Publications

Available Online

J. Sci. Res. **7** (3), 33-42 (2015)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

Vertex Equitable Labeling of Cycle and Star Related Graphs

P. Jeyanthi[1*], A. Maheswari[2], M. Vijayalakshmi[3]

[1]Research Centre, Department of Mathematics Govindammal Aditanar College for Women, Tiruchendur-628 215, Tamilnadu, India

[2]Department of Mathematics, Kamaraj College of Engineering and Technology Virudhunagar-626001, Tamilnadu, India

[3]Department of Mathematics, Dr. G. U. Pope College of Engineering, Sawyerpuram, Thoothukudi District, Tamil Nadu, India.

Received 30 March 2015, accepted in final revised form 29 May 2015

Abstract

Let G be a graph with p vertices and q edges and $A = \left\{0,1,2,\ldots,\left\lceil \frac{q}{2} \right\rceil\right\}$. A vertex labeling $f: V(G) \rightarrow A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv. For $a \in A$, let $v_f(a)$ be the number of vertices v with $f(v) = a$. A graph G is said to be vertex equitable if there exists a vertex labeling f such that for all a and b in A, $\left|v_f(a) - v_f(b)\right| \leq 1$ and the induced edge labels are 1, 2, 3,…, q. In this paper, we prove that jewel graph J_n, jelly fish graph $(JF)_n$, balanced lobster graph $BL(n,2,m)$, $L_n \odot \overline{K_m}$ and $\langle L_n \hat{o} K_{1,m} \rangle$ are vertex equitable graphs.

Keywords: Vertex equitable labeling; Vertex equitable graph; Jewel graph; Jelly fish; Balanced lobster.

doi: http://dx.doi.org/10.3329/jsr.v7i3.22810 J. Sci. Res. **7** (3), 33-42 (2015)

1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminologies of graph theory as in [1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and a detailed survey of graph labeling can be found in [2]. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. The concept of vertex equitable labeling was due to Lourdusamy and Seenivasan [3] and

* *Corresponding author*: jeyajeyanthi@rediffmail.com

further studied in [4-7]. This is the further extension work on vertex equitable labeling and we prove that the jewel graph J_n , jelly fish graph $(JF)_n$ **,** the lobster $L(n,2,m)$**,** $L_n \odot K_m$ and $\langle L_n \hat{o} K_{1,m} \rangle$ are vertex equitable. The following definitions have been used in the subsequent section.

Definition 1.1: The jewel graph J_n is a graph with vertex set $V(J_n)=\{u, v, x, y, u_i : 1 \le i \le n\}$ and edge set $E(J_n)=\{ux, uy, xy, xv, yv, uu_i, vu_i : 1 \le i \le n\}$.

Definition 1.2: The jelly fish graph $(JF)_n$ is a graph with vertex set $V((JF)_n)=\{u, v, u_i, v_j : 1 \le i \le n, 1 \le j \le n-2\}$ and edge set $E((JF)_n)=$ $\{uu_i : 1 \le i \le n\} \cup \{vv_j : 1 \le j \le n-2\} \cup \{u_{n-1}u_n, vu_n, vu_{n-1}\}$.

Definition 1.3: The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is defined as a graph obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2.

Definition 1.4: A tree, which yields a path when its pendant vertices are removed, is called a *caterpillar*. A tree, which yields a caterpillar when its pendant vertices are removed, is called a *lobster*. Let $L(n, 2, k)$ be the lobster constructed as follows. Let $a_1, a_2, \ldots, a_n$ be the vertices of the path P_n and a_{i1} and a_{i2} be the vertices adjacent to a_i, $1 \le i \le n$. Join a_{ij} with the pendant vertices $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^k$, $1 \le i \le n$, $j = 1,2$.

Definition 1.5: A graph $\langle L_n \hat{o} K_{1,m} \rangle$ is the graph obtained from ladder L_n and $2n$ copies of $K_{1,m}$ by identifying a non central vertex of i^{th} copy of $K_{1,m}$ with i^{th} vertex of L_n .

