JSR Publications

Available Online

J. Sci. Res. **1** (3), 551-557 (2009)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

Short Communication

On Minimally Nonouterplanarity of the Semi Total (Point) Graph of a Graph

V. R. Kulli[1] and K. M. Niranjan[2]

[1]Department of Mathematics, Gulbarga-585106, Karnataka, India

[2]Department of Computer Science and Engineering, S.J.M.I.T, NH4 Bye Pass Road, Chitradurga-577502, Karnataka, India

Received 17 March 2009, accepted in final revised form 4 July 2009

Abstract

We present here characterizations of graphs whose semi total (point) graphs are outerplanar and k-minimally nonouterplanar (k = 1, 2 or 3).

Keywords: Semi total (Point); Minimally nonouterplanarity; Outerplnanar; Block.

DOI: 10.3329/jsr.v1i3.2578 J. Sci. Res. **1** (3), 551-557 (2009)

1. Introduction

All graphs considered are finite, undirected and without loops or multiple lines. For standard terminology and notations we follow Harary [1]. The semi total (point) graph $T_2(G)$ of a graph G is the graph whose point set is $V(G)\cup X(G)$ where two points are adjacent if and only if (i) they are adjacent points of G, or (ii) one is a point and the other is a line of G incident with it. This concept was introduced by Sampathkumar and Chikkodimath [2, 3]. In 1975 Kulli [4] introduced the concept minimally nonouterplanar graph. The inner point number $i(G)$ of a planar graph G is the minimum possible number of points not belonging to the boundary of the exterior region in any embedding of G in the plane. Obviously G is outerplanar if and only if $i(G)$=0. A graph G is minimally nonouterplanar if $i(G)$=1, and G is k-minimally ($k \geq 2$) nonouterplanar if $i(G)=k$.

Definition 1.1: A block of a graph G is a maximal nonseperable subgraph.

Definition 1.2: A line joining two nonadjacent points of a cycle is called a chord of the cycle.

[2] Corresponding author: niru_km@yahoo.com

Definition 1.3: Let C be a cycle with $p\geq 4$ points. If a path of length two joins two nonadjacent points of C then it is called a theta$^+$ block.

The following will be useful in the proof of our results:

Theorem A [2]: For a graph G, let $T_2(G)$ be the semi total (point) graph of G. Then $T_2(G)$ is planar if and only if G is planar.

2. Prerequisites

We first prove two lemmas which are useful to prove our results.

Lemma 1: If G is a cycle with $n(n\geq 1)$ chords, such that no two chords intersect when all points lie on the exterior region, then $i(T_2(G))=n$.

Proof: To prove the result we use mathematical induction on n.
Suppose $n=1$. Then graph G is a cycle with one chord, when all points lie on the exterior region. Then by Theorem A, $T_2(G)$ is planar. On drawing of G in the plane as shown in Fig. 1a, in any plane embedding of $T_2(G)$ has one inner point (see Fig. 1b). Hence the result is true for n=1.

Suppose n=2. Then graph G is a cycle with two chords, such that no two chords intersect when all points lie on the exterior region. Then by Theorem A, $T_2(G)$ is planar. On drawing of G in the plane as shown in Fig. 2(a), in any plane embedding of $T_2(G)$ has two inner points (see Fig.2b). Hence the result is true for n=2.

Assume the result is true for $n=m$ chords.

Now we prove the result for $n=m+1$ chords. Let e_j be the chord of G. Delete from G the chord e_j, let $G'=G-e_j$, which has m chords. Then by inductive hypothesis, $i(T_2(G'))=m$.

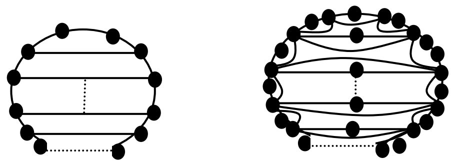
Now again rejoin the chord e_j to G', resulting the graph G, which produces an inner point in $i(T_2(G))$.On drawing of G in the plane as shown in Fig. 3a, in any plane embedding of $T_2(G)$ has m+1 inner points. (see Fig. 3b). Hence the result. □

Lemma 2. If G is a theta $^+$ block, then $i(T_2(G))$=3.

