Publications

Block Split Subdivision Domination in Graphs

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Abstract

A dominating set $D \subseteq V[BS(G)]$ is a split dominating set in $[BS(G)]$, if the induced subgraph $\langle V[BS(G)] - D \rangle$ is disconnected in $[BS(G)]$. The split domination number of $[BS(G)]$ is denoted by $\gamma_{sbs}(G)$, is the minimum cardinality of a split dominating set in $[BS(G)]$. In this paper, some results on $\gamma_{sbs}(G)$ were obtained interms of vertices, blocks and other different parameters of G but not the members of $[BS(G)]$. Further we developed its relationship with other different domination parameters of G .

Keywords: Domination; Subdivision graph; Block subdivision graph; Split domination number.

1. Introduction

All graphs, subdivision graphs and block subdivision graphs considered here are simple, finite, nontrivial, undirected and connected, as usual p, q, n denotes the number of vertices, edges and blocks of a graph *G* respectively. In this paper, the study of domination in graphs started by C. Berge [1] and O. Ore [2]. The graph theoretical terminology undefined or notations can be found in G. Chartrand and P. Zhang [3] and F. Harary [4]. The domination in graphs is discussed by T. W. Haynes [5].

     As usual, the minimum degree and maximum degree of a graph *G* are denoted by $\delta(G)$ and $\Delta(G)$ respectively. A vertex v is called a cut vertex if removing it from *G*

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increases the number of components of G. For any real number x, $\lfloor x \rfloor$ denotes the greatest integer not greater than x, $\lceil x \rceil$ denotes the smallest integer not less than x.

The complement $\overline{G}$ of a graph G has V as its vertex set, but two vertices are adjacent in $\overline{G}$, if they are not adjacent in G. The vertex independence number $\beta_0(G)$ is the maximum cardinality of an independence set of vertices. An edge cover of G is the set of edges that covers all the vertices. The edge covering number $\alpha_1(G)$ is minimum cardinality of an edge cover. The edge independence number $\beta_1(G)$ is the minimum cardinality of an independent set of edges. A set of vertices $D \subseteq V(G)$ is a dominating set, if every vertex in $V - D$ is adjacent to some vertex in D. The dominance number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G [6]. A dominating set D of G is a cototal dominating set, if the induced sub graph $\langle V - D \rangle$ has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ of G is the minimum cardinality of a cototal dominating set [6]. A dominating set D of a graph G is a split dominating set, if the induced subgraph $\langle V - D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ of a graph G is the minimum cardinality of a split dominating set. This concept was introduced by Kulli [6,7]. A dominating set D is a connected dominating set, if the induced sub graph $\langle D \rangle$ is connected. The connected domination number $\gamma_c(G)$ of a connected graph G is the minimum cardinality of a connected dominating set [8]. The dominating set D is a total dominating set, if the induced subgraph $\langle D \rangle$ has no isolated vertices. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set [9].

The following figures illustrate the formation of $[BS(G)]$ of a graph G.
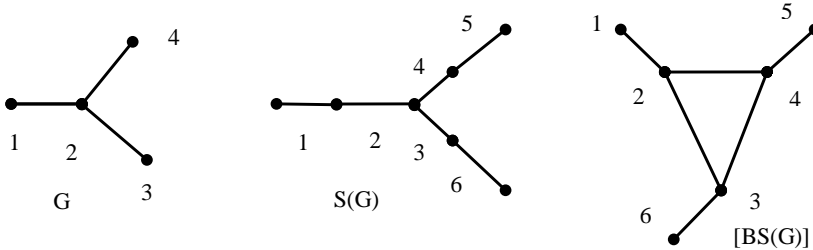
Blocks of G = n = 3

The domination of block split subdivision domination in graph is denoted by $\gamma_{sbs}(G)$. In this paper some results on $\gamma_{sbs}(G)$ were obtained interms of vertices, blocks and other different parameters of G.

One application of block split subdivision domination is that of prisoners and guards. For security and their over all development of prisoners, each prisoner must be seen by some guards, the concept is that of domination. However in order to develop good attitudes of prisoners, we may require that of subdivision graph, also require each prisoner is taken care by the higher official for overall development, the concept is that of block split subdivision domination.

We need the following theorems for our further results.

2. Main Results

Here, we will consider several domination parameters and their relation to the block split subdivision domination number. The following results appeared in [6].