2. Main Results

Theorem 2.1: Let $G_1(p_1, q_1)$, $G_2(p_2, q_2)$,…, $G_m(p_m, q_m)$ be a vertex equitable graphs with vertex equitable labeling f_i where $q_i \, (1 \le i \le m)$ is even and let $e_i = x_i u_i$ and $e_i' = y_i v_i$ be an edge with $f_i(x_i) = f_i(u_i) = \frac{q_i}{2}$ and $f_i(y_i) = 0, f_i(v_i) = 1$ respectively of the graph G_i $(1 \le i \le m)$. If G is a graph obtained by joining the vertex x_i with y_{i+1} and u_i with v_{i+1} $(1 \le i \le m-1)$ by an edge, then G is a vertex equitable graph.

Proof: The graph G has $p_1 + p_2 + \ldots + p_m$ vertices and $\sum_{i=1}^{m} q_i + 2(m-1)$ edges.

Let $A = \left\{0,1,2,\ldots,\dfrac{\sum_{i=1}^{m} q_i + 2(m-1)}{2}\right\}$. Define a vertex labeling $f: V(G) \to A$ as follows.

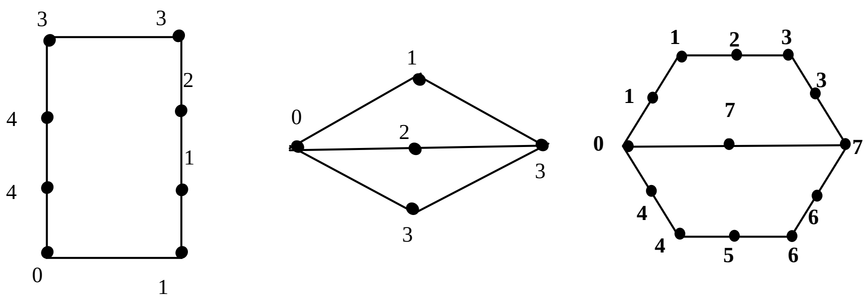
$$f(x) = f_1(x) \;\; if \;\; x \in V(G_1),$$

$$f(x) = f_i(x) + (i-1) + \frac{\sum_{k=1}^{i-1} q_k}{2} \;\; if \;\; x \in V(G_i) \;\; for \;\; 2 \le i \le m$$

The edge labels of the graph G_1 will remain fixed, the edge labels of the graph G_i $(2 \le i \le m)$ are

$q_1+3, q_1+4, \ldots, q_1+q_2+2\,; q_1+q_2+5, q_1+q_2+6, \ldots, q_1+q_2+q_3+4\,; \ldots,$ $\sum_{i=1}^{m-1} q_i + 2m-1, \sum_{i=1}^{m-1} q_i + 2m, \ldots, \sum_{i=1}^{m} q_i + 2(m-1)$. The bridges between the two graphs G_i, G_{i+1} will get the label $\sum_{k=1}^{i} q_i + 2i - 1, \sum_{k=1}^{i} q_i + 2i$ $(1 \le i \le m-1)$. Hence the edge labels of G are distinct and is $\left\{0,1,2,\ldots,\sum_{i=1}^{m} q_i + 2(m-1)\right\}$. Also $\left|v_f(a) - v_f(b)\right| \le 1$ for all $a, b \in A$. Hence G is a vertex equitable graph.

Example 1: The vertex equitable labeling of G_1, G_2 and G_3 are given below:



The vertex equitable labeling of the graph obtained by the above construction is given in Fig. 1.

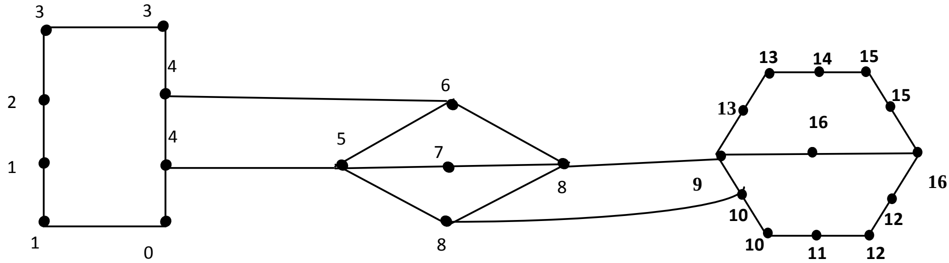


Fig. 1.

