

Cordial and 3-Equitable Labeling for Some Shell Related Graphs

S. K. Vaidya¹, N. A. Dani, K. K. Kanani, and P. L. Vihol

Department of Mathematics, Saurashtra University, Rajkot-360005, Gujrat, India

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Abstract

We present here cordial and 3-equitable labeling for the graphs obtained by joining apex vertices of two shells to a new vertex. We extend these results for k copies of shells.

Keywords: Cordial labeling; 3-equitable labeling; Shell.

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1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V, E)$. For all standard terminology and notations we follow Harary [1]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 A shell S_n is the graph obtained by taking $n-3$ concurrent chords in a cycle C_n on n vertices. The vertex at which all the chords are concurrent is called the *apex vertex*. The shell is also called fan F_{n-1} . i.e. $S_n = F_{n-1} = P_{n-1} + K_1$.

Definition 1.2: Consider two shells $S_n^{(1)}$ and $S_n^{(2)}$ then graph $G = \langle S_n^{(1)}; S_n^{(2)} \rangle$ obtained by joining apex vertices of shells to a new vertex x .

Definition 1.3: Consider k copies of shells namely $S_n^{(1)}, S_n^{(2)}, S_n^{(3)}, \dots, S_n^{(k)}$. Then the graph $G = \langle S_n^{(1)}; S_n^{(2)}; S_n^{(3)}; \dots; S_n^{(k)} \rangle$ obtained by joining apex vertex of each $S_n^{(p)}$ and apex of $S_n^{(p-1)}$ to a new vertex x_{p-1} where $2 \leq p \leq k$.

Definition 1.4: If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

Most interesting graph labeling problems have following three important ingredients.

- (i) a set of numbers from which the labels are chosen;
- (ii) a rule that assigns a value to each edges;
- (iii) a condition that these values must satisfy.

¹Corresponding Author : samirkvaidya@yahoo.co.in

For detail survey on graph labeling one can refer Gallian [2]. Vast amount of literature is available on different types of graph labeling. According to Beineke and Hegde [3] graph labeling serves as a frontier between number theory and structure of graphs. Labeled graph have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-ray crystallography, communication network and to determine optimal circuit layouts. A detail study of variety of applications of graph labeling is given by Bloom and Golomb[4].

Definition 1.5: Let $G = (V, E)$ be a graph. A mapping $f: V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f .

For an edge $e = uv$, the induced edge labeling $f^*: E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f while $e_f(0)$, $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^* .

Definition 1.6: A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [5]. Many researchers have studied cordiality of graphs. Cahit [5] proved that tree is cordial. In the same paper he proved that K_n is cordial if and only if $n \leq 3$. Ho *et al.* [6] proved that unicyclic graph is cordial unless it is C_{4k+2} . Andar *et al.* [7] have discussed the cordiality of multiple shells. Vaidya *et al.* [8, 9, 10] have also discussed the cordiality of various graphs.

Definition 1.7: Let $G = (V, E)$ be a graph. A mapping $f: V(G) \rightarrow \{0, 1, 2\}$ is called ternary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f .

For an edge $e = uv$, the induced edge labeling $f^*: E(G) \rightarrow \{0, 1, 2\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ and $v_f(2)$ be the number of vertices of G having labels 0, 1 and 2 respectively under f while $e_f(0)$, $e_f(1)$ and $e_f(2)$ be the number of edges having labels 0, 1 and 2 respectively under f^* .

Definition 1.8: A vertex labeling of a graph G is called a 3-equitable labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph G is 3-equitable if it admits 3-equitable labeling.

The concept of 3-equitable labeling was introduced by Cahit [11]. Many researchers have studied 3-equitability of graphs. For example Cahit [11] proved that C_n is 3-equitable except $n \equiv 3(mod 6)$. In the same paper he proved that an Eulerian graph with number of edges congruent to $3(mod 6)$ is not 3-equitable. Youssef [12] proved that W_n is 3-equitable for all $n \leq 4$. In the present investigations we prove that graphs $\langle S_n^{(1)}: S_n^{(2)} \rangle$ and $\langle S_n^{(1)}: S_n^{(2)}: S_n^{(3)}: \dots: S_n^{(k)} \rangle$ cordial as well as 3-equitable.

2. Main Results

Theorem 2.1: Graph $\langle S_n^{(1)}: S_n^{(2)} \rangle$ is cordial.