Theorem 1: A split dominating set D of G is minimal of each vertex.
$v \in D$, One of the following conditions holds.
i) There exists a vertex $u \in V - D$, such that $N(u) \cap D = \{v\}$.
ii) v is an isolated vertex in $\langle D \rangle$.
iii) $\langle (V - D) \cup \{v\} \rangle$ is connected.

Theorem 2: For any graph, $\gamma_s(G) \leq \frac{p \cdot \Delta(G)}{1 + \Delta(G)}$.

3. Upper bound for $\gamma_{sbs}(G)$

In this section we compare the parameter $\gamma_{sbs}(G)$ with the blocks, domination and end blocks of G.

Theorem 3: For any connected graph G with $n - blocks$ and $\quad [BS(G)] \neq K_p$, then $\gamma_{sbs}(G) \leq n$.

Proof: Suppose $[BS(G)]$ is a complete graph. Then by definition, split domination does not exists. Hence $[BS(G)] \neq K_p$.

Let $S = \{ B_1, B_2, B_3, \ldots \ldots B_n \}$ be the number of blocks in G and $S_1 = \{ B_1{}^1, B_2{}^1, B_3{}^1, \ldots \ldots B_m{}^1 \}$ be the blocks in $S(G)$. We consider the following cases.

Case 1: Suppose $\forall B_i \in S$ there exists atleast one $B_i, 1 \leq i \leq n$ as an edge. Let $V[BS(G)] = \{ b_1{}^1, b_2{}^1, b_3{}^1, \ldots \ldots b_m{}^1 \}$ be a set of vertices corresponding to the blocks of $S(G)$. In S_1 , $\forall B_i \in S$ generates $B_i{}^1$ and $B_j{}^1$ as blocks in $S(G)$ such that $\forall B_i{}^1, B_j{}^1 \in S$. Clearly $b_i{}^1, b_j{}^1 \in V[BS(G)]$. Consider $V_1 \subset V[BS(G)]$ and $V_2 = V[BS(G)] - V_1$. If every vertex of V_2 is adjacent to at least one vertex in V_1 and if there exist more than one component in V_2, then V_1 is a split dominating set. Further if $V_2 \cup \{b_k\}$ where $b_k \in V_1$ gives one component in $[BS(G)] - V_1$. Then V_1 is a minimal split dominating set of $[BS(G)]$. Hence $|V_1| = \gamma_{sbs}(G) - set$ which gives $\gamma_{sbs}(G) \leq n$.

Case 2: Suppose $\forall B_i \in S$ such that each B_i is an edge. Then $S_1 = S + \{q\}$ where $\{q\}$ is the number of edges in $S(G)$. As in case 1, which gives $\gamma_{sbs}(G) \leq n$.

Next we discuss in case 3, about the blocks of G such that each block contains at least three vertices.

Case 3: Suppose $\forall B_i \in S$ such that each B_i is not an edge. Let $V[BS(G)] = \left\{ b_1^{\ 1}, b_2^{\ 1}, b_3^{\ 1}, \ldots \ldots \ldots b_m^{\ 1} \right\}$ be the set of vertices corresponding to the blocks of $S(G)$. Consider $V_1 \subset V[BS(G)]$ and $V_2 = V[BS(G)] - V_1$. If every vertex of V_2 adjacent to at least one vertex in V_1 and V_2 has more than one component, then $|V_1|$ gives the split domination number by the definition of block graph which is always less than the number of blocks in G. Hence $|V_1| \leq n$ which gives $\gamma_{sbs}(G) \leq n$.

Theorem 4: For any graph (p,q) with *N-end* blocks and $[BS(G)] \neq K_p$ then $\gamma_{sbs}(G) \leq \gamma(G) + N$.

Proof: In view of the definition, we consider only those graphs which are not complete in $[BS(G)]$. Let $V = \left\{ v_1, v_2, v_3, \ldots \ldots \ldots v_n \right\}$ be the vertices in G and $V_1 \subset V$, every vertex of $V - V_1$ is adjacent to atleast one vertex of V_1, then $|V_1| = \gamma(G)$. To establish the upper bound for $\gamma_{sbs}(G)$ in terms of $\gamma(G)$ and the number of end blocks in G. We consider the following cases.

