JSR Publications

Available Online

J. Sci. Res. **2** (1), 87-90 (2009)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

Short Communication

Pseudocomplementation on the Lattice of Convex Sublattices of a Lattice

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Received 12 May 2009, accepted in final revised form 13 July 2009

Abstract

By a new partial ordering relation " ≤ " the set of convex sublattices CS(L) of a lattice L is again a lattice. In this paper we establish some results on the pseudocomplementation of (CS(L); ≤). We show that a lattice L with 0 is dense if and only if CS(L) is dense. Then we prove that a finite distributive lattice is a Stone lattice if and only if CS(L) is Stone. We also prove that an upper continuous lattice L is a Stone lattice if and only if CS(L) is Stone.

Keywords: Upper continuous lattice; Pseudocomplemented lattice; Dense lattice; Stone lattice.

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DOI: 10.3329/jsr.v2i1.2485 J. Sci. Res. **2** (1), 87-90 (2010)

Set of all convex sublattices of a lattice L is denoted by CS(L). Lavanya and Bhatta [1] have defined the partial order " ≤ " on CS(L) as follows:

For A, B ∈ CS(L), A ≤ B if and only if "for every a ∈ A there exists a b ∈ B, such that that a ≤ b and for every b∈ B there exists an a ∈ A, Such that b ≥ a." It is easy to see that '≤' is clearly a partial order and (CS(L); ≤) forms a lattice, where for A, B ∈ CS(L),

$$\text{Inf}\{A, B\} = A \dot{\wedge} B = \langle\{a\wedge b \mid a\in A,\ b\in B\}\rangle = \{x\in L \mid a\wedge b\le x\le a_1\wedge b_1 \text{ for some } a, a_1\in A \text{ and } b, b_1\in B\}$$

$$\text{Sup}\{A, B\} = A \dot{\vee} B = \langle\{a\vee b \mid a\in A,\ b\in B\}\rangle = \{x\in L \mid a\vee b\le x\le a_1\vee b_1 \text{ for some } a, a_1\in A \text{ and } b, b_1\in B\}$$

where for any non-empty subset H of L, ⟨H⟩ denotes the convex sublattice generated by H. Note that $A \dot{\wedge} B$ and $A \dot{\vee} B$ have also been studied by Nieminen [2]. In this paper we establish some result on the pseudocomplementation of **"**(CS(L); ≤)**"**. We show that a lattice L with 0 is dense if and only if CS(L) is dense. Then we prove that a finite

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distributive lattice is a Stone lattice if and only if the lattice of its convex sublattices is Stone. We also prove that an upper continuous lattice L is a Stone lattice if and only if CS(L) is Stone.

A lattice L is called *upper continuous* if for any $a \in L$, $A \wedge \vee D = \vee (a \wedge x \mid x \in D)$ holds for every directed above subset D of L. In a lattice L with 0 and 1, an element $b \in L$ is called a *pseudocomplement* of $a \in L$ if $a \wedge b = 0$ and for any $x \in L$, $x \wedge a = 0$ implies $x \leq b$. We denote the pseudocomplement of a by a^*. A lattice L with 0 and 1 is called *pseudocomplemented* if its every element has a pseudocomplement. Clearly $0^* = 1$ and $1^* = 0$. It is well known that a lattice L is called *complete* if for any $H \subseteq L$ both $\wedge H$ and $\vee H$ exist in L. An element a of a complete lattice L is called a *compact element* if $a \leq \vee X$ for some $X \subseteq L$ implies that $a \leq \vee X_1$ for some finite $X_1 \subseteq X$. A complete lattice L is called *algebraic* if its every element is the join of compact elements. It is well known that every distributive algebraic lattice is pseudocomlemented. Therefore any finite distributive lattice is pseudocomlemented. One may ask the question for a pseudocomplemented lattice L "Is CS(L) pseudocomplemented?" Lavanya and Bhatta [1] have proved the following result.

Theorem 1. *Let L be an upper continuous lattice then L is pseudocomplemented if and only if* CS(L) *is pseudocomplemented.* ∎

Thus for any distributive algebraic lattice L, CS(L) is pseudocomplemented. In particular, if L is finite distributive, then CS(L) is also a finite distributive lattice and hence is pseudocomplemented. Following result will be needed for the development of the paper.

Proposition 2. *Let* L *and* CS(L) *be pseudocomplemented lattices. If* a^* *is the pseudocomplement of a in* L, *then* $\{a^*\}$ *is the pseudocomplement of* $\{a\}$ *in* CS(L).