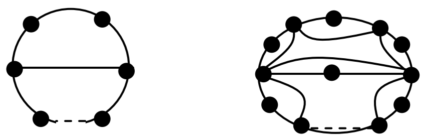
Proof. Suppose G is a theta$^+$ block. Then by the definition of theta$^+$ block, G contains path of length two joining a pair of nonadjacent points. Then by Theorem A, $T_2(G)$ is planar. On drawing of G in the plane as shown in Fig.4(a), in any plane embedding of $T_2(G)$ it has three inner points (see Fig.4b).Thus $i(T_2(G))$=3. □

(a) Fig.1. (b)

(a) Fig. 2. (b)



(a) Fig. 3. (b)



(a) Fig. 4. (b)

3. Main Results

Theorem 2 in the paper [4] is incorrect. Theorem 2 of [4] states that for a graph *G*, let $T_2(G)$ be the semi total (point) graph of *G*. Then $T_2(G)$ is planar (outerplanar) if and only if *G* is planar (outerplanar). As per the above theorem statement graph shown below (see Fig 5), now *G* is outerplanar but $T_2(G)$ is not a outerplanar.Therefore Theorem 2 in the paper [4] is incorrect.

Fig. 5.

Now we improve the above stated theorem for outerplanarity.

Theorem 1: The semi total (point) graph $T_2(G)$ of a graph *G* is outerplanar if and only if every block of *G* is either a line or a cycle.

Proof: Suppose $T_2(G)$ is outerplanar. Then it is planar and by Theorem A, G is planar, which implies that every block of G is planar. Assume G has a cyclic block b_i other than cycle. Then the block b_i has a subgraph homeomorphic from K_4-x. Let $G' = K_4-x$, which has a chord, by Lemma 1, $i(T_2(G'))=1$. Therefore $T_2(G)$ has at least one inner point. Thus $i(T_2(G))\geq 1$. Hence $T_2(G)$ is nonouterplanar, a contradiction. Thus every block of G is either a line or a cycle.

Conversely, suppose every block of G is either a line or a cycle. Then it is easy to see that,$T_2(G)$ is outer planar. This completes the proof of the theorem. □

We now characterize graphs whose semi total (point) graphs are minimally nonouterplanar.

Theorem 2: The semi total (point) graph $T_2(G)$ of a graph G is minimally nonouterplanar if and only if G has exactly one block such that it is a cycle with one chord, when all point lie on the exterior region and every other block of G is either a line or a cycle.

Proof: Suppose $T_2(G)$ is minimally nonouterplanar.Then by Theorem A, G is planar. We now consider the following cases.

Case 1. Suppose every block of G is either a line or a cycle. Then G is an outerplanar graph and by Theorem 1, $T_2(G)$ is outerplanar, a contradiction.

Case 2. Suppose G has exactly one block which is a cycle with at least two chords, such that no two chords intersect when all points lie on the exterior region and every other block of G is either a line or a cycle. Then by Lemma 1, $i(T_2(G))\geq 2$, a contradiction.

Case 3. Suppose G has exactly one block such that it is a theta$^+$ block and every other block of G is either a line or a cycle. Then by Lemma 2, $i(T_2(G))=3$, a contradiction.

Case 4. Suppose G has exactly two blocks each of which is a cycle with at least one chord, when all points lie on the exterior region and every other block of G is either a line or a cycle. Then by Lemma 1, $i(T_2(G))\geq 2$, a contradiction.

We have exhausted all possibilities. In each case we found that $T_2(G)$ is not minimally nonouterplanar. Thus we conclude that G holds the condition.

Conversely, suppose G has exactly one block such that it is a cycle with one chord, when all points lie on the exterior region and every other block of G is either a line or a cycle. Then by Lemma 1, $i(T_2(G))=1$, since every other block of $T_2(G)$ is outerplanar.Hence $T_2(G)$ is minimally nonouterplanar. □

In the following theorem, we establish a characterization of graphs whose semi total (point) graphs are 2-minimally nonouterplanar.

