.

Lemma 2.3. Let d be a σ-derivation on a 2-torsion free Γ-near-ring N. Then for all $x, y, z\in N$, $\alpha, \beta\in\Gamma$,

(i) $(\sigma(x)\alpha d(y) + d(x)\alpha y)\beta z = \sigma(x)\alpha d(y)\beta z + d(x)\alpha y\beta z$.

(ii) $(d(x)\alpha y + \sigma(x)\alpha d(y))\beta z = d(x)\alpha y\beta z + \sigma(x)\alpha d(y)\beta z$.

Proof. (i) Let $x, y, z\in N$, $\alpha, \beta\in\Gamma$. Then

$$d(x\alpha(y\beta z)) = \sigma(x)\alpha d(y\beta z) + d(x)\alpha(y\beta z)$$

$$= \sigma(x)\alpha(\sigma(y)\beta d(z) + d(y)\beta z) + d(x)\alpha(y\beta z)$$

$$= (\sigma(x)\alpha\sigma(y))\beta d(z) + \sigma(x)\alpha d(y)\beta z + d(x)\alpha(y\beta z)$$

$$= \sigma(x\alpha y)\beta d(z) + \sigma(x)\alpha d(y)\beta z + d(x)\alpha y\beta z. \quad (1)$$

Also,

$$d((x\alpha y)\beta z) = \sigma(x\alpha y)\beta d(z) + d(x\alpha y)\beta z$$

$$= \sigma(x\alpha y)\beta d(z) + (\sigma(x)\alpha d(y) + d(x)\alpha y)\beta z. \quad (2)$$

From (1) and (2), we get

$$(\sigma(x)\alpha d(y) + d(x)\alpha y)\beta z = \sigma(x)\alpha d(y)\beta z + d(x)\alpha y\beta z.$$

(ii) By using Lemma 2.2, we obtain (ii).

Lemma 2.4. Let d be a σ-derivation of a 2-torsion free prime Γ-near-ring N and let $a\in N$ such that $a\Gamma d(x) = 0$ (or $d(x)\Gamma a = 0$) for all $x\in N$. Then $a = 0$ or $d = 0$.

Proof. For all $x, y\in N$, $\alpha, \beta\in\Gamma$,

$$0 = a\beta d(x\alpha y) = a\beta(\sigma(x)\alpha d(y) + d(x)\alpha y) = a\beta\sigma(x)\alpha d(y) + a\beta d(x)\alpha y$$

$$= a\beta\sigma(x)\alpha d(y) + 0 = a\beta\sigma(x)\alpha d(y) = a\beta\sigma(x)\alpha d(y),$$

since σ is an automorphism. Thus $a\Gamma N\Gamma d(y) = 0$. Since N is prime, we get $a = 0$ or $d = 0$.

To prove the case when $d(x)\Gamma a = 0$, we need Lemma 2.2. So if $d(x)\Gamma a = 0$ for all $x\in N$, then for all $x, y\in N$, $\alpha, \beta\in\Gamma$,we have

$0 = d(y\alpha x)\beta a = (\sigma(y)\alpha d(x) + d(y)\alpha x)\beta a = \sigma(y)\alpha d(x)\beta a + d(y)\alpha x\beta a$, by Lemma 2.2,

$= 0 + d(y)\alpha x\beta a = d(y)\alpha x\beta a$.

Thus $d(y)\Gamma N\Gamma a = 0$. By the primeness of N implies that $d = 0$ or $a = 0$.

Theorem 2.5. Let N be a 2-torsion-free prime Γ-near-ring. Let d be a σ-derivation on N such that $d\sigma = \sigma d$. Then $d^2 = 0$ implies $d = 0$.

Proof. Suppose that $d^2 = 0$. Let $x, y\in N$, $\alpha\in\Gamma$. Then

$d^2(x\alpha y) = 0 = d(d(x\alpha y)) = d(\sigma(x)\alpha d(y) + d(x)\alpha y)$

$= d(\sigma(x)\alpha d(y)) + d(d(x)\alpha y)$

$= \sigma^2(x)\alpha d^2(y) + d(\sigma(x))\alpha d(y) + \sigma(d(x))\alpha d(y) + d^2(x)\alpha y$

$= d(\sigma(x))\alpha d(y) + \sigma(d(x))\alpha d(y)$

$= 2d(\sigma(x))\alpha d(y)$.

