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Comparative Study of the Volume Charge Density in Most General Lorentz Transformation and Quaternion Lorentz Transformation

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Abstract

Lorentz transformation is the relation of space and time coordinates of one inertial frame relative to another inertial frame in special relativity. In this paper we have studied the volume charge density in most general and quaternion Lorentz transformations for different angles with different velocities of the moving frame. We have also used numerical data to see the comparative situation.

Keywords: Special relativity; Lorentz Transformation [LT]; Most General Lorentz Transformation [MGLT]; Quaternion Lorentz Transformation [QLT]; Volume Charge Density [VCD].

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1. Introduction

In most cases of Special Relativity [1] the line of motion is along *x* axis. In that case *y* and *z* coordinates remain invariant under the LTs. But practically in some cases the line of motion does not coincide anyone of the axes. For example when an air plane landing or take off at an airport. The ground is a natural coordinate system and the *x*-axis is parallel to the ground. It is our interest to study the situations of special relativity when the line of motion is along any arbitrary direction. In that cases there are different types of LT. In this paper we want to study the comparative situation of the Volume Charge Density (VCD) in most general and quaternion LTs.

The charge density is the amount of electric charge per unit length, area or volume. Rafiq and Alam [2] have studied the surface charge density in mixed number LT. Bhuiyan *et al*. [3] have studied the surface charge density in different types of LTs and also derived the volume charge density [4] in MGLT.

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1.1. *Most general Lorentz transformation*

Consider two inertial frames of references S and S' where the frame S is at rest and S' is moving along arbitrary direction then the space and time coordinates of S and S' is known as MGLT [1-7] can be written as

$$\vec{X}' = \vec{X} + \vec{V}\left[\left\{\frac{\vec{X}.\vec{V}}{V^2}(\gamma - 1)\right\} - t\gamma\right]$$

$$t' = \gamma\left(t - \vec{V}.\vec{X}\right) \tag{1}$$

where $\gamma = \dfrac{1}{\sqrt{1-\dfrac{V^2}{c^2}}}$ & $c = 1$ and the inverse MGLT can be written as

$$\vec{X} = \vec{X}' + \vec{V}\left[\left\{\frac{\vec{X}'.\vec{V}}{V^2}(\gamma - 1)\right\} - t'\gamma\right]$$

$$t = \gamma\left(t' + \vec{V}.\vec{X}'\right) \tag{2}$$

where $\vec{X}$ & $\vec{X}'$ are the space parts of rest and moving frames respectively.

1.2. *Quaternion Lorentz transformation*

Irish physicist, astronomer and mathematician **Sir William Rowan Hamilton** discovered quaternion in 1843. The quaternion [8] is the sum of a scalar and a vector

$$\underline{A} = x + \vec{A} \tag{3}$$

The product of two quaternions $\underline{A} = x + \vec{A}$ and $\underline{B} = y + \vec{B}$ is given by

$$\left.\begin{aligned} \underline{AB} &= (x+\vec{A})(y+\vec{B}) = xy - \vec{A}.\vec{B} + x\vec{B} + y\vec{A} + \vec{A}\times\vec{B} \\ \underline{BA} &= (y+\vec{B})(x+\vec{A}) = yx - \vec{B}.\vec{A} + y\vec{A} + x\vec{B} + \vec{B}\times\vec{A} \end{aligned}\right\} \tag{4}$$

when $x = y = 0$, we have from equation (4)

$$\left.\begin{aligned} \vec{A}\vec{B} &= -\vec{A}.\vec{B} + \vec{A}\times\vec{B} \\ \vec{B}\vec{A} &= -\vec{B}.\vec{A} + \vec{B}\times\vec{A} \end{aligned}\right\}$$