Case 1: Suppose each block of G is an edge. Then in $S(G)$ the number the blocks is $2q$. Let $S = \left\{ B_1, B_2, B_3, \ldots \ldots \ldots B_n \right\}$ be the block in G. In $S(G)$, $\left\{ B_1^{\ 1}, B_2^{\ 1}, B_3^{\ 1}, \ldots \ldots \ldots B_m^{\ 1} \right\} \cup \left\{ B_1, B_2, B_3, \ldots \ldots \ldots B_n \right\}$ are the blocks. Now $V[BS(G)] = \left\{ b_1, b_2, b_3, \ldots \ldots \ldots b_n \right\} \cup \left\{ b_1^{\ 1}, b_2^{\ 1}, b_3^{\ 1}, \ldots \ldots \ldots b_m^{\ 1} \right\}$ corresponds to the blocks of $S(G)$. Let $H_1 = \left\{ b_1, b_2, b_3, \ldots \ldots \ldots b_i \right\}$, $1 \leq i \leq n$, $H_2 = \left\{ b_1^{\ 1}, b_2^{\ 1}, b_3^{\ 1}, \ldots \ldots \ldots b_j^{\ 1} \right\}$ $1 \leq j \leq m$ and $\{H_1 \cup H_2\} \subseteq V[BS(G)]$ are the cut vertices. Suppose $H_1^1 \subseteq H_1$ and $H_2^1 \subseteq H_2$ and $D = V[BS(G)] - \{H_1^1 \cup H_2^1\}$. Then $\forall b_i \in V[BS(G)] - \{H_1^1 \cup H_2^1\}$ are adjacent at least one vertex of D gives $\langle V[BS(G)] - \{H_1^1 \cup H_2^1\} \rangle$ a disconnected graph. Hence $|D|$ is a $\gamma_{sbs}(G) - set$. In $S(G)$ the number of blocks is more than that of G. On comparing the cardinality of $V_1(G)$ and the number of end blocks N, we have $|D| \leq |V_1| + |N|$ which gives $\gamma_{sbs}(G) \leq \gamma(G) + N$.

Case 2: Suppose at least one block of G is not an edge. Then there exists a block with atleast three vertices, for this block $|V_1|$ is more than as in case 1. Again $|\langle V[BS(G)] - \{H_1^1 \cup H_2^1\} \rangle|$ remains constant.
Hence $|D| \leq |V_1| + |N|$ gives $\gamma_{sbs}(G) \leq \gamma(G) + N$.
    Further we compared our concept to the domination and cut vertices of subdivision of G.

Theorem 5: For any connected (p,q) *graph G and* $[BS(G)] \neq K_p$, then $\gamma_{sbs}(G) \leq \gamma[S(G)]$.

Proof: For any connected graph G, by the definition of split domination $[BS(G)] \neq K_p$. We have the following cases.

Case 1: Suppose each block of G is an edge. Then $V[S(G)] = V(G) + \{q\}$. Let $V(G) = \{v_1, v_2, v_3, \ldots \ldots v_n\}$ and $[S(G)] - V(G) = \{v_1{}^1, v_2{}^1, v_3{}^1, \ldots \ldots v_m{}^1\}$. We consider $V_1(G) = \{v_1, v_2, v_3, \ldots \ldots v_i\}$, $1 \le i \le n$ and $V_2(G) = \{v_1{}^1, v_2{}^1, v_3{}^1, \ldots \ldots v_j{}^1\}$, $1 \le j \le m$. If there exists a set $D = V[S(G)] - \{V_1(G) \cup V_2(G)\}$ such that $\forall v_k \in D$ is adjacent to at least one vertex of $\{V_1(G) \cup V_2(G)\}$, then $\{V_1(G) \cup V_2(G)\}$ is $\gamma[S(G)] - set$. Since each edge is a block in G, then $V[BS(G)] = E(G) + \{q\}$ and $V[BS(G)] \le V[S(G)]$. Let $H = \{v_1, v_2, v_3, \ldots \ldots v_k\}$ be the block vertices corresponding to the blocks of $S(G)$ such that $H \subset V[BS(G)]$. Suppose $J = V[BS(G)] - H$ and $\forall v_j \in H$ are adjacent to at least one vertex of J and generates more than one component in J. Then H is a $\gamma_{sbs}(G) - set$. Hence $|H| \le |V_1(G) \cup V_2(G)|$ gives $\gamma_{sbs}(G) \le \gamma[S(G)]$.