Theorem 2.2: Let $G_1(p_1, q_1), G_2(p_2, q_2), \ldots, G_m(p_m, q_m)$ be a vertex equitable graphs with vertex equitable labeling f_i where $q_i\,(1 \le i \le m)$ is odd and let $e_i = x_i u_i$ and $e_i^{'} = y_i v_i$ be an edge with $f_i(x_i) = \left\lfloor \frac{q_i}{2} \right\rfloor$, $f_i(u_i) = \left\lceil \frac{q_i}{2} \right\rceil$ and $f_i(y_i) = 0,\ f_i(v_i) = 1$ respectively of the graph G_i $(1 \le i \le m)$. If G is a graph obtained by identifying x_i with y_{i+1} and u_i with v_{i+1} (that is identifying the edge e_i with $e_{i+1}^{'}$) $(1 \le i \le m-1)$, then G is a vertex equitable graph.

Proof: The graph G has $p_1 + p_2 + \ldots + p_m$ vertices and $\sum_{i=1}^{m} q_i - (m-1)$ edges. Let $A = \left\{0, 1, 2, \ldots, \left\lceil \frac{\sum_{i=1}^{m} q_i - (m-1)}{2} \right\rceil\right\}$. Define a vertex labeling f: $V(G) \to A$ as follows.

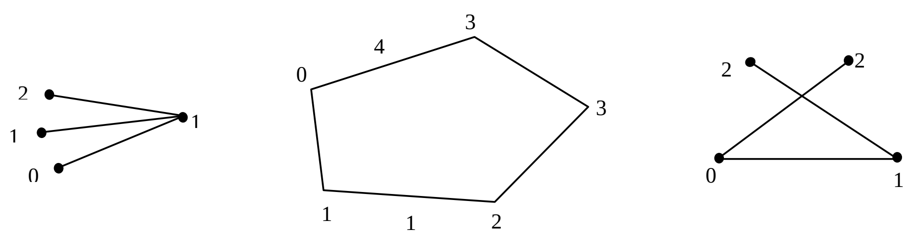
$f(x) = f_1(x)$ *if* $x \in V(G_1)$ and

$$f(x) = f_i(x) + \left\lceil \frac{\sum_{k=1}^{i-1} q_k - (i-2)}{2} \right\rceil - 1 \quad if \quad x \in V(G_i) \quad for \quad 2 \le i \le m .$$

The edge labels of the graph G_1 will remain fixed, the edge labels of the graph G_i (except $e_i^{'}$) $(2 \le i \le m)$ are

$q_1 + 1, q_1 + 2, \ldots, q_1 + q_2 - 1;\ q_1 + q_2, q_1 + q_2 + 1, \ldots, q_1 + q_2 + q_3 - 2;\ \sum_{i=1}^{m-1} q_i - m + 3,$ $\sum_{i=1}^{m-1} q_i - m + 4, \ldots, \sum_{i=1}^{m} q_i - m + 1$. Also $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$. Hence G is a vertex equitable graph.

Example 2: The vertex equitable labeling of G_1, G_2 and G_3 are given below:



The vertex equitable labeling of the graph obtained by the above construction is given in Fig. 2.

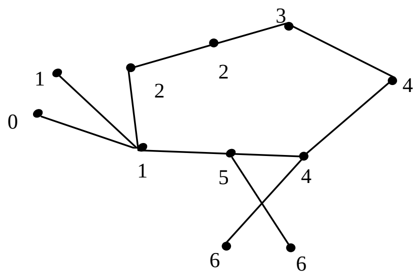


Fig. 2

Theorem 2.3: The jewel graph J_n is a vertex equitable graph.