Hence, $2d(\sigma(x))\alpha d(y) = 0$. Since N is 2-torsion-free, we have $d(\sigma(x))\alpha d(y) = 0$. Since σ is onto, we get $d(x)\alpha d(y) = 0$ and hence by Lemma 2.4, $d = 0$.

The following theorem displays the commutativity of automorphisms of N and the derivation which we are considering on N.

Theorem 2.6. Let d be a σ-derivation on a Γ-near-ring N. Let τ be an automorphism of N which commutes with d. Then $(\sigma\tau)(x)\alpha(d\tau)(y) = (\tau\sigma)(x)\alpha(\tau d)(y)$ for all $x, y\in N$, $\alpha\in\Gamma$.

Proof. Let $x, y\in N$, $\alpha, \beta\in\Gamma$. Then

$$(\tau d)(x\alpha y) = \tau(\sigma(x)\alpha d(y) + d(x)\alpha y) = (\tau\sigma)(x)\alpha(\tau d)(y) + (\tau d)(x)\alpha\tau(y). \quad (3)$$

and,

$$(d\tau)(x\alpha y) = d(\tau(x)\alpha\tau(y)) = (\sigma\tau)(x)\alpha d(\tau(y)) + d(\tau(x))\alpha\tau(y). \quad (4)$$

Since $d\tau = \tau d$, equations (3) and (4) imply that

$(\sigma\tau)(x)\alpha(d\tau)(y) = (\tau\sigma)(x)\alpha(\tau d)(y)$.

Theorem 2.7. Let d_1 be a σ-derivation and d_2 be a τ-derivation on a 2-torsion free prime Γ-near-ring N such that $d_1\sigma = \sigma d_1$ and $d_2\tau = \tau d_2$. Then d_1d_2 is a $\sigma\tau$-derivation if and only if $d_1 = 0$ or $d_2 = 0$.

Proof. Let d_1d_2 be a $\sigma\tau$-derivation. For $x, y\in N$, $\alpha\in\Gamma$, we have

$(d_1d_2)(x\alpha y) = (\sigma\tau)(x)\alpha d_1d_2(\mathrm{y}) + (d_1d_2)(x)\alpha y.$ (5)

Also,

$(d_1d_2)(x\alpha y) = d_1(d_2(x\alpha y))$

$= d_1(\tau(x)\alpha d_2(y) + d_2(x)\alpha y)$

$= d_1(\tau(x)\alpha d_2(y)) + d_1(d_2(x)\alpha y)$

$= (\sigma\tau)(x)\alpha d_1d_2(y) + (d_1\tau)(x)\alpha d_2(y) + (\sigma d_2)(x)\alpha d_1(y) + d_1d_2(x)\alpha y$ (6)

From (5) and (6), we get

$(d_1\tau)(x)\alpha d_2(y) + (\sigma d_2)(x)\alpha d_1(y) = 0.$ (7)

Replacing x by $x\beta d_2(z)$ in (7), we get

$(d_1\tau)(x\beta d_2(\mathrm{z}))\alpha d_2(y) + (\sigma d_2)(x\beta d_2(\mathrm{z}))\alpha d_1(y) = 0,$

and so,

$(\tau d_1)(x\beta d_2(\mathrm{z}))\alpha d_2(y) + (\sigma d_2)(x\beta \mathrm{d}_2(\mathrm{z}))\alpha d_1(y) = 0.$ (8)

Using Lemma 2.2, Eq. (8) becomes

$\tau(d_1(x)\beta d_2(z) + \sigma(x)\beta d_1d_2(\mathrm{z}))\alpha d_2(y) + \sigma(\tau(x)\beta d_2^2(z) + d_2(x)\beta d_2(\mathrm{z}))\alpha d_1(y) = 0,$

$(\tau d_1(x)\beta\tau d_2(z) + \tau\sigma(x)\beta\tau d_1d_2(\mathrm{z}))\alpha d_2(y) + (\sigma\tau(x)\beta\sigma d_2^2(\mathrm{z}) + \sigma d_2(x)\beta\sigma d_2(\mathrm{z}))\alpha d_1(y) = 0$

(9)