Case 2: Suppose G has at least one block which is not an edge. Then there exists a block B with at least three edges. Clearly $V[S(G)] > V[BS(G)]$. In this view and as the argument in case 1, we have $\gamma_{sbs}(G) \le \gamma[S(G)]$.

Theorem 6: Let G be a connected graph with C number of cut vertices of $S(G)$ and $[BS(G)] \ne K_p$, then $\gamma_{sbs}(G) \le \left\lceil \frac{C[S(G)]}{2} \right\rceil$.

Proof: Suppose $[BS(G)] \ne K_p$. Then we considered the following cases.

Case 1: Suppose G is acyclic graph. Let $S = \{B_1, B_2, B_3, \ldots \ldots B_n\}$ be the blocks of G. In $S(G)$, $S_1 = \{B_1, B_2, B_3, \ldots \ldots B_n, B_1{}^1, B_2{}^1, B_3{}^1, \ldots \ldots B_m^1\}$ are the blocks. Let $\{B_1, B_2, B_3, \ldots \ldots B_i, B_1{}^1, B_2{}^1, B_3{}^1, \ldots \ldots B_j^1\}$, $1 \le i, j \le n, m$ are the non end blocks of $S(G)$. Then in $B[S(G)]$, the set $H = \{b_1, b_2, b_3, \ldots \ldots b_i, b_1{}^1, b_2{}^1, b_3{}^1, \ldots \ldots b_j{}^1\}$ is the cut vertex set. Now there exists a set $H_1 \subset H$ such that $H_2 = V[BS(G)] - H_1$ and $\forall b_i \in H_1$ are adjacent to all the vertices of $V[BS(G)] - H_1$. Since $\forall b_i \in H$ is a cut vertex, then $\langle V[BS(G)] - H_1 \rangle$ is disconnected. Hence H_2 is a $\gamma_{sbs}(G) - set$. Clearly $|H_2| \le \left\lceil \frac{|H|}{2} \right\rceil$ gives $\gamma_{sbs}(G) \le \left\lceil \frac{C[S(G)]}{2} \right\rceil$.

Case 2: Suppose G is cyclic graph. Then it contains at least one block which is cyclic. In $S(G)$ the blocks remains same as in G and $V[S(G)] \ge V(G)$. Let $M \subset V[BS(G)]$ is a cut vertex set, there exist a set $M_1 \subset M$ such that $M_2 = V[BS(G)] - M_1 \forall b_i \in M_1$ is adjacent to all the vertices of $V[BS(G)] - M_1$. Since $\forall b_i \in M$ is a cut vertex, then $\langle V[BS(G)] - M_1 \rangle$ has more than one component. Hence M_2 is a $\gamma_{sbs}(G) - set$. Clearly $|M_2| \le \left\lceil \frac{|M|}{2} \right\rceil$ gives $\gamma_{sbs}(G) \le \left\lceil \frac{C[S(G)]}{2} \right\rceil$.

Now we proceed to establish the upper bound interms of $\Delta(G)$, minimum, vertex and edge covering of G and maximum independence number of G for block split subdivision domination of G.

Theorem 7: For any graph G, $[BS(G)] \ne K_p$ then $\gamma_{sbs}(G) \le \left\lceil \frac{p \Delta(G)}{1 + \Delta(G)} \right\rceil$.

Proof: To establish the required results, we have the graphs $[BS(G)] \neq K_p$. Let $V = \{v_1, v_2, v_3, \ldots \ldots v_n\}$ be the vertices of G and $S_1 = \left\{ B_1^{\ 1}, B_2^{\ 1}, B_3^{\ 1}, \ldots \ldots B_m^{\ 1} \right\}$ be the blocks in $S(G)$. Let $V[BS(G)] = \left\{ b_1^{\ 1}, b_2^{\ 1}, b_3^{\ 1}, \ldots \ldots b_m^{\ 1} \right\}$ be the vertices in $[BS(G)]$ which corresponds to the blocks of S_1. Let D be a dominating of $[BS(G)]$, suppose $\forall v_i \in D$, there exists a vertex $u \in V[BS(G)] - D$ such that $N(u) \cap D = \{v_i\}$ and $V[BS(G)] - D$ has more than one component.