The above products are known as quaternion product [9-11] of two vectors. Using this quaternion product M. S. Alam *et al*. [12] have derived a type of MGLT is known as QLT can be written as

$$\vec{Z}' = \gamma\left(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V}\right)$$

$$t' = \gamma\left(t + \vec{Z}.\vec{V}\right) \tag{5}$$

and

$$\vec{Z} = \gamma\left(\vec{Z}' + t'\vec{V} + \vec{Z}'\times\vec{V}\right)$$

$$t = \gamma\left(t' - \vec{V}.\vec{X}'\right) \tag{6}$$

where $\vec{Z} = x\hat{i} + y\hat{j} + z\hat{k}$ & $\vec{Z}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$ are the space parts of the QLT. Since $\vec{V}$ has three components V_x, V_y & V_z, the above transformations can also be written as

$$x' = \gamma\left\{x - tV_x - \left(yV_z - zV_y\right)\right\} \tag{7}$$

$$y' = \gamma\left\{y - tV_y - \left(zV_x - xV_z\right)\right\}$$

$$z' = \gamma\left\{z - tV_z - \left(xV_y - yV_x\right)\right\}$$

$$t' = \gamma(t + xV_x + y\,V_y + zV_z)$$

and

$$x = \gamma\left\{x' + t'\,V_x + \left(y'V_z - z'V_y\right)\right\}$$

$$y = \gamma\left\{y' + t'V_y + \left(z'V_x - x'V_z\right)\right\}$$

$$z = \gamma\left\{z' + t'V_z + \left(x'V_y - y'V_x\right)\right\} \tag{8}$$

$$t = \gamma(t' - x'V_x - y'V_y - z'V_z)$$

2. Volume Charge Density

The amount of electric charge density per unit volume is known as VCD and is denoted by

$$\rho = \frac{q}{V} \tag{9}$$

Our research group has derived the formula for VCD in most general LT.

2.1. *Volume charge density in most general Lorentz transformation*

Considering two inertial frames of references S and S' where the frame S is at rest and S' is moving with velocity $\vec{V}$ along arbitrary direction [Fig. 1] then $\vec{V}$ has three components V_x, V_y & V_z. Consider a stationary cube of side L_0 with uniform charge density $+\rho\left(c/m^3\right)$ is at rest in S frame. The cube is placed parallel to *xy*-plane. The observer in S' frame will observe that the frame S is moving with velocity $\vec{V}$ along opposite direction. Then the volume charge density of this cube in MGLT have derived by our research group Bhuiyan *et al.* [4] can be written as

$$\rho' = [1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2]^{\frac{3}{2}} \times \rho \tag{10}$$

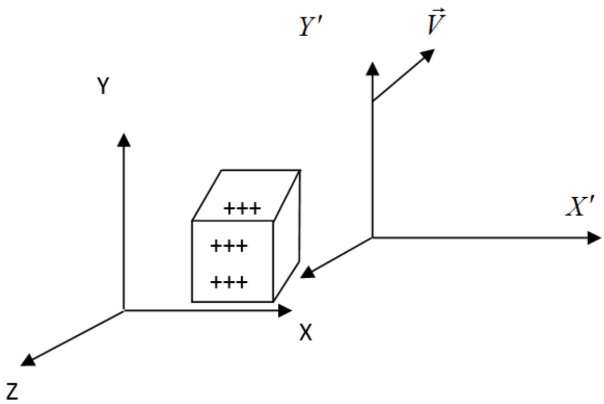

Fig. 1. Volume charge density in MGLT [4].