Using Theorem 2.5 and the hypothesis, equation (9) becomes

$(d_1\tau(x)\beta d_2\tau(z) + \sigma\tau(x)\beta d_1(\tau d_2(\mathrm{z})))\alpha d_2(y) + (\sigma\tau(x)\beta d_2^2(\sigma(z) + d_2\sigma(x)\beta d_2(\sigma(\mathrm{z})))\alpha d_1(y) = 0.$ (10)

Using Lemma 2.2, Eq. (10) becomes

$d_1\tau(x)\beta d_2\tau(z)\alpha d_2(y) + \sigma\tau(x)\beta d_1(\tau d_2(\mathrm{z}))\alpha d_2(y) + \sigma\tau(x)\beta d_2^2(\sigma(\mathrm{z}))\alpha d_1(y) +$

$d_2(\sigma(x))\beta d_2(\sigma(\mathrm{z}))\alpha d_1(y) = 0,$

$d_1\tau(x)\beta d_2\tau(z)\alpha d_2(y) + (\sigma\tau)(x)\beta(d_1(\tau d_2(\mathrm{z}))\alpha d_2(y) + d_2^2(\sigma(\mathrm{z}))\alpha d_1(y)) +$

$d_2(\sigma(x))\beta d_2(\sigma(\mathrm{z}))\alpha d_1(y) = 0.$ (11)

Replacing x by $d_2(z)$ in (7), we get

$(d_1\tau)(d_2(\mathrm{z}))\alpha d_2(y) + (\sigma d_2)(d_2(\mathrm{z}))\alpha d_1(y) = 0,$

or

$d_1(\tau d_2(\mathrm{z}))\alpha d_2(y) + d_2^2(\sigma(\mathrm{z}))\alpha d_1(y) = 0.$ (12)

Since N is zero symmetric, Eqs. (11) and (12) imply that

$d_1\tau(x)\alpha d_2\tau(z)\beta d_2(y) + d_2(\sigma(x))\alpha d_2(\sigma(\mathrm{z}))\beta d_1(y) = 0.$ (13)

Replacing now x by z in (7), we get

$$(d_1\tau)(z)\alpha d_2(y) + (\sigma d_2)(z)\alpha d_1(y) = 0,$$

or

$$\sigma d_2(z)\alpha d_1(y) = -\, d_1(\tau(z))\alpha d_2(y). \qquad (14)$$

Replacing y by $\tau(z)$ in (7), we get

$$(d_1\tau)(x)\alpha d_2(\tau(z)) + (\sigma d_2)(x)\alpha d_1(\tau(z)) = 0.$$

So,

$$d_1(\tau(x))\alpha d_2(\tau(z)) = -\, d_2(\sigma(x))\alpha d_1(\tau(z)). \qquad (15)$$

Combining (13), (14) and (15) we get

$$(-\,(d_2(\sigma(x))\alpha d_1(\tau(z))))\beta d_2(y) + d_2(\sigma(x))\alpha(-\,(d_1(\tau(z))\beta d_2(y))) = 0. \qquad (16)$$

To simplify notations, we put $u = d_2(\sigma(x))$, $v = d_1(\tau(z))$, and $w = d_2(y)$. Then

$$-\quad (u\alpha v)\beta w + u\alpha(-\,v\beta w) = 0,$$

$$\Rightarrow u\alpha(-\,v)\beta w + u\alpha(-\,v\beta w) = 0,$$

$$\Rightarrow u\alpha(-\,v)\beta w - u\alpha(v\beta w) = 0,$$

$$\Rightarrow -\,u\alpha v\beta w - u\alpha v\beta w = 0,$$

$$\Rightarrow u\alpha v\beta w + u\alpha v\beta w = 0,$$

$$\Rightarrow u\alpha(2v\beta w) = 0.$$

If $u \neq 0$ (i.e. $d_2 \neq 0$), then by Lemma 2.4, $2v\beta w = 0$, that is, $v\beta(2w) = 0$. Again if $w \neq 0$ (i.e. $d_2 \neq 0$), then by hypothesis $2w \neq 0$, and then by Lemma 2.4 we have $v = 0$, that is $d_1 = 0$. This shows that if $d_2 \neq 0$ then $d_1 = 0$ which completes the proof.

Remarks 2.8. In the above theorem, the composition that N is a 2-torsion free may be weakened if we do not take the existence of an element y in N such that $2d_2(\mathrm{y}) \neq 0$. The same proof will lead to the conclusion that $d_1 = 0$.

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