Theorem 3: The semi total (point) graph $T_2(G)$ of a graph G is 2-minimally nonouterplanar if and only if G holds (1) or (2):

(i) G has exactly one block which is a cycle with two chords, such that no two chords intersect when all points lie on the exterior region and every other block of G is either a line or a cycle.

(ii) G has exactly two blocks each of which is a cycle with one chord, when all points lie on the exterior region and every other block of G is either a line or a cycle.

Proof: Suppose $T_2(G)$ is 2-minimally nonouterplanar.Then by Theorem A, *G* is planar. We now consider the following cases.

Case 1. Suppose every block of *G* is either a line or a cycle. Then *G* is an outerplanar graph and by Theorem 1, $T_2(G)$ is outerplanar, a contradiction.

Case 2. Suppose *G* has exactly one block which is a cycle with one chord, when all points lie on the exterior region and every other block of *G* is either a line or a cycle. Then by Theorem 2, $i(T_2(G))=1$, a contradiction.

Case 3. Suppose *G* has exactly one block which is a cycle with at least three chords, such that no two chords intersect when all points lie on the exterior region and every other block of *G* is either a line or a cycle. Then by Lemma 1, $i(T_2(G))\geq 3$, a contradiction.

Case 4. Suppose *G* has exactly one block such that it is a theta$^+$ block and every other block of *G* is either a line or a cycle. Then by Lemma 2, $i(T_2(G))=3$, a contradiction.

Case 5. Suppose *G* has exactly two blocks one of which is a cycle with at least two chords and other is a cycle with at least one chord, such that no two chords intersect when all points lie on the exterior region and every other block of *G* is either a line or a cycle. Then by Lemma 1, $i(T_2(G))\geq 3$, a contradiction.

Case 6. Suppose *G* has exactly three blocks each of which is a cycle with at least one chord, when all points lie on the exterior region and every other block of *G* is either a line or a cycle. Then by Lemma 1, $i(T_2(G))\geq 3$, a contradiction.

We have exhausted all possibilities. In each case we found that $T_2(G)$ is not 2-minimally nonouterplanar. Thus we conclude that *G* holds the conditions.

Conversely, suppose *G* has exactly one block which is a cycle with two chords, such that no two chords intersect when all points lie on the exterior region and every other block of *G* is either a line or a cycle. Then by Lemma 1, $i(T_2(G))=2$, since every other block of $T_2(G)$ is outerplanar.

Again suppose *G* has exactly two blocks each of which is a cycle with one chord, when all points lie on the exterior region and every other block of *G* is either a line or a cycle. Then by Lemma 1, $i(T_2(G))=2$, since every other block of $T_2(G)$ is outerplanar.Hence $T_2(G)$ is 2-minimally nonouterplanar. □

Now we establish a characterization of graphs whose semi total (point) graphs are 3-minimally nonouterplanar.

Theorem 4: The semi total (point) graph $T_2(G)$ of a graph *G* is 3-minimally nonouterplanar if and only if

(i) *G* has exactly one block which is a cycle with three chords, such that no two chords intersect when all points lie on the exterior region and every other block of *G* is either a line or a cycle, or

(ii) G has exactly one theta $^{+}$ block and every other block of G is either a line or a cycle, or

(iii) G has exactly two blocks one of which is a cycle with two chords and other is a cycle with one chord, such that no two chords intersect when all points lie on the exterior region and every other block of G is either a line or a cycle, or

(iv) G has exactly three blocks each of which is a cycle with one chord, when all points lie on the exterior region and every other block of G is either a line or a cycle.

Proof: Suppose $T_2(G)$ is 3-minimally nonouterplanar.Then by Theorem A, G is planar. We now consider the following cases.

Case 1. Suppose every block of G is either a line or a cycle. Then G is an outerplanar graph and by Theorem 1, $T_2(G)$ is outerplanar, a contradiction.