2.2. *Volume charge density in quaternion Lorentz transformation*

Consider same case as MGLT in figure 1. If L_0 be the length of one side of the cube in S frame then from length contraction [13] of QLT, we can write

$$\vec{L}_O = \gamma\left(\vec{L} - \vec{L}\times\vec{V}\right)$$

$$or, \vec{L}_O{}^2 = \gamma^2\left\{L^2 - \left(\vec{L}\times\vec{V}\right)^2 - \vec{L}\left(\vec{L}\times\vec{V}\right) - \left(\vec{L}\times\vec{V}\right).\vec{L} + \left(\vec{L}\times\vec{V}\right)\times\left(\vec{L}\times\vec{V}\right)\right\}$$

$$= \gamma^2\left\{L^2 - (LV\sin\theta)^2\right\}$$

$$= \gamma^2\left\{L^2 - L^2V^2\sin^2\theta\right\}$$

$$= \gamma^2\left\{L^2 - L^2V^2\left(1-\cos^2\theta\right)\right\}$$

$$\therefore \quad L^2 = \frac{L_0{}^2}{\gamma^2\left\{1 - V^2\left(1-\cos^2\theta\right)\right\}}$$

$$\& \quad L_x{}^2 + L_y{}^2 + L_z{}^2 = \frac{L_{0x}{}^2 + L_{0y}{}^2 + L_{0z}{}^2}{\gamma^2\left\{1 - V^2\left(1-\cos^2\theta\right)\right\}}$$

$$L_x{}^2 = \frac{L_{0x}{}^2}{\gamma^2\left\{1 - V^2\left(1-\cos^2\theta\right)\right\}}$$

$$L_y{}^2 = \frac{L_{0y}{}^2}{\gamma^2\left\{1 - V^2\left(1-\cos^2\theta\right)\right\}} \qquad (11)$$

$$L_z{}^2 = \frac{L_{0z}{}^2}{\gamma^2\left\{1 - V^2\left(1-\cos^2\theta\right)\right\}}$$

The total charge observed by an observer in the moving frame

$$Q' = L_x L_y L_z \rho$$

$$= \sqrt{\frac{L_{0x}^{\ 2}}{\gamma^2\{1-V^2(1-\cos^2\theta)\}}} \times \sqrt{\frac{L_{0y}^{\ 2}}{\gamma^2\{1-V^2(1-\cos^2\theta)\}}} \times \sqrt{\frac{L_{0z}^{\ 2}}{\gamma^2\{1-V^2(1-\cos^2\theta)\}}}$$

$$\therefore Q' = \frac{L_0^{\ 3}}{\left[\gamma^2\{1-V^2(1-\cos^2\theta)\}\right]^{\frac{3}{2}}}$$

According to the principle of conservation of charge $Q' = Q_o$

$$or,\ \frac{L_0^{\ 3}}{\left[\gamma^2\{1-V^2(1-\cos^2\theta)\}\right]^{\frac{3}{2}}}\rho' = L_0^{\ 3}\rho$$

$$\therefore \rho' = \rho\left[\gamma^2\{1-V^2(1-\cos^2\theta)\}\right]^{\frac{3}{2}}$$

which is the VCD of this cube in QLT.

3. Comparative Study

We have compared the formulae of the VCD in MGLT and QLT by the following tables.

Table 1. Transformation equations of VCD in MGLT and QLT.

	MGLT	QLT
Space	$\vec{X}' = \vec{X} + \vec{V}\left[\left\{\frac{\vec{X}.\vec{V}}{V^2}(\gamma-1)\right\} - t\gamma\right]$	$x' = \gamma\{x - tV_x - (yV_z - zV_y)\}$ $y' = \gamma\{y - tV_y - (zV_x - xV_z)\}$ $z' = \gamma\{z - tV_z - (xV_y - yV_x)\}$
Time	$t' = \gamma(t - \vec{V}.\vec{X})$	$t' = \gamma(t + xV_x + y\,V_y + zV_z)$
VCD	$\rho' = [1 + 2\cos^2\theta(\gamma-1) + \cos^2\theta(\gamma-1)^2]^{\frac{3}{2}} \times \rho$	$\rho' = \rho\left[\gamma^2\{1-V^2(1-\cos^2\theta)\}\right]^{\frac{3}{2}}$

Table 2. Numerical values of VCDs in MGLT and QLT (considering the resultant velocity $V = \sqrt{V_x^{\ 2} + V_y^{\ 2} + V_z^{\ 2}}$ of the three components of the velocity of the moving frame in unit of c).

