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Some More Results on Total Equitable Bondage Number of A Graph

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Abstract

The bondage number $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $E_0 \subseteq E(G)$ for which $\gamma(G - E_0) > \gamma(G)$. An equitable dominating set D is called a total equitable dominating set if the induced subgraph $< D >$ has no isolated vertices. The total equitable domination number $\gamma_t^e(G)$ of G is the minimum cardinality of a total equitable dominating set of G. If $\gamma_t^e(G) \neq |V(G)|$ and $< G - E_0 >$ contains no isolated vertices then the total equitable bondage number $b_t^e(G)$ of a graph G is the minimum cardinality among all sets of edges $E_0 \subseteq E(G)$ for which $\gamma_t^e(G - E_0) > \gamma_t^e(G)$. In the present work we prove some characterizations and investigate total equitable bondage number of Ladder and degree splitting of path.

Keywords: Dominating set; Equitable dominating set; Total dominating set; Bondage number.

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1. Introduction

We begin with simple, finite and undirected graph $G = (V(G), E(G))$. We denote the degree of a vertex v in a graph G by $d_G(v)$. The maximum degree among all the vertices of G is denoted by $\Delta(G)$. For any real number n, $\lceil n \rceil$ denotes the smallest integer not less than n and $\lfloor n \rfloor$ denotes the greatest integer not greater than n.

The domination in graph is one of the fastest growing concepts in graph theory. Many variants of domination models are available in literature: Restrained Domination, Equitable Domination, Total Domination and Total Equitable Domination are among worth to mention.

Definition 1.1: A set $D \subseteq V(G)$ of vertices in a graph G is called dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. The minimum cardinality of a dominating set is called the domination number of G which is denoted by $\gamma(G)$.

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Definition 1.2: The bondage number $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $E_0 \subseteq E(G)$ for which $\gamma(G - E_0) > \gamma(G)$ [1] .

The concept of bondage number was introduced by Fink *et al.* [1] which is useful for measuring the vulnerability of the network under link failure. Variety of bondage numbers are introduced on the basis of characteristic of dominating sets. Some of them are total bondage number, equitable bondage number, total equitable bondage number are really noteworthy.

Definition 1.3: A subset D of $V(G)$ is called an equitable dominating set if for every $v \in V(G) - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(v) - d_G(u)| \leq 1$ [2]. The minimum cardinality of D is called the equitable domination number of G which is denoted by $\gamma^e(G)$.

Definition 1.4: An equitable bondage number $b^e(G)$ of a graph G is the cardinality of a smallest set $E_0 \subseteq E(G)$ of edges for which $\gamma^e(G - E_0) > \gamma^e(G)$ [3].

The concept of equitable domination was introduced by Swaminathan and Dharmalingam [2] while the concept of equitable bondage number was introduced by Deepak *et al*. [3].

Definition 1.5: A subset D of $V(G)$ is called a total dominating set of G if $N(D) = V(G)$ or equivalently if every vertex $v \in V(G)$ is adjacent to at least one element in D [4]. The minimum cardinality of total dominating set is called total domination number which is denoted by $\gamma_t(G)$.

Definition 1.6: If $\gamma_t \neq |V(G)|$ and $< G - E_0 >$ contains no isolated vertices then the total bondage number $b_t(G)$ of a graph G is the minimum cardinality among all sets of edges $E_0 \subseteq E(G)$ for which $\gamma_t(G - E_0) > \gamma_t(G)$ [5].

The concept of total domination was introduced by Cockayne *et al.* [4] while the concept of total bondage number was introduced by Kulli and Patwari [5].

Definition 1.7: An equitable dominating set D is called a total equitable dominating set if the induced subgraph $< D >$ has no isolated vertices [6]. The total equitable domination number $\gamma_t^e(G)$ of G is the minimum cardinality of a total equitable dominating set of G.