Proof: Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \dots, v_n^{(1)}$ be the vertices $S_n^{(1)}$ of $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \dots, v_n^{(2)}$ be the vertices of $S_n^{(2)}$. Let $v_1^{(1)}$ and $v_1^{(2)}$ be the apex vertices of $S_n^{(1)}$ and $S_n^{(2)}$, respectively. Let $G = \langle S_n^{(1)}; S_n^{(2)} \rangle$. We define binary vertex labeling $f: V(G) \rightarrow \{0, 1\}$ as follows.

- For $j=1, 2$
- $f(v_i^{(j)}) = 0$; if $i \equiv 2, 3 \pmod{4}$
- $f(v_i^{(j)}) = 1$; if $i \equiv 0, 1 \pmod{4}$
- $f(x) = 0$; if $n \equiv 1 \pmod{4}$
- $f(x) = 1$; if $n \equiv 0, 2, 3 \pmod{4}$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. as shown in Table 1. i.e. G admits cordial labeling. □

Let $n=4a+b, a \in \mathbb{N}, b \in \{0\}$.

Table 1. Table showing vertex and edge conditions.

b	Vertex condition	Edge condition
0,1,2	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
3	$v_f(0) = v_f(1) + 1$	$e_f(0)=e_f(1)$

Illustration 2.2: Consider a graph $G = \langle S_7^{(1)}; S_7^{(2)} \rangle$. Here $n = 7$. The cordial labeling is as shown in Fig. 1.

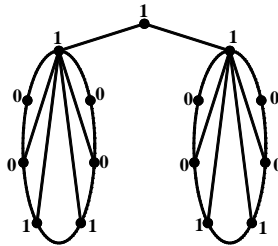


Fig. 1. Cordial labeling of the graph G .

Theorem 2.3: Graph $\langle S_n^{(1)}; S_n^{(2)}; S_n^{(3)}; \dots; S_n^{(k)} \rangle$ is cordial.

Proof: Let $S_n^{(j)}$ be the shells. Let $v_i^{(j)}$ be the vertices of $S_n^{(j)}$ and $v_1^{(j)}$ be the apex vertices of $S_n^{(j)}$. Let $x_j (j \neq k)$ be the new vertices. Let $G = \langle S_n^{(1)}; S_n^{(2)}; S_n^{(3)}; \dots; S_n^{(k)} \rangle$. We define binary vertex labeling $f: V(G) \rightarrow \{0, 1\}$ as follows.

- For $j \equiv 1, 2 \pmod{4}$
- $f(v_i^{(j)}) = 0$; if $i \equiv 2, 3 \pmod{4}$
- $f(v_i^{(j)}) = 1$; if $i \equiv 0, 1 \pmod{4}$
- For $j \equiv 0, 3 \pmod{4}$
- $f(v_i^{(j)}) = 0$; if $i \equiv 0, 1 \pmod{4}$

$$\begin{aligned}
 &f(v_i^{(0)}) = 1; \text{ if } i \equiv 2, 3 \pmod{4} \\
 &\text{For } n \equiv 0, 2, 3 \pmod{4} \\
 &f(x_j) = 0; \text{ if } j \equiv 2, 3 \pmod{4} \\
 &f(x_j) = 1; \text{ if } j \equiv 0, 1 \pmod{4}, j \neq k \\
 &\text{For } n \equiv 1 \pmod{4} \\
 &f(x_j) = 0; \text{ if } j \equiv 1, 2 \pmod{4} \\
 &f(x_j) = 1; \text{ if } j \equiv 0, 3 \pmod{4}, j \neq k
 \end{aligned}$$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in Table 2. i.e. G admits cordial labeling. \square

Let $n=4a+b, k=4c+d, a, c \in \mathbb{N} \setminus \{0\}$

Table 2. Table showing vertex and edge conditions.

b	d	Vertex condition	Edge condition
0	0	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	1,3	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
	2	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1)$
1	0	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	1	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1) + 1$
	2	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1)$
	3	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1) + 1$
2	0	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	1,3	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
	2	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1)$
3	0,2	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	1,3	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$

Illustration 2.4: Consider a graph $G = \langle S_5^{(1)} : S_5^{(2)} : S_5^{(3)} \rangle$. Here $n = 5$. The cordial labeling is as shown in Fig. 2.

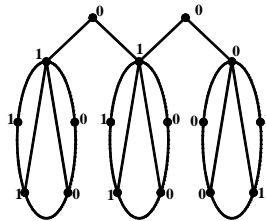


Fig. 2. Cordial labeling of the graph G.