Now for a vertex $v \in V(G)$ with $\deg(v) = \Delta(G)$, $\{D\}\{1 + \Delta(G)\} \leq V(G).\Delta(G)$. Further $\{D\} \leq \frac{V(G).\Delta(G)}{[1+\Delta(G)]}$ and $|D| \leq \left| \frac{V(G).\Delta(G)}{1+\Delta(G)} \right|$ gives $\gamma_{sbs}(G) \leq \left| \frac{p.\Delta(G)}{1+\Delta(G)} \right|$.

Theorem 8: For any graph G, $[BS(G)] \neq K_p$ then $\gamma_{sbs}(G) \leq \alpha_1(G) + \Delta(G) - 1$ where $\alpha_1(G)$ is the minimum edge covering number of G.

Proof: By the definition of split domination, $[BS(G)] \neq K_p$. Let $S = \{B_1, B_2, B_3, \ldots \ldots B_n\}$ the blocks of G and $S_1 = \left\{ B_1^{\ 1}, B_2^{\ 1}, B_3^{\ 1}, \ldots \ldots B_m^{\ 1} \right\}$ be the blocks in $S(G)$. Let $V = \left\{ b_1^{\ 1}, b_2^{\ 1}, b_3^{\ 1}, \ldots \ldots b_m^{\ 1} \right\}$ be the set of vertices in $[BS(G)]$ which corresponds to the blocks of S_1. Let $B = \{e_1, e_2, e, \ldots \ldots e_n\} \subset E(G)$ and $V = \{v_1, v_2, v_3, \ldots \ldots v_n\}$ be the set of vertices which are incident with the edges of B and if $|V| = P$, then B itself is an edge covering number. Otherwise, consider the minimum number of edges $\{e_m\} \subset E(G) - B$, such that $B_1 = B \cup \{e_m\}$ forms a minimal edge covering set of G. Suppose $V_1[BS(G)] \subset V[BS(G)], \forall v_i \in V_1[BS(G)]$ which are cut vertices in $[BS(G)]$. Again $V_2[BS(G)] \subset V_1[BS(G)] \forall v_j \in V_2[BS(G)]$. Let $V_3[BS(G)]$ be the end vertices in $[BS(G)]$. Then $N(V_2) \cap N(V_3) \neq \emptyset$, $V[BS(G)] - \{V_2[BS(G)] \cup V_3[BS(G)]\} = H$, every vertex in $(V_2 \cup V_3)$ is adjacent to atleast one vertex in H and $(V_2 \cup V_3)$ has more than one component. Then $\langle H \rangle$ is a split dominating set, $|H| = \gamma_{sbs}(G)$. Clearly $|H| \leq |B_1| + \Delta(G) - 1$ which gives $\gamma_{sbs}(G) \leq \alpha_1(G) + \Delta(G) - 1$.

Theorem 9: Let G be a graph with $[BS(G)] \neq K_p$, then $\gamma_{sbs}(G) + \gamma_c(G) \leq \alpha_0(G) + \beta_0(G) + \gamma(G) + 1$. Equality holds if $G \cong P_9$.

Proof: To establish the required result we have the graph $[BS(G)] \neq K_p$. Let $V_1 = \{v_1, v_2, v_3, \ldots \ldots v_n\} \subseteq V(G)$ be the set of vertices with $\deg(v_i) \geq 2, \forall v_i \in V_1, 1 \leq i \leq n$, which are at distance at most two, covers all the edges in G. Clearly $|V_1| = \alpha_0(G)$. Further, if for any vertex $x \in V_1, N(x) \in V(G) - V_1$, then V_1 itself is an independent vertex set, otherwise $V_1^1 \cup V_2$, where $V_1^1 \subseteq V_1$ and $V_2 \subseteq V(G) - V_1$ forms a maximum independent set of G with $|V_1^1 \cup V_2| = \beta_0(G)$. Let $S = H \cup H_1$, where $H \subseteq V_1$ and $H_1 \subseteq V(G) - V_1$ be the minimal set of vertices which covers all the vertices in G. Clearly S forms a minimal $\gamma - set$ of G. Suppose the subgraph $\langle S \rangle$ has more than one component, then attach the minimum number of vertices $\{W_i\} \in V(G) - S$, where $\deg(W_i) \geq 2$ which are between the vertices of S such that $S_1 = S \cup \{W_i\}$ forms exactly one component in the sub graph $\langle S_1 \rangle$, clearly S_1 forms a minimal $\gamma_c - set$ of G. Let $M = \{v_1, v_2, v_3, \ldots \ldots v_n\}$ be the set of vertices in $[BS(G)]$. If $J = \{v_1, v_2, v_3, \ldots \ldots v_m\}$ where $1 \leq m \leq n$, such that $J \subset M$ and