Proof: Let vertex set $V(J_n)=\{u, v, x, y, u_i{:}1\le i\le n\}$ and edge set $E(J_n)=\{ux, uy, xy, xv, yv, uu_i, vu_i : 1 \le i \le n\}$. Then J_n has n+4 vertices and 2n+5 edges. Let $A = \left\{0,1,2,\ldots,\left\lceil \frac{2n+5}{2} \right\rceil\right\}$. Define a vertex labeling f: $V(G) \rightarrow A$ as follows. $f(u_i) = i+1$ if $1 \le i \le n$, $f(u) = 0$, $f(v) = \left\lceil \frac{2n+5}{2} \right\rceil$, $f(x) = n+2$, $f(y) = 1$. It can be verified that the induced edge labels of J_n are 1, 2,…, 2n+5 and $\left|v_f(i) - v_f(j)\right| \le 1$ for all i, $j \in A$. Clearly f is a vertex equitable labeling of J_n. Thus, Jewel graph J_n is a vertex equitable graph.

Example 3: The vertex equitable labeling of J_3 is given in Fig. 3.

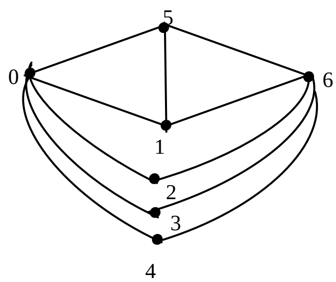


Fig. 3

Theorem 2.4: The jelly fish graph $(JF)_n$ is a vertex equitable graph.

Proof: Let vertex set *V*((*JF)$_n$*)=$\{u, v, u_i, v_j : 1 \le i \le n, 1 \le j \le n-2\}$ and edge set *E*((*JF)$_n$*) = $\{uu_i : 1 \le i \le n\} \cup \{u_{n-1}u_n, vu_n, vu_{n-1}\} \cup \{vv_j : 1 \le j \le n-2\}$. Then (*JF)$_n$* has 2*n* vertices and 2*n*+1 edges. Let *A* = $\left\{0,1,2,\ldots,\left\lceil \frac{2n+1}{2} \right\rceil\right\}$. Define a vertex labeling *f*: *V* (*G*) → *A* as follows. $f(v) = 0,\ f(v_j) = j \text{ if } 1 \le j \le n-2,\ f(u) = \left\lceil \frac{2n+1}{2} \right\rceil$, $f(u_i) = \begin{cases} i-1 & \text{if } 1 \le i \le n-2 \\ i & \text{if } n\text{-}1 \le i \le n \end{cases}$. It can be verified that the induced edge labels of (*JF)$_n$* are 1, 2,…, 2*n*+1 and $|v_f(i) - v_f(j)| \le 1$ for all *i*, $j \in A$. Clearly *f* is a vertex equitable labeling of (*JF)$_n$*. Thus, Jelly fish graph (*JF)$_n$* is a vertex equitable graph.

Example 4: The vertex equitable labeling of (*JF)$_6$* is given in Fig. 4.

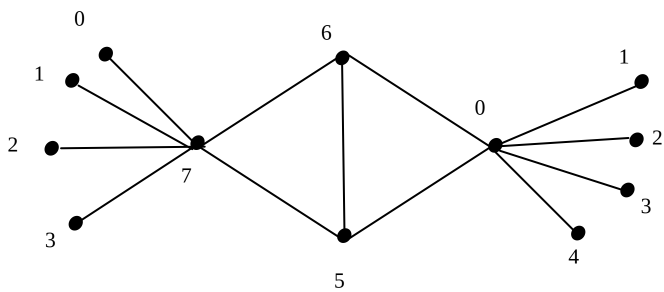


Fig. 4

Theorem 2.5: The balanced lobster *BL*(*n*, 2, *m*) is a vertex equitable graph.