Parameters	MGLT	QLT
$\theta = 20^\circ, V_x = .1c, V_y = .1c, V_z = .1c, V = 0.1732c$	1.04125ρ	1.04125ρ
$\theta = 20^\circ, V_x = .2c, V_y = .2c, V_z = .2c, V = 0.3464c$	1.18591ρ	1.1859ρ
$\theta = 20^\circ, V_x = .3c, V_y = .3c, V_z = .3c, V = 0.5196c$	1.5279ρ	1.5279ρ
$\theta = 20^\circ, V_x = .4c, V_y = .4c, V_z = .4c, V = 0.6928c$	2.44515ρ	2.4499ρ
$\theta = 20^\circ, V_x = .5c, V_y = .5c, V_z = .5c, V = 0.866c$	6.96862ρ	6.96861ρ

$\theta = 30^\circ, V_x = .1c, V_y = .1c, V_z = .1c, V = 0.1732c$	1.035ρ	1.035ρ
$\theta = 30^\circ, V_x = .2c, V_y = .2c, V_z = .2c, V = 0.3464c$	1.15723ρ	1.57226ρ
$\theta = 30^\circ, V_x = .3c, V_y = .3c, V_z = .3c, V = 0.5196c$	1.4437ρ	1.4437ρ
$\theta = 30^\circ, V_x = .4c, V_y = .4c, V_z = .4c, V = 0.6928c$	2.20129ρ	2.205576ρ
$\theta = 30^\circ, V_x = .5c, V_y = .5c, V_z = .5c, V = 0.866c$	5.8574ρ	5.85738ρ
$\theta = 40^\circ, V_x = .1c, V_y = .1c, V_z = .1c, V = 0.1732c$	1.02735ρ	1.02736ρ
$\theta = 40^\circ, V_x = .2c, V_y = .2c, V_z = .2c, V = 0.3464c$	1.12238ρ	1.12236ρ
$\theta = 40^\circ, V_x = .3c, V_y = .3c, V_z = .3c, V = 0.5196c$	1.34261ρ	1.34262ρ
$\theta = 40^\circ, V_x = .4c, V_y = .4c, V_z = .4c, V = 0.6928c$	1.9141ρ	1.917787ρ
$\theta = 40^\circ, V_x = .5c, V_y = .5c, V_z = .5c, V = 0.866c$	4.58526ρ	4.58524ρ
$\theta = 50^\circ, V_x = .1c, V_y = .1c, V_z = .1c, V = 0.1732c$	1.01923ρ	1.01924ρ
$\theta = 50^\circ, V_x = .2c, V_y = .2c, V_z = .2c, V = 0.3464c$	1.08567ρ	1.085658ρ
$\theta = 50^\circ, V_x = .3c, V_y = .3c, V_z = .3c, V = 0.5196c$	1.23776ρ	1.23776ρ
$\theta = 50^\circ, V_x = .4c, V_y = .4c, V_z = .4c, V = 0.6928c$	1.62349ρ	1.62663ρ
$\theta = 50^\circ, V_x = .5c, V_y = .5c, V_z = .5c, V = 0.866c$	3.35072ρ	3.35069ρ
$\theta = 60^\circ, V_x = .1c, V_y = .1c, V_z = .1c, V = 0.1732c$	1.011623ρ	1.011633ρ
$\theta = 60^\circ, V_x = .2c, V_y = .2c, V_z = .2c, V = 0.3464c$	1.05156ρ	1.05134ρ
$\theta = 60^\circ, V_x = .3c, V_y = .3c, V_z = .3c, V = 0.5196c$	1.14185ρ	1.141849ρ
$\theta = 60^\circ, V_x = .4c, V_y = .4c, V_z = .4c, V = 0.6928c$	1.36536ρ	1.36799ρ
$\theta = 60^\circ, V_x = .5c, V_y = .5c, V_z = .5c, V = 0.866c$	2.31464ρ	2.314599ρ
$\theta = 70^\circ, V_x = .1c, V_y = .1c, V_z = .1c, V = 0.1732c$	1.00543ρ	1.00545ρ
$\theta = 70^\circ, V_x = .2c, V_y = .2c, V_z = .2c, V = 0.3464c$	1.024ρ	1.02399ρ
$\theta = 70^\circ, V_x = .3c, V_y = .3c, V_z = .3c, V = 0.5196c$	1.06559ρ	1.06559ρ
$\theta = 70^\circ, V_x = .4c, V_y = .4c, V_z = .4c, V = 0.6928c$	1.16624ρ	1.168486ρ
$\theta = 70^\circ, V_x = .5c, V_y = .5c, V_z = .5c, V = 0.866c$	1.57ρ	1.569985ρ
$\theta = 80^\circ, V_x = .1c, V_y = .1c, V_z = .1c, V = 0.1732c$	1.00139ρ	1.001413ρ
$\theta = 80^\circ, V_x = .2c, V_y = .2c, V_z = .2c, V = 0.3464c$	1.00617ρ	1.00615ρ
$\theta = 80^\circ, V_x = .3c, V_y = .3c, V_z = .3c, V = 0.5196c$	1.011677ρ	1.016777ρ
$\theta = 80^\circ, V_x = .4c, V_y = .4c, V_z = .4c, V = 0.6928c$	1.04203ρ	1.04404ρ
$\theta = 80^\circ, V_x = .5c, V_y = .5c, V_z = .5c, V = 0.866c$	1.13868ρ	1.138641ρ