Theorem 2.5: Graph $\langle S_n^{(1)} : S_n^{(2)} \rangle$ is 3-equitable.

Proof: Let $v_1^{(1)}, v_2^{(1)}, \dots, v_n^{(1)}$ be the vertices $S_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \dots, v_n^{(2)}$ be the vertices $S_n^{(2)}$. Let $v_1^{(1)}$ and $v_1^{(2)}$ be the apex vertices of $S_n^{(1)}$ and $S_n^{(2)}$, respectively. Let $G = \langle S_n^{(1)} : S_n^{(2)} \rangle$. We define ternary vertex labeling $f: V(G) \rightarrow \{0, 1, 2\}$ as follows.

Case-1: For $n \equiv 0, 5 \pmod{6}$

$$\begin{aligned} f(v_i^{(1)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \\ f(v_i^{(1)}) &= 1; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(1)}) &= 2; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_i^{(2)}) &= 0; \text{ if } i \equiv 0, 3 \pmod{6} \\ f(v_i^{(2)}) &= 1; \text{ if } i \equiv 4, 5 \pmod{6} \\ f(v_i^{(2)}) &= 2; \text{ if } i \equiv 1, 2 \pmod{6} \\ f(x) &= 0; \end{aligned}$$

Case-2: For $n \equiv 1 \pmod{6}$

$$\begin{aligned} f(v_i^{(1)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6}, i \neq n \\ f(v_i^{(1)}) &= 1; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(1)}) &= 2; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_n^{(1)}) &= 1; \\ f(v_i^{(2)}) &= 0; \text{ if } i \equiv 0, 3 \pmod{6} \\ f(v_i^{(2)}) &= 1; \text{ if } i \equiv 4, 5 \pmod{6} \\ f(v_i^{(2)}) &= 2; \text{ if } i \equiv 1, 2 \pmod{6} \\ f(x) &= 0; \end{aligned}$$

Case-3: For $n \equiv 2 \pmod{6}$

$$\begin{aligned} f(v_i^{(1)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6}, i \neq n-1 \\ f(v_i^{(1)}) &= 1; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(1)}) &= 2; \text{ if } i \equiv 2, 3 \pmod{6}, i \neq n \\ f(v_i^{(2)}) &= 0; \text{ if } i \equiv 0, 3 \pmod{6} \\ f(v_i^{(2)}) &= 1; \text{ if } i \equiv 4, 5 \pmod{6} \\ f(v_i^{(2)}) &= 2; \text{ if } i \equiv 1, 2 \pmod{6}, i \neq n \\ f(v_{n-1}^{(1)}) &= 1; \\ f(v_n^{(1)}) = f(v_n^{(2)}) &= 0; \\ f(x) &= 2; \end{aligned}$$

Case-4: For $n \equiv 3 \pmod{6}$

$$\begin{aligned} f(v_i^{(1)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6}, i \neq n-2 \\ f(v_i^{(1)}) &= 1; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(1)}) &= 2; \text{ if } i \equiv 2, 3 \pmod{6}, i \neq n-1, n \\ f(v_i^{(2)}) &= 0; \text{ if } i \equiv 0, 3 \pmod{6}, i \neq n \\ f(v_i^{(2)}) &= 1; \text{ if } i \equiv 4, 5 \pmod{6} \\ f(v_i^{(2)}) &= 2; \text{ if } i \equiv 1, 2 \pmod{6}, i \neq n-1, n-2 \\ f(v_{n-2}^{(1)}) = f(v_n^{(2)}) &= 1; \\ f(v_{n-1}^{(1)}) = f(v_{n-1}^{(2)}) &= 2; \\ f(v_n^{(1)}) = f(v_{n-2}^{(2)}) &= 0; \\ f(x) &= 0; \end{aligned}$$

Case-5: For $n \equiv 4 \pmod{6}$

$$\begin{aligned} f(v_i^{(1)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \\ f(v_i^{(1)}) &= 1; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(1)}) &= 2; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_i^{(2)}) &= 0; \text{ if } i \equiv 0, 3 \pmod{6}, i \neq n-1 \\ f(v_i^{(2)}) &= 1; \text{ if } i \equiv 4, 5 \pmod{6} \\ f(v_i^{(2)}) &= 2; \text{ if } i \equiv 1, 2 \pmod{6}, i \neq n-2 \\ f(v_{n-2}^{(2)}) = f(v_{n-1}^{(2)}) &= 1; \\ f(x) &= 0; \end{aligned}$$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_j(i) - v_j(j)| \leq 1$ and $|e_j(i) - e_j(j)| \leq 1$, where $0 \leq i, j \leq 2$ as shown in Table 3. i.e. G admits 3-equitable labeling. \square

Let $n = 6a + b, a \in \mathbb{N}, b \in \{0\}$.