Proof: Let the vertex set *V*(*BL*(*n*, 2, *m*))=$\{u_i, v_i, w_i, v_{ij}, w_{ij} : 1 \le i \le n, 1 \le j \le m\}$ and edge set
E(*BL*(*n*,2,*m*))=$\{u_iu_{i+1} : 1 \le i \le n-1\} \cup \{u_iv_i, u_iw_i : 1 \le i \le n\} \cup \{v_iv_{ij}, w_iw_{ij} : 1 \le i \le n,$
$1 \le j \le m \text{ and } v_iv_{in} = v_iu_i,\ w_iw_{in} = w_iu_i\}$. Then *BL*(*n*, 2, *m*) has 2*nm*+3*n* vertices and 2*mn*+3*n*-1 edges. Let *A* = $\left\{0,1,2,\ldots,\left\lceil \frac{2mn+3n-1}{2} \right\rceil\right\}$. Define a vertex labeling *f*: *V* (*G*) → *A* as follows. $f(v_{2i-1}) = (2m+3)(i-1),\ f(u_{2i-1}) = f(w_{2i-1}) = (2m+3)i - (m+2)$

if $1 \le i \le \left\lceil \frac{n}{2} \right\rceil$,

$f(u_{2i}) = f(v_{2i}) = (2m+3)i - (m+1),\ f(w_{2i}) = (2m+3)i$ if $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$,

$f(v_{2i-1,j}) = f(w_{2i-1,j}) = (2m+3)(i-1)+j$ if $1 \le i \le \left\lceil \frac{n}{2} \right\rceil$, $1 \le j \le m$,

$f(v_{2i,j}) = f(w_{2i,j}) = (2m+3)i-(m+1)+j$ if $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, $1 \le j \le m$. It can be verified that the induced edge labels of $BL(n, 2, m)$ are 1, 2,…, 2*mn*+3*n*-1 and $|v_f(i) - v_f(j)| \le 1$ for all $i, j \in A$. Clearly *f* is a vertex equitable labeling of *L(n, 2, m)*. Thus, balanced lobster $BL(n, 2, m)$ is a vertex equitable graph.

Example 5: The vertex equitable labeling of $L(4,2,4)$ is given in Fig. 5.

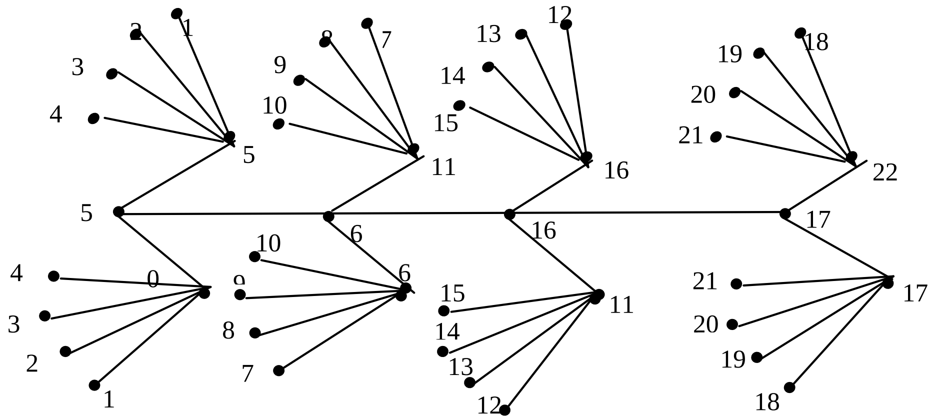


Fig. 5

Theorem 2.6: The graph $L_n \odot \overline{K_m}$ is a vertex equitable graph.

Proof: Let $v_1, v_2, ..., v_n, u_1, u_2, ..., u_n$ be the vertices of the ladder L_n. Let v_{ij}, u_{ij} $(1 \le i \le n, 1 \le j \le m)$ be the vertices of *n* copies of $\overline{K_m}$. Clearly $L_n \odot \overline{K_m}$ has 2*n*+2*mn* vertices and 2*mn*+3*n*-2 edges. Let $A = \left\{0,1,2,..., \left\lceil \frac{2mn+3n-2}{2} \right\rceil \right\}$. Define a vertex labeling f : $V(G) \to A$ as follows.