4. Conclusion

The transformation equation of volume charge density in MGLT and QLT are shown in Table 1. We have seen that the formula of VCD in QLT is easier than the formula of MGLT. It is observed that (Table 2) for same angle between the rest and moving frame, when the velocity of the moving frame increases the VCD increases. On the other hand, for same velocity of the moving frame, if the angle between the rest and moving frames increases the VCD deceases. The calculation is easier in QLT than the calculation in MGLT. Therefore, it is more suitable to use QLT than MGLT when the line of motion does not coincide with the coordinate axes.

References

1. R. Resnick, Introduction to Special Relativity (Wileyastern Limited, 1994).
2. S. B. Rafiq and M. S. Alam, Sri Lankan J. Phys. **13**, 17 (2012). https://doi.org/10.4038/sljp.v13i1.3546
3. S. A. Bhuiyan and A. R. Baizid, Appl. Math. **5**, 57 (2015).
4. S. A. Bhuiyan and A. R. Baizid, J. Sci. Res. **8**, 259 (2016. https://doi.org/10.3329/jsr.v8i3.27133
5. C. Moller, The Theory of Relativity (Oxford Univ. Press, London, 1972).
6. A. R. Baizid and M. S. Alam, Am. J. Math. Stat. **2**, 153 (2012). https://doi.org/10.5923/j.ajms.20120205.08
7. A. R. Baizid and M. S. Alam, J. Sci. Res. **8**, 249 (2016). https://doi.org/10.3329/jsr.v8i3.27033
8. A. Kyrala, Theoretical Physics: Applications of Vectors, Matrices, Tensors and Quaternions (Toppon Company Limited, Tokyo, Japan. 1967).
9. http://mathworld.wolfram.com/Quaternion.html
10. http://www.cs.appstate.edu/~sjg/class/3110/mathfestalg2000/quaternions1.html
11. M. S. Alam, News Bull. Cal. Math. Soc. **26**, 21 (2003).
12. M. S. Alam and S. Bauk, Phys. Essays, **24**, 2, 2011. https://doi.org/10.4006/1.3556536
13. M. S. Alam and K. Begum, Jahangirnagar Phys. Stud. 15 (2009).