Table 3. Table showing vertex and edge conditions.

b	Vertex condition	Edge condition
0,3	$v_j(0) = v_j(1) + 1 = v_j(2) + 1$	$e_j(0) + 1 = e_j(1) = e_j(2)$
1,4	$v_j(0) = v_j(1) = v_j(2)$	$e_j(0) = e_j(1) = e_j(2)$
2,5	$v_j(0) = v_j(1) + 1 = v_j(2)$	$e_j(0) + 1 = e_j(1) + 1 = e_j(2)$

Illustration 2.6: Consider a graph $G = \langle S_6^{(1)} : S_6^{(2)} \rangle$. Here $n = 6$. The 3-equitable labeling is as shown in Fig. 3.

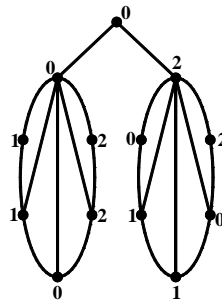


Fig. 3. 3-equitable labeling of the graph G.

Theorem-2.7: Graph $\langle S_n^{(1)} : S_n^{(2)} : S_n^{(3)} : \dots : S_n^{(k)} \rangle$ is 3-equitable.

Proof: Let $S_n^{(j)}$ be the shells. Let $v_i^{(j)}$ be the vertices $S_n^{(j)}$ and $v_1^{(j)}$ be the apex vertices of $S_n^{(j)}$. Let $x_j (j \neq k)$ be the new vertices. Let $G = \langle S_n^{(1)} : S_n^{(2)} : S_n^{(3)} : \dots : S_n^{(k)} \rangle$. We define vertex labeling $f: V(G) \rightarrow \{0, 1, 2\}$ as follows.

Case-1: For $n \equiv 0 \pmod{6}$

Subcase 1.1: $k \equiv 0 \pmod{3}$

For $j \equiv 1 \pmod{3}$

$$f(v_i^{(j)}) = 0; \text{ if } i \equiv 1, 4 \pmod{6}$$

$$f(v_i^{(j)}) = 1; \text{ if } i \equiv 0, 5 \pmod{6}$$

$$f(v_i^{(j)}) = 2; \text{ if } i \equiv 2, 3 \pmod{6}$$

$$f(x_j) = 0;$$

For $j \equiv 0, 2 \pmod{3}$

$$f(v_i^{(j)}) = 0; \text{ if } i \equiv 0, 3 \pmod{6} \text{ } i \neq n$$

$$f(v_i^{(j)}) = 1; \text{ if } i \equiv 4, 5 \pmod{6}$$

$$f(v_i^{(j)}) = 2; \text{ if } i \equiv 1, 2 \pmod{6}$$

$$f(v_n^{(j)}) = 2; \text{ if } j \equiv 2 \pmod{3}$$

$$f(v_n^{(j)}) = 1; \text{ if } j \equiv 0 \pmod{3}$$

$$f(x_j) = 0; j \neq k$$

Subcase 1.2: $k \equiv 1 \pmod{3}$

For first $k-1$ copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow.

$$f(v_i^{(k)}) = 0; \text{ if } i \equiv 1, 4 \pmod{6}$$

$$f(v_i^{(k)}) = 1; \text{ if } i \equiv 0, 5 \pmod{6}$$

$$f(v_i^{(k)}) = 2; \text{ if } i \equiv 2, 3 \pmod{6}$$

$$f(x_{k-1}) = 0;$$

Subcase 1.3: $k \equiv 2 \pmod{3}$

For first $k-2$ copies of shells use the pattern of subcase 1.1 and for $k-1$ and k^{th} copy define function as follow:

$$\begin{aligned} f(v_i^{(k-1)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \\ f(v_i^{(k-1)}) &= 1; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(k-1)}) &= 2; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_i^{(k)}) &= 0; \text{ if } i \equiv 0, 3 \pmod{6} \\ f(v_i^{(k)}) &= 1; \text{ if } i \equiv 4, 5 \pmod{6} \\ f(v_i^{(k)}) &= 2; \text{ if } i \equiv 1, 2 \pmod{6} \\ f(x_{k-2}) &= f(x_{k-1}) = 0; \end{aligned}$$