$f(v_1) = 0$, $f(u_1) = m+1$, $f(u_{1j}) = f(v_{1j}) = j$ *if* $1 \le j \le m$, $f(v_{2i+1}) = (2m+3)i+1$,

$f(u_{2i+1}) = m + (2m+3)i$ *if* $1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$,

$f(v_{2i}) = (2m+3)i-1$, $f(u_{2i}) = (2m+3)i-m-1$ *if* $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$,

$f(v_{2i+1,j}) = (2m+3)i + j - 1,\ f(u_{2i+1,j}) = (2m+3)i + m + 2 - j$

$if\ \ 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor, 1 \le j \le m.$

$f(v_{2i,j}) = (2m+3)i - j,\ f(u_{2i,j}) = (2m+3)i - m - 2 + j \quad if\ \ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,\ 1 \le j \le m.$

It can be verified that the induced edge labels of $L_n \odot \overline{K_m}$ are 1, 2,…, 2*mn*+3*n*-2 and $|v_f(i) - v_f(j)| \le 1$ for all *i*, $j \in A$. Clearly *f* is a vertex equitable labeling of $L_n \odot \overline{K_m}$. Thus, $L_n \odot \overline{K_m}$ is a vertex equitable graph.

Example 6: The vertex equitable labeling of $L_5 \odot \overline{K_4}$ is given in Fig. 6.

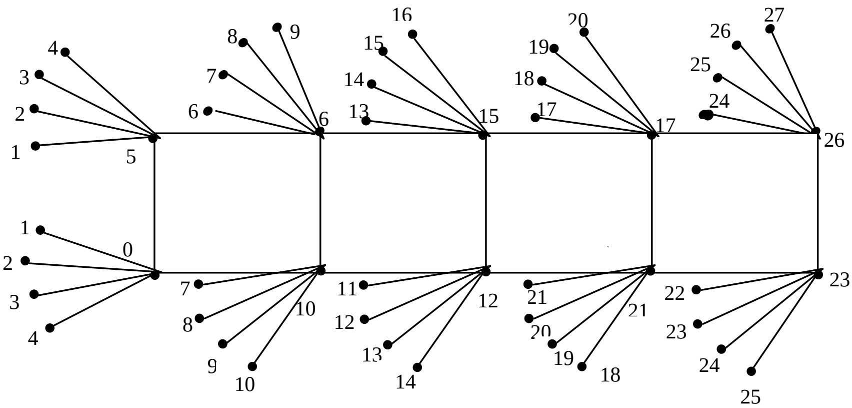


Fig. 6

Theorem 2.7: The graph $\langle L_n \hat{o} K_{1,m} \rangle$ is a vertex equitable graph.

Proof: Let $v_1, v_2, ..., v_n, u_1, u_2, ..., u_n$ be the vertices of the ladder L_n .Let vertex set $V(\langle L_n \hat{o} K_{1,m} \rangle) = \{u_i, v_i, u_{i0}, v_{i0}, v_{ij}, u_{ij} : 1 \le i \le n, 1 \le j \le m\}$ and edge set $E(\langle L_n \hat{o} K_{1,m} \rangle)$ $= \{u_i v_i : 1 \le i \le n\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \le i \le n-1\} \cup \{u_i u_{i0}, v_i v_{i0} : 1 \le i \le n\} \cup$ $\{v_{i0} v_{ij}, u_{i0} u_{ij} : 1 \le i \le n,\ 1 \le j \le m\ \ and\ v_{in} = u_i,\ \ w_{in} = u_i\}$. Clearly $\langle L_n \hat{o} K_{1,m} \rangle$ has 2*n*+2*mn* vertices and 2*mn*+3*n*-2 edges. Let $A = \left\{0,1,2,...,\left\lceil \frac{2mn+3n-2}{2} \right\rceil\right\}$. Define a vertex labeling *f*: *V*(*G*) →*A* as follows. $f(v_1) = 1$, $f(u_1) = m$, $f(v_{10}) = 0$, $f(u_{10}) = m+1$,

$f(u_{1j}) = j,\ f(v_{1j}) = j+1$ *if* $1 \le j \le m-1$, $f(v_{2i+1}) = (2m+3)i,\ f(u_{2i+1}) = (2m+3)i + m+1$ *if* $1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$, $f(v_{(2i)0}) = f(v_{2i}) = (2m+3)i - 1,$

$f(u_{(2i)0}) = f(u_{2i}) = (2m+3)i - m - 1$ *if* $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$,

$f(v_{(2i+1)0}) = (2m+3)i + 1,\ f(u_{(2i+1)0}) = (2m+3)i + m$ *if* $1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$,

$f(v_{2i+1,j}) = (2m+3)i + j,\ f(u_{2i+1,j}) = (2m+3)i + m + 1 - j$

if $1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor, 1 \le j \le m-1$.