The concepts of total dominating set and total equitable dominating set are useful for the formation of any committee. It is desirable that each committee member might feel comfortable knowing at least one member of the committee. In this situation, total domination is useful while there is no difference of opinion between any two members or they differ on at most one issue then the concept of equitable domination is applicable. A concept with both the blends is termed as total equitable dominating set was introduced by Basavanagoud *et al.* [6] and further formalized by Vaidya and Parmar [7-10]. Using the framework of total equitable dominating sets, we have introduced a new concept called total equitable bondage number [11].

The total equitable bondage number is a bondage number with the additional property that removal of an edge subset from the given graph results in a graph with larger total equitable domination number. This concept is also useful for measuring the link failure of network.

Definition 1.8: If $\gamma_t^e(G) \neq |V(G)|$ and $< G - E_0 >$ contains no isolated vertices then the total equitable bondage number $b_t^e(G)$ of a graph G is the minimum cardinality among all sets of edges $E_0 \subseteq E(G)$ for which $\gamma_t^e(G - E_0) > \gamma_t^e(G)$ [11].

In the following Fig.1(a) for the graph G, $\gamma_t^e(G) = 2$ but $\gamma_t^e(G - e) = 3$ as shown in Fig. 1(b). Thus $b_t^e(G) = 1$. The vertices of the respective sets are shown by solid vertices.

Fig. 1(a). $\gamma_t^e(G) = 2$ Fig. 1(b). $\gamma_t^e(G - e) = 3$

We also introduce the concept of total equitable domination edge critical graph as follows.

Definition 1.9: A graph G is called total equitable domination edge critical or γ_t^e −critical graph if the removal of any edge in the graph changes the total equitable domination number. *i.e.* $\gamma_t^e(G - e) \neq \gamma_t^e(G)$ for every edge $e \in E(G)$.

In the following Fig. 2(a) for the graph G, $\gamma_t^e(G) = 8$ but $\gamma_t^e(G - e) = 9$ as shown in Fig. 1(b). Thus $\gamma_t^e(G) = 8 \neq 9 = \gamma_t^e(G - e)$. Therefore G is γ_t^e - critical graph. The vertices of the respective sets are shown by solid vertices.

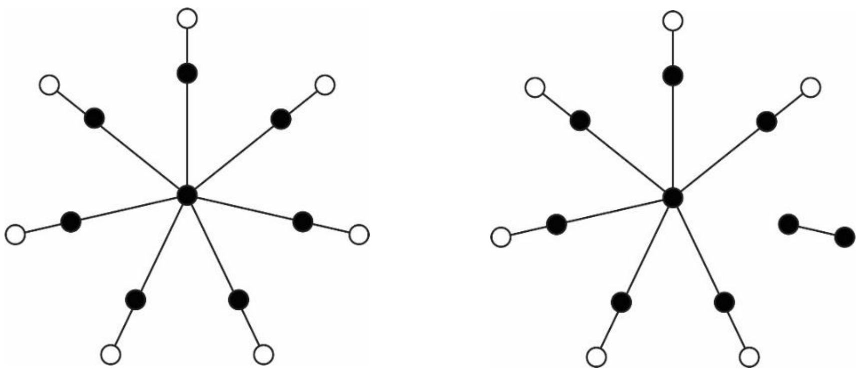

Fig. 2(a). $\gamma_t^e(G) = 8$ Fig. 2(b). $\gamma_t^e(G - e) = 9$

Definition 1.10: The Cartesian product of G and H is a graph, denoted as $G \times H$ whose vertex set $V(G) \times V(H)$ [12]. Two vertices (g, h) and (g', h') are adjacent precisely if

$g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$. Thus $V(G \times H) = \{(g, h) \,/\, g \in V(G) \; and \; h \in V(H)\}$ and $E(G \times H) = \{(g, h)(g', h') / \, g = g', hh' \in E(H) \; or \; gg' \in E(G), h = h'\}$.

Definition 1.11: The ladder L_n is defined as $P_n \times P_2$ [12].