Case-2: For $n \equiv 1 \pmod{6}$

Subcase 2.1: $k \equiv 0 \pmod{3}$

For $j \equiv 1, 2 \pmod{3}$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6}, i \neq n \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 0, 2, 3, 5 \pmod{6} \text{ and } j \equiv 1 \pmod{3} \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 0, 2, 3, 5 \pmod{6} \text{ and } j \equiv 2 \pmod{3} \\ f(v_n^{(j)}) &= 1; \text{ if } j \equiv 1 \pmod{3} \\ f(v_n^{(j)}) &= 0; \text{ if } j \equiv 2 \pmod{3} \\ f(x_j) &= 0; \text{ if } j \equiv 1 \pmod{3} \\ f(x_j) &= 2; \text{ if } j \equiv 2 \pmod{3} \end{aligned}$$

For $j \equiv 0 \pmod{3}$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 0, 3 \pmod{6} \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 4, 5 \pmod{6} \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 1, 2 \pmod{6} \\ f(x_j) &= 0; j \neq k \end{aligned}$$

Subcase 2.2: $k \equiv 1 \pmod{3}$

For first $k-1$ copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow:

$$\begin{aligned} f(v_i^{(k)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \\ f(v_i^{(k)}) &= 1; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(k)}) &= 2; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(x_{k-1}) &= 2; \end{aligned}$$

Subcase 2.3: $k \equiv 2 \pmod{3}$

For first $k-2$ copies of shells use the pattern of subcase 1.1 and for $k-1$ and k^{th} copy define function as follow:

For $j = k-1, k$;

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \text{ and } j \neq k, i \neq n \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_n^{(k)}) &= 1; \\ f(x_{k-2}) &= 0; \\ f(x_{k-1}) &= 2; \end{aligned}$$

Case-3: For $n \equiv 2 \pmod{6}$

Subcase 3.1: $k \equiv 0 \pmod{3}$

For $j \equiv 1 \pmod{3}$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(x_j) &= 2; \end{aligned}$$

For $j \equiv 2 \pmod{3}$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6}, i \neq n-1 \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 2, 3 \pmod{6}, i \neq n \\ f(v_{n-1}^{(j)}) &= 1; \text{ if } j \equiv 1 \pmod{3} \\ f(v_n^{(j)}) &= 0; \\ f(x_j) &= 2; \end{aligned}$$

For $j \equiv 0 \pmod{3}$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 1, 4 \pmod{6} \\ f(x_j) &= 0; j \neq k \end{aligned}$$

Subcase 3.2: $k \equiv 1 \pmod{3}$

For first $k-1$ copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow:

$$\begin{aligned} f(v_i^{(k)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \\ f(v_i^{(k)}) &= 1; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_i^{(k)}) &= 2; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(x_{k-1}) &= 2; \end{aligned}$$

Subcase 3.3: $k \equiv 2 \pmod{3}$

For first $k-2$ copies of shells use the pattern of subcase 1.1 and for $k-1$ and k^{th} copy define function as follow:

For $j = k-1, k$;

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \text{ and } j \neq k, i \neq 1 \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_1^{(k)}) &= 2; \\ f(x_{k-2}) &= 2; \\ f(x_{k-1}) &= 0; \end{aligned}$$

Case-4: For $n \equiv 3 \pmod{6}$

Subcase 4.1: $k \equiv 0 \pmod{3}$

For $j \equiv 1, 2 \pmod{3}$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 0, 2, 3, 5 \pmod{6} \text{ and } j \equiv 2 \pmod{3} \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 0, 2, 3, 5 \pmod{6} \text{ and } j \equiv 1 \pmod{3} \\ f(x_j) &= 1; \text{ if } j \equiv 1 \pmod{3} \\ f(x_j) &= 2; \text{ if } j \equiv 2 \pmod{3} \end{aligned}$$

For $j \equiv 0 \pmod{3}$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 2, 3 \pmod{6} \text{ and } i \neq n-1 \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 1, 4 \pmod{6} \text{ and } i \neq n-2 \\ f(v_{n-2}^{(j)}) &= 0; \\ f(v_{n-1}^{(j)}) &= 2; \\ f(x_j) &= 0; j \neq k \end{aligned}$$