Case 2. Suppose G has exactly one block which is a cycle with one chord, when all points lie on the exterior region and every other block of G is either a line or a cycle. Then by Theorem 2, $i(T_2(G))=1$, a contradiction.

Case 3. Suppose G has exactly one block which is a cycle with two chords, such that no two chords intersect when all points lie on the exterior region and every other block of G is either a line or a cycle. Then by Theorem 3, $i(T_2(G))=2$, a contradiction.

Case 4. Suppose G has exactly one block which is a cycle with at least four chords, such that no two chords intersect when all points lie on the exterior region and every other block of G is either a line or a cycle. Then by Lemma 1, $i(T_2(G))\geq 4$, a contradiction.

Case 5. Suppose G has exactly two blocks each of which is a cycle with one chord, when all points lie on the exterior region and every other block of G is either a line or a cycle. Then by Theorem 3, $i(T_2(G))=2$, a contradiction.

Case 6. Suppose G has exactly two blocks each of which is a cycle with at least two chords, such that no two chords intersect when all points lie on the exterior region and every other block of G is either a line or a cycle. Then by Lemma 1, $i(T_2(G))\geq 4$, a contradiction.

Case 7. Suppose G has exactly two blocks each of which is a theta^{+} block and every other block of G is either a line or a cycle. Then by Lemma 2, $i(T_2(G))\geq 4$, a contradiction.

Case 8. Suppose G has exactly two blocks one of which is a cycle with at least three chords and other is a cycle with at least one chord, such that no two chords intersect when all points lie on the exterior region and every other block of G is either a line or a cycle. Then by Lemma 1, $i(T_2(G))\geq 4$,a contradiction.

Case 9. Suppose G has exactly two blocks one of which is a cycle with at least one chord and other is a theta^{+} block, such that no two chords intersect when all points lie on the exterior region. Then by Lemma 2 and Lemma 1, $i(T_2(G))\geq 4$, a contradiction.

Case 10. Suppose G has exactly three blocks in which two blocks each of which is a cycle one with at least chord and other is a cycle with at least two chords, such that no two chords intersect when all points lie on the exterior region. Then by Lemma 1, $i(T_2(G))\geq 4$, a contradiction.

Case 11. Suppose *G* has exactly four blocks each of which is a cycle with at least one chord, when all points lie on the exterior region. Then by Lemma 1, $i(T_2(G))\geq 4$, a contradiction.

We have exhausted all possibilities. In each case we found that $T_2(G)$ is not 3-minimally nonouterplanar. Thus we conclude that *G* holds the conditions.

Conversely, suppose *G* has exactly one block which is cycle with three chords, such that no two chords intersect when all points lie on the exterior region and every other block of *G* is either a line or a cycle. Then by Lemma 1, $i(T_2(G))=3$, since every other block of $T_2(G)$ is outerplanar.

Suppose *G* has exactly one block which is a theta $^+$ block and every other block of *G* is either a line or a cycle. Then by Lemma 2, $i(T_2(G))=3$, since every other block of $T_2(G)$ is outerplanar.

Suppose *G* has exactly two blocks one of which is a cycle with two chords and other is a cycle with one chord, such that no two chords intersect when all points lie on the exterior region and every other block of *G* is either a line or a cycle. Then by Lemma 1, $i(T_2(G))=3$, since every other block of $T_2(G)$ is outerplanar.

Suppose *G* has exactly three blocks each of which is a cycle with one chord, when all points lie on the exterior region and every other block of *G* is either a line or a cycle. Then by Lemma 1, $i(T_2(G))=3$, since every other block of $T_2(G)$ is outerplanar. Hence $T_2(G)$ is 3-minimally nonouterplanar. □

4. Conclusions

We present here characterizations of graphs whose semi total (point) graphs are outerplanar and *k*-minimally nonouterplanar (*k*=1, 2 or 3).

We further find characterizations of graphs whose semi total (point) graphs are outerplanar in terms of forbidden subgraphs and k-minimally nonouterplanar (*k*=1, 2 or 3) in terms of forbidden subgraphs.

Acknowledgements

The authors are grateful to the reviewers for their critical comments and valuable suggestions.

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