suppose that there exists a set $J_1 \subset J$, then $\{v_i\} \in J_1$ which gives $M - J_1$ is a disconnected graph. Suppose $J_1 \cup J$ has the minimum number of vertices, such that $N(J_1 \cup J) = V[BS(G)] - (J_1 \cup J)$ gives a minimal split dominating in $[BS(G)]$. Hence $|J_1 \cup J| = \gamma_{sbs}(G)$. Clearly $|J_1 \cup J| + |S_1| \leq |V_1| + |V_1^1 \cup V_2| + |S| + 1$ which gives $\gamma_{sbs}(G) + \gamma_c(G) \leq \alpha_0(G) + \beta_0(G) + \gamma(G) + 1$.

For equality, If $G = P_9$, then $\gamma(P_9) = 3, \alpha_0(P_9) = 4, \beta_0(P_9) = 5, \gamma_c(P_9) = 7$. But we have $\gamma_{sbs}(P_9) = 6$, which satisfies the equality of the statement of theorem.

Theorem 10: For any connected graph G with $n - blocks$ and $[BS(\overline{G})] \neq K_p$, then $\gamma_{sbs}(\overline{G}) \leq n$.

Proof: using Theorem 1, the above result follows.

4. *Lower bound for* $\gamma_{sbs}(G)$

In this section we will show that the sharp lower bound to the cototal domination, total domination and maximum independence (vertex/edge) number of G.

Theorem 11: For any connected graph G and $[BS(G)] \neq K_p$, then $\gamma_{sbs}(G) + \Delta(G) \geq \gamma_{cot}(G) + 2$.

Proof: We consider only $[BS(G)] \neq K_p$. For $\gamma_{cot}(G)$, we consider the following cases.

Case 1: Suppose G has end vertices, Let $H = \{v_1, v_2, v_3, \dots \dots v_s\}$ be the set of vertices in G and $H_1 = \{v_1, v_2, v_3, \dots \dots v_n\}$ be the set of all end vertices and $H_2 = \{v_1, v_2, v_3, \dots \dots v_k\}$ be the set of all vertices at a distance three for each of $v_n \in H_1$. Further consider $H_2^1 \subseteq H_2$ such that $\forall v_i, v_j \in H_2^1$, $N(v_i) \cap N(v_j) = \emptyset$. Now $\langle V(G) - \{H_1 \cup H_2^1\}\rangle$ has no isolates and every vertex of $\langle V(G) - \{H_1 \cup H_2^1\}\rangle$ is adjacent to at least one vertex of $\{H_1 \cup H_2^1\}$. Hence $\{H_1 \cup H_2^1\}$ is a $\gamma_{cot}(G) - set$. Let $\{v_i\} \in V[BS(G)]$ is a set of cut vertices in $[BS(G)]$, $V_1 \subseteq V[BS(G)]$ then V_1 is a split dominating set. Otherwise $u_i \subset V[BS(G)]$ which are end vertices in $[BS(G)]$, then $V_1 \cup \{u_i\}$ is a $\gamma - set, \forall v_i, v_j \in V_1$ which are at a distance of three, $N(v_i) \cap N(v_j) = \emptyset$. Now $\langle V[BS(G)] - (V_1 \cup \{u_i\})\rangle$ is disconnected. Every vertex is adjacent to at least one vertex of $V_1 \cup \{u_i\}$. Then $|V_1 \cup \{u_i\}| = \gamma_{sbs}(G)$. Clearly $|V_1 \cup \{u_i\}| + \Delta(G) \geq |H_1 \cup H_2^1| + 2$ gives $\gamma_{sbs}(G) + \Delta(G) \geq \gamma_{cot}(G) + 2$.