Subcase 4.2: $k \equiv 1 \pmod{3}$

For first $k-1$ copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow:

$$\begin{aligned} f(v_i^{(k)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \text{ and } i \neq n-2 \\ f(v_i^{(k)}) &= 1; \text{ if } i \equiv 2, 3 \pmod{6} \text{ and } i \neq n-1 \\ f(v_i^{(k)}) &= 2; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_{n-2}^{(k)}) &= 2; \\ f(v_{n-1}^{(k)}) &= 0; \\ f(x_{k-1}) &= 0; \end{aligned}$$

Subcase 4.3: $k \equiv 2 \pmod{3}$

For first $k-2$ copies of shells use the pattern of subcase 1.1 and for $k-1$ and k^{th} copy define function as follow:

For $j=k-1, k$;

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \text{ and } i \neq n-2, j \neq k-1 \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 2, 3 \pmod{6} \text{ and } i \neq n-1 \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_{n-2}^{(k-1)}) &= 2; \\ f(v_{n-1}^{(k-1)}) &= 0; \\ f(v_{n-1}^{(k)}) &= 2; \\ f(x_{k-2}) = f(x_{k-1}) &= 0; \end{aligned}$$

Case-5: For $n \equiv 4 \pmod{6}$

Subcase 5.1: $k \equiv 0 \pmod{3}$

For $j \equiv 1, 2 \pmod{3}$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 0, 2, 3, 5 \pmod{6} \text{ and } j \equiv 2 \pmod{3} \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 0, 2, 3, 5 \pmod{6} \text{ and } j \equiv 1 \pmod{3} \\ f(x_j) &= 2; \end{aligned}$$

For $j \equiv 0 \pmod{3}$

If $1 \leq i \leq n-4$

$$\begin{aligned} f(v_i^{(j)}) &= 0; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_i^{(j)}) &= 1; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_i^{(j)}) &= 2; \text{ if } i \equiv 1, 4 \pmod{6} \\ f(v_{n-3}^{(j)}) = f(v_{n-2}^{(j)}) = f(v_{n-1}^{(j)}) &= 1; \\ f(v_n^{(j)}) &= 0; \\ f(x_j) &= 2; j \neq k \end{aligned}$$

Subcase 5.2: $k \equiv 1 \pmod{3}$

For first $k-1$ copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow:

$$\begin{aligned} f(v_i^{(k)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \text{ and } i \neq n \\ f(v_i^{(k)}) &= 1; \text{ if } i \equiv 2, 3 \pmod{6} \\ f(v_i^{(k)}) &= 2; \text{ if } i \equiv 0, 5 \pmod{6} \\ f(v_n^{(k)}) &= 1; \\ f(x_{k-1}) &= 2; \end{aligned}$$

Subcase 5.3: $k \equiv 2 \pmod{3}$

For first $k-2$ copies of shells use the pattern of subcase 1.1 and for $k-1$ and k^{th} copy define function as follow:

$$\begin{aligned} f(v_i^{(k-1)}) &= 0; \text{ if } i \equiv 1, 4 \pmod{6} \text{ and } i \neq n \\ f(v_i^{(k-1)}) &= 1; \text{ if } i \equiv 0, 5 \pmod{6} \end{aligned}$$

$$\begin{aligned}
 f(v_i^{(k-1)}) &= 2; \text{ if } i \equiv 2,3 \pmod{6} \\
 f(v_i^{(k)}) &= 0; \text{ if } i \equiv 1,4 \pmod{6} \text{ and } i \neq n \\
 f(v_i^{(k)}) &= 1; \text{ if } i \equiv 2,3 \pmod{6} \text{ and } i \neq n-2 \\
 f(v_i^{(k)}) &= 2; \text{ if } i \equiv 0,5 \pmod{6} \\
 f(v_{n-2}^{(k)}) &= f(x_{k-2}) = 2; \\
 f(v_n^{(k)}) &= f(v_n^{(k-1)}) = 1; \\
 f(x_{k-1}) &= 0;
 \end{aligned}$$