Proof : Suppose $A \in CS(L)$ such that $\{a\} \dot{\wedge} A = 0$. Then $\langle \{a \wedge p \mid p \in A\} \rangle = \{0\}$, thus, $a \wedge p = 0$ for all $p \in A$. This implies $p \leq a^*$ for all $p \in A$, and so $A \leq \{a^*\}$ in CS(L) therefore $\{a^*\}$ is the pseudocomplement of $\{a\}$ in CS(L). ∎

To establish our next theorem, we need the following lemma.

Lemma 3. *Let* A, D *and* B, E *are the convex sublattices of the lattices* L *and* K *respectively. Then,*

$(A \times B) \dot{\vee} (D \times E) = (A \dot{\vee} D) \times (B \dot{\vee} E)$ and

$(A \times B) \dot{\wedge} (D \times E) = (A \dot{\wedge} D) \times (B \dot{\wedge} E)$

Proof: Clearly $A \leq A \dot{\vee} D$ and $B \leq B \dot{\vee} E$, so $A \times B \leq (A \dot{\vee} D) \times (B \dot{\vee} E)$. Similarly $D \times E \leq (A \dot{\vee} D) \times (B \dot{\vee} E)$.Hence, $(A \times B) \dot{\vee} (D \times E) \leq (A \dot{\vee} D) \times (B \dot{\vee} E)$.

For reverse inequality, let $(x, y) \in (A \dot{\vee} D) \times (B \dot{\vee} E)$.Then $x \in A \dot{\vee} D$ and $y \in B \dot{\vee} E$, which implies that $a \vee d \leq x \leq a_1 \vee d_1$ and $b \vee e \leq y \leq b_1 \vee e_1$ for some $a, a_1 \in A$, $b, b_1 \in B$, $d, d_1 \in D$ and $e, e_1 \in E$. Then $(a \vee d, b \vee e) \leq (x, y) \leq (a_1 \vee d_1, b_1 \vee e_1)$ and so $(a, b) \vee (d, e) \leq (x, y) \leq (a_1, b_1) \vee (d_1, e_1)$. But $(a_1, b_1) \vee (d_1, e_1) \in (A \times B) \dot{\vee} (D \times E)$.

Again, if $(p, q) \in (A \times B) \dot{\vee} (D \times E)$, Then $(r_1, s_1) \vee (m_1, n_1) \leq (p, q) \leq (r_2, s_2) \vee (m_2, n_2)$ for some $r_1, r_2 \in A$, $s_1, s_2 \in B$, $m_1, m_2 \in D$ and $n_1, n_2 \in E$, But $(r_1, s_1) \vee (m_1, n_1) = (r_1 \vee m_1,$

$s_1 \vee n_1) \in (A \dot{\vee} D)\times(B \dot{\vee} E)$. Hence by definition of '≤' in CS(L), $(A\dot{\vee}D)\times(B\dot{\vee}E) \leq (A\times B) \dot{\vee} (D\times E)$.

Therefore, $(A\times B) \dot{\vee} (D\times E)=(A\dot{\vee}D)\times(B\dot{\vee}E)$.

Similarly, $(A\times B) \dot{\wedge} (D\times E)=(A\dot{\wedge}D)\times(B\dot{\wedge}E)$. ∎

Theorem 4. *For any lattices* L *and* K, CS(L×K) ≅ CS(L) ×CS(K).

Proof: Let the map f: CS(L×K) → CS(L) ×CS(K) be defined by f(C) =⟨A , B⟩, where C∈ CS(L×K), and so by Koh [3] C= A×B for A∈ CS(L) and B∈CS(K) and the representation is unique. Clearly f is well defined .Let C,F ∈ CS(L×K), then C= A×B and F =D×E for A,D ∈ CS(L) and B,E∈ CS(K).

Suppose f(C) =f(F). Then ⟨A , B⟩=⟨D , E⟩ and so A=D and B= E which imply C= A×D= D×E= F Hence f is one-one . f is obviously onto.

Now $f(C\dot{\vee}F) = f((A\times B) \dot{\vee} (D\times E))$

$=f((A\dot{\vee}D)\times(B\dot{\vee}E))$ by Lemma 3

$=\langle A\dot{\vee}D,\ B\dot{\vee}E\rangle$

$=\langle A, B\rangle \dot{\vee} \langle D , E\rangle$

$=f(C) \dot{\vee} f(F)$

Similarly, $f(C \dot{\wedge} F) =f(C) \dot{\wedge} f(F)$

Hence f is isomorphism ∎

In a lattice L an element x is said to *cover* an element y if x ≥ y and for any t∈L with x ≥ t ≥ y, implies either x=t or y=t. In a lattice with 0, an element which covers 0 is called an *atom*.Any lattice with a single atom is called a *dense lattice*.That is, a lattice with 0 is dense if and only if 0 is meet irreducible.