Case 2: Suppose G has no end vertices and each block is not an edge. Let $H \subset V(G)$ be a set of cut vertices in G and $H_1 \subset H, \forall v_i \in H_1$ which are at a distance two from each $v_i, v_j \in H_1$. Let $H_2 = \{v_s\}$ be the set of non cutvertices in G. Again $H_2^1 \subseteq H_2, \forall v_t \in H_2$ such that $N(v_i) \cap N(v_t) = \emptyset$. Now $\langle V(G) - \{H_1 \cup H_2^1\}\rangle$ has no isolates and every vertex of $\langle V(G) - \{H_1 \cup H_2^1\}\rangle$ is adjacent to at least one vertex of $\{H_1 \cup H_2^1\}$. Hence $\{H_1 \cup H_2^1\}$ is a $\gamma_{cot}(G) - set$. Since each block of G has no edge, then $V[S(G)] > V[BS(G)]$ as in case 1 remains same, $|V_1 \cup \{u_i\}| = \gamma_{sbs}(G)$ which gives $\gamma_{sbs}(G) + \Delta(G) \geq \gamma_{cot}(G) + 2$.

Theorem 12: Suppose G be a graph and $[BS(G)] \neq K_p$, then $\gamma_{sbs}(G) + \Delta(G) \geq \gamma_t(G)$.

Proof: For split domination, we considered the graphs with the property such that $[BS(G)] \neq K_p$, we have the following cases.

Case 1: Suppose each block of G is an edge. Then $P = q + 1$. Let V be the set of vertices in G and $V_1 \subset V$ which are non end vertices in G. Again consider a subset $V_2 \subset V_1$. If $V_1 - V_2 = D$ has no isolated vertices and $v_i \in V(G) - \{V_1 - V_2\}$ has atleast one vertex adjacent to a vertex of D. Then D is a total dominating set which gives $|D| = \gamma_t(G)$. Let $H = \{v_{1_1} v_2, v_{3_1}, \dots \dots v_n\}$ be the cut vertices in $[BS(G)]$ which are non end blocks in $S(G)$. Consider $H_1 \subset H$, $H - H_1 = D_1$, $v_i \in H$. Then $V[BS(G)] - \{H - H_1\}$ is adjacent to at least one vertex in $V[BS(G)]$ and gives $\langle V[BS(G)] - \{H - H_1\}\rangle$ a disconnected graph. Hence $|D_1|$ is $\gamma_{sbs}(G) -$ set. Clearly $|D_1| + \Delta(G) \geq |D|$ which gives $\gamma_{sbs}(G) + \Delta(G) \geq \gamma_t(G)$.

Case 2: Suppose each block of G is not an edge. Then there exists at least one block which contains at least three vertices in G. Let $D_2 \subset V(G)$ such that $\forall v_i \in V(G) - D_2$ is adjacent to at least one vertex of D_2 and $\forall v_i \in D_2$, $\deg(v_i) \geq 1$ which gives a total dominating set. Hence $|D_2| = \gamma_t(G)$. In $S(G)$, since each block contains atleast three vertices of G the cardinality of blocks remains same as in G. As in case 1, again $\langle V[BS(G)] - \{H - H_1\}\rangle$ is disconnected. Hence $|D_1| + \Delta(G) \geq |D_2|$ which gives $\gamma_{sbs}(G) + \Delta(G) \geq \gamma_t(G)$.

Theorem 13: For any graph G, $[BS(G)] \neq K_p$ then $\gamma_{sbs}(G) + \gamma(G) \geq \beta_0(G) - 1$ where $\beta_0(G)$ is a maximum independence number of G.

Proof: By the definition of split domination, $[BS(G)] \neq K_p$. Let $S = \{B_1, B_2, B_3, \dots \dots B_n\}$ be the blocks of G and $S_1 = \{B_1^{\ 1}, B_2^{\ 1}, B_3^{\ 1}, \dots \dots B_n^{\ 1}\}$ be the blocks in $S(G)$. Consider $\{b_1^{\ 1}, b_2^{\ 1}, b_3^{\ 1}, \dots \dots b_n^{\ 1}\}$ be the set of vertices in $[BS(G)]$ which corresponds to the blocks of S_1. We have the following cases.