Case-6: For $n \equiv 5 \pmod{6}$

Subcase 6.1: $k \equiv 0 \pmod{3}$

For $j \equiv 1, 2 \pmod{3}$

$$\begin{aligned}
 f(v_i^{(j)}) &= 0; \text{ if } i \equiv 1,4 \pmod{6} \\
 f(v_i^{(j)}) &= 1; \text{ if } i \equiv 0,2,3,5 \pmod{6} \text{ and } j \equiv 2 \pmod{3} \\
 f(v_i^{(j)}) &= 2; \text{ if } i \equiv 0,2,3,5 \pmod{6} \text{ and } j \equiv 1 \pmod{3} \\
 f(x_j) &= 2; \text{ if } j \equiv 1 \pmod{3} \\
 f(x_j) &= 0; \text{ if } j \equiv 2 \pmod{3}
 \end{aligned}$$

For $j \equiv 0 \pmod{3}$

If $1 \leq i \leq n-2$

$$\begin{aligned}
 f(v_i^{(j)}) &= 0; \text{ if } i \equiv 0,5 \pmod{6} \\
 f(v_i^{(j)}) &= 1; \text{ if } i \equiv 2,3 \pmod{6} \\
 f(v_i^{(j)}) &= 2; \text{ if } i \equiv 1,4 \pmod{6} \\
 f(v_{n-1}^{(j)}) &= 1; \\
 f(v_n^{(j)}) &= 2; \\
 f(x_j) &= 0; j \neq k
 \end{aligned}$$

Subcase 6.2: $k \equiv 1 \pmod{3}$

For first $k-1$ copies of shells use the pattern of subcase 1.1 and for k^{th} copy define function as follow:

$$\begin{aligned}
 f(v_i^{(k)}) &= 0; \text{ if } i \equiv 1,4 \pmod{6} \\
 f(v_i^{(k)}) &= 1; \text{ if } i \equiv 2,3 \pmod{6} \\
 f(v_i^{(k)}) &= 2; \text{ if } i \equiv 0,5 \pmod{6} \\
 f(x_{k-1}) &= 2;
 \end{aligned}$$

Subcase 6.3: $k \equiv 2 \pmod{3}$

For first $k-2$ copies of shells use the pattern of subcase 1.1 and for $k-1$ and k^{th} copy define function as follow:

$$\begin{aligned}
 f(v_i^{(k-1)}) &= 0; \text{ if } i \equiv 1,4 \pmod{6} \\
 f(v_i^{(k-1)}) &= 1; \text{ if } i \equiv 2,3 \pmod{6} \\
 f(v_i^{(k-1)}) &= 2; \text{ if } i \equiv 0,5 \pmod{6} \\
 f(v_i^{(k)}) &= 0; \text{ if } i \equiv 0,5 \pmod{6} \\
 f(v_i^{(k)}) &= 1; \text{ if } i \equiv 2,3 \pmod{6} \\
 f(v_i^{(k)}) &= 2; \text{ if } i \equiv 1,4 \pmod{6} \\
 f(x_{k-2}) &= 2; \\
 f(x_{k-1}) &= 0;
 \end{aligned}$$

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, where $0 \leq i, j \leq 2$ as shown in Table 4. i.e. G admits 3-equitable labeling. \square

Let $n=6a+b$, and $k=3c+d$ where $a, c \in \mathbb{N} \setminus \{0\}$.

Table 4. Vertex and edge conditions.

b	d	Vertex condition	Edge condition
0	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
1	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)+1$
	2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
2	0,1,2	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
3	2	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)=e_f(1)+1=e_f(2)$
	$0(n \neq 4)$	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	$0(n=4)$	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)=e_f(1)+1=e_f(2)+1$
4	1	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	0,1,2	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$

Illustration 2.8: Consider a graph $G = \langle S_4^{(1)} : S_4^{(2)} : S_4^{(3)} \rangle$. Here $n = 4$. The 3-equitable labeling is as shown in Fig. 4.

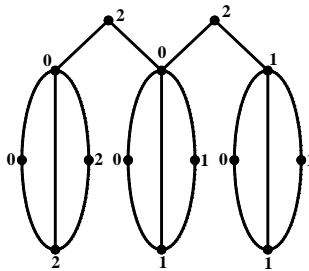


Fig. 4. 3-equitable labeling of the graph G.

3. Concluding Remarks

Labeled graph is the topic of current interest for many researchers as it has diversified applications. We discuss here cordial labeling and 3-equitable labeling of some shell related graphs. This approach is novel and contributes four new results. The derived

labeling pattern is demonstrated by means of elegant illustrations which provide better understanding of the derived results. The results reported here are new and expected to add new dimension to the theory of cordial and 3-equitable graphs.

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