$f(v_{2i,j}) = (2m+3)i - j - 1,\ f(u_{2i,j}) = (2m+3)i - m - 1 + j$

if $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, $1 \le j \le m-1$.

It can be verified that the induced edge labels of $\langle L_n \hat{o} K_{1,m} \rangle$ are 1, 2,…, 2*mn*+3*n*-2 and $|v_f(i) - v_f(j)| \le 1$ for all $i, j \in A$. Clearly f is a vertex equitable labeling of $\langle L_n \hat{o} K_{1,m} \rangle$. Thus, $\langle L_n \hat{o} K_{1,m} \rangle$ is a vertex equitable graph.

Example 7: The vertex equitable labeling of $\langle L_4 \hat{o} K_{1,5} \rangle$ is given in Fig. 7.

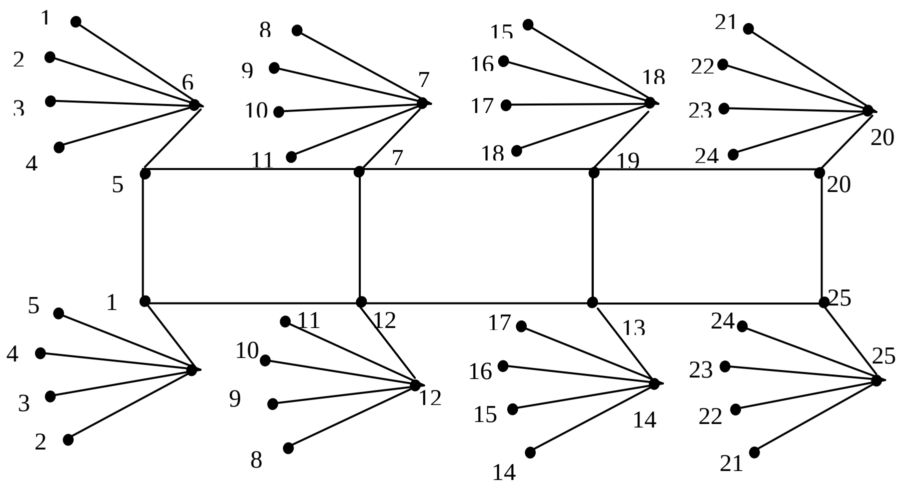


Fig. 7

3. Conclusion

Lourdusamy and Seenivasan introduced the concept of vertex equitable labeling in 2008 and established that some families of graphs admit vertex equitable labeling. In this paper, we prove that jewel graph *Jn*, jelly fish graph *(JF)n*, balanced lobster graph $BL(n,2,m)$, $L_n \odot \overline{K_m}$ and $\langle L_n \hat{o} K_{1,m} \rangle$ are vertex equitable graphs.

References

1. F. Harary, Graph theory (Addison Wesley, Massachusetts, 1972).
2. J. A. Gallian, Electronic J. Combinatorics 17 (2014).
3. A. Lourdusamy and M.Seenivasan, J. Discrete Math. Sci. Cryptography **11**, 727 (2008). http://dx.doi.org/10.1080/09720529.2008.10698401
4. P. Jeyanthi and A.Maheswari, Open J. Discrete Math. **2**, 51 (2012).
5. P. Jeyanthi and A. Maheswari, Utilitas Mathematica 98 (2015).
6. P. Jeyanthi and A. Maheswari, J. Algorithms Comput. **44**, 9 (2013).
7. P. Jeyanthi and A. Maheswari, J. Sci. Res. **6**, 79 (2014). http://dx.doi.org/10.3329/jsr.v6i1.15044