Case 1: Suppose G is a tree. Let $V_1 = \{v_{1_1} v_2, v_{3_1}, \dots \dots v_n\}$ be the set of vertices in G and $V_2 \subset V_1 \forall v_i \in V_2$ such that, $\exists e_i \in E(G)$ is joining two vertices of V_2. Then $\forall u, v \in \langle V(G) - V_2\rangle$, so $d(u, v) \geq 2$ and hence $|V(G) - V_2| = \beta_0(G)$. Suppose a set $H \subset V_1(G)$ be a minimal dominating set of G such that, if every vertex in $V_1(G) - H$ is adjacent to atleast one vertex in H. Then $|H| = \gamma(G)$. Consider $H \subset V[BS(G)]$ which are cut vertices. Again $H_1 \subset H \forall v_i \in H_1$ then there exists $\langle V[BS(G)] - H_1\rangle = J$ a disconnected graph and every vertex in J is adjacent to atleast one vertex of H_1. Then $|H_1|$ is a split domination number of $[BS(G)]$. Then $|H_1| = \gamma_{sbs}(G)$. Clearly $|H_1| + |H| \geq |V(G) - V_2| - 1$ which gives $\gamma_{sbs}(G) + \gamma(G) \geq \beta_0(G) - 1$.

Case 2: Suppose G is not a tree. Then G has atleast one block which is not an edge. Let $V_1(G)$ be the vertices in G and $V_2^1 \subset V_1 \forall v_t \in V_2^1$ and $V_3^1 \subset V_1 \forall v_i \in V_3^1$. Since $V_2^1 \cap V_3^1 = \emptyset$, then for every vertex in V_3^1 and V_2^1 which are incident to exactly one vertex of V_1. Hence $V_2^1 \cup V_3^1$

is a independent set in G which gives $|V_2^1 \cup V_3^1| = \beta_0(G)$. As in case 1, which gives $|H| = \gamma(G)$ and $|H_1| = \gamma_{sbs}(G)$. Clearly $|H_1| + |H| \geq |V_2^1(G) \cup V_3^1(G)| - 1$ which gives the above result.

Theorem 14: For any connected graph G, $[BS(G)] \neq K_p$, then $\gamma_{sbs}(G) + \Delta(G) \geq \beta_1(G)$ where $\beta_1(G)$ is the maximum edge independence number of G.

Proof: By the definition of split domination, we considered the graphs with the property that $[BS(G)] \neq K_p$. Let $E = \{e_1, e_2, e, \ldots \ldots e_n\} \subseteq E(G)$ be the maximal set of edges with $N(e_i) \cap N(e_j) = e$, for every $e_i, e_j \in E, 1 \leq i, j \leq n$ and $e \in E(G) - E$. Clearly E forms a maximal independent edge set in G. Then $|E| = \beta_1(G)$. Let $S = \{B_1, B_2, B_3, \ldots \ldots B_n\}$ be the blocks in G and $S_1 = \{B_1^1 B_2^1, B_3^1, \ldots \ldots B_m^1\}$ be the blocks in $S(G)$. Let $V[BS(G)] = \{b_1^1, b_2^1, b_3^1, \ldots \ldots b_m^1\}$ be the set of vertices which corresponds to the blocks of S_1. Let D be a dominating set of $[BS(G)]$, $\forall v_i \in D$, then there exist $H = V[BS(G)] - D$ a disconnected graph. Hence D is minimal split dominating set in $[BS(G)]$. Hence $|D| = \gamma_{sbs}(G)$. Now for a vertex $v \in V(G)$ with $\deg(v) = \Delta(G)$ then $|D| + \Delta(G) \geq |E|$ which gives $\gamma_{sbs}(G) + \Delta(G) \geq \beta_1(G)$.

5. Nordhaus-Gaddum type results

Theorem 15: For any graph G with $n - blocks$, $[BS(G)] \neq K_p$ and $[BS(\bar{G})] \neq K_p$, then
i) $\gamma_{sbs}(G) + \gamma_{sbs}(\bar{G}) \leq 2n$.   ii) $\gamma_{sbs}(G) \times \gamma_{sbs}(\bar{G}) \leq n^2$.

6.   Conclusion

In this paper we found an upper bound for block split subdivision domination in graphs in terms of blocks, domination, end blocks, cut vertices, minimum number of vertices, edge covering of G and maximum independence number of G. Further the sharp lower bounds were found interms of cototal domination, total domination and maximum independence (vertex/edge) number of G. Finally Nordhaus-Gaddum type results are also found.

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