Now we prove the following result, which will be needed to prove our next two theorems.

Theorem 5. *Let* L *be a lattice with* 0. L *is dense if and only if* CS(L) *is dense*

Proof : Let L be dense. Then it has only one atom , say a. Then a≤x for all x∈L, with x ≠ 0. This implies {0,a} ≤ A for all A ∈CS(L), where A ≠ {0}.That is {0,a} is the only atom in CS(L). Hence CS(L) is a dense lattice.

Conversely, Suppose CS(L) is dense. Let A be the only atom of CS(L). As A is the atom so A ≠ {0}. Let t ≠ 0 be any element of L-A, then A ≤ {t} implies that x ≤ t, for all x ∈ A. Since A ≠ {0}, so A must contain non-zero elements. For any x ∈ A, Since the interval [0, x] ≤ {x}, so 0 ∈ A. Now, we prove that A cannot contain more than one non-zero elements. If not, let $x_1 \neq 0$, $x_2 \neq 0$ such that $x_1, x_2 \in A$. Then obviously the intervals $[0,x_1] \leq A$ and $[0,x_2] \leq A$. Since $x_1 \neq x_2$ so at least one of them, say $[0,x_1] \neq A$. This contradicts the atom property of A, Hence A must be of the form {0, a} where a ≠ 0. Since A is convex, so A= {0, a} = [0, a]. This implies that a an atom of L. Moreover, a ≤ t for all t ∈ L-A, implies that a is the only atom of A and hence L is a dense Lattice ∎

A distributive pseudocomplemented lattice L is called a *Stone lattice* if a* ∨ a** = 1 for all a ∈ L. It should be noted that in any distributive pseudocomplemented lattice L, (a ∨ b)* = a* ∧ b* always holds for all a, b ∈ L. Moreover, L is a Stone lattice if and only if (a ∧ b)* = a* ∨ b* for all a, b ∈ L. Equivalently, we can say that a distributive lattice L with pseudocomplementation is a Stone algebra if and only if $S(L)=\{a^* \mid a \in L\}$ is a subalgebra of L. In a dense lattice L with 1, notice that for all $a \in L$ with $a \neq 0$, $a^* = 0$ and so $a^* \vee a^{**} = 0 \vee 1=1$. Therefore every distributive dense lattice is a Stone lattice. Moreover by Grätzer [4], we know that a finite distributive lattice is a Stone lattice if and only if it is the direct product of finite distributive dense lattices. Therefore we have the following result.

Theorem 6. *A finite distributive lattice L is Stone if and only if* CS(L) *is Stone.*

Proof: Suppose L is a Stone lattice. Since CS(L) is a finite distributive lattice so CS(L) is also pseudocomplemented Since L is Stone, so by corollary 2.6.4 of [2], $L= L_1 \times L_2 \times \dots \times L_n$, where L_i are finite distributive dense lattices. Then by Theorem 4, $CS(L) \cong CS(L_1) \times CS(L_2) \times \dots \times CS(L_n)$. Now by Theorem 5, $CS(L_i)$ are dense lattices and so by G. Grätzer [4] again, CS(L) is Stone.

Conversely, let CS(L) be Stone, Let $a \in L$, then by Proposition 2, $\{a^* \vee a^{**}\}= \{a^*\} \dot{\vee} \{a^{**}\} = \{a\}^* \dot{\vee} \{a\}^{**} = \{1\}$. This implies $a^* \vee a^{**} = 1$ and so L is also Stone. ∎

We conclude the paper with a similar type of result when L is upper continuous.

Theorem 7. *If L is an upper continuous lattice then L is a Stone lattice if and only if* CS(L) *is Stone.*

Proof: Suppose L is a Stone lattice. Then by Theorem 1, CS(L) is pseudocomplemented. Let $A \in CS(L)$, then by Lavanya and Bhatta [1] $A^*=\{(\vee A)^*\}$; that is, $A^*=\{t\}$ for some $t \in L$. Then $A^{**} = \{t\}^*=\{t^*\}$ by Proposition 2. Therefore, again by Proposition 2, $A^*=A^{***} = \{t^*\}^* = \{t^{**}\}$. Hence $A^* \dot{\vee} A^{**}=\{t^*\} \dot{\vee} \{t^{**}\}=\{t^* \vee t^{**}\} = \{1\}$ as L is Stone. This implies CS(L) is also Stone.

The proof of its converse is exactly same as the proof of the converse part of Theorem 6 ∎

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