Definition 1.12: Let $G = (V(G), E(G))$ be a graph with $V(G) = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_t \cup T$, where each S_i is a set of all the vertices having same degree (at least 2 vertices) and $T = V(G) \setminus \cup_{i=1}^{t} S_i$ [13]. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $w_1, w_2, w_3, \ldots, w_t$ and joining to each vertex of S_i for $1 \leq i \leq t$.

Proposition 1.13: For any k −regular or $(k, k+1)$ −biregular graph G, $\gamma_t(G) = \gamma_t^e(G)$ [6].

Proposition 1.14: For any k −regular or $(k, k+1)$ −biregular graph G, $b(G) = b^e(G)$ [2].

Proposition 1.15: $\gamma_t(P_n \times P_2) = \gamma_t^e(P_n \times P_2) = 2\left\lceil \frac{n}{3} \right\rceil$ [9].

Proposition 1.16: $b_t(L_n) = \begin{cases} 1 & ; \text{ if } n \equiv 0 (\text{mod } 3) \\ 2 & ; \text{ if } n \equiv 2 (\text{mod } 3) \\ 3 & ; \text{ otherwise} \end{cases}$ [14]

Proposition 1.17: $\gamma_t(DS(P_n)) = \begin{cases} 1; & \text{if } n = 4 \\ 2; & \text{otherwise} \end{cases}$ [8]

For the various graph theoretic notation and terminology we follow West [15] while for any undefined terms related to the concept of domination we refer to Haynes *et al*. [16].

Here we contribute some characterizations and also investigate total equitable bondage number of some graph families.

2. Main Results

Theorem 2.1: Let G be any graph with $\gamma_t(G) \neq |V(G)|$ and $\gamma_t^e(G) \neq |V(G)|$. Let E_0 be a total bondage set with minimum cardinality. If $G - E_0$ be k −regular or $(k, k+1)$ −biregular then $b_t(G) = b_t^e(G)$.

Proof: Let G be any graph with $\gamma_t(G) \neq |V(G)|$ and $\gamma_t^e(G) \neq |V(G)|$. Let E_0 be a total bondage set of G with minimum cardinality. If $G - E_0$ be k −regular or $(k, k+1)$ −biregular then from Proposition 1.13, $\gamma_t(G - E_0) = \gamma_t^e(G - E_0)$. Hence, $b_t(G) = b_t^e(G)$.

We have an immediate corollary as follows:

Corollary 2.2: Let G be a graph. Let E_0 be a bondage set with minimum cardinality. If $G - E_0$ be k −regular or $(k, k+1)$ −biregular then $b(G) = b^e(G)$.

Proof: Let G be a graph. Let E_0 be a bondage set with minimum cardinality. If $G - E_0$ be k −regular or $(k, k+1)$ −biregular then from Proposition 1.13, $\gamma(G - E_0) = \gamma^e(G - E_0)$. Hence, $b(G) = b^e(G)$.

The exact values of b_t^e for L_n and b_t for $DS(P_n)$ are obtained in next two results.

Theorem 2.3: $b_t^e(L_n) = \begin{cases} 1 & ;\text{ if } n \equiv 0(\text{mod } 3) \\ 2 & ;\text{ if } n \equiv 2(\text{mod } 3) \\ 3 & ;\text{otherwise} \end{cases}$

Proof: Let $V(L_n) = \{u_i, v_i \ / \ 1 \le i \le n\}$ be the vertex set with $|V(L_n)| = 2n$ and $E(L_n) = \{e_i, e_i', f_i \ / \ 1 \le i \le n\}$ where $d_{L_n}(u_1) = d_{L_n}(v_1) = d_{L_n}(u_n) = d_{L_n}(v_n) = n$ and $d_{L_n}(u_i) = d_{L_n}(v_i) = 3$ for all $i \in \{2,3,\dots,n-1\}$. Moreover L_n is a (2,3) – biregular graph.

To prove the result we consider the following cases:
Case I: For $n \not\equiv 1(\text{mod } 3)$
From Proposition 1.15, there exist $E_0 = \{e_1, e_1'\} \subseteq E(G)$. Further $H = L_n - E_0$ is a graph with two components namely P_2 and L_{n-1}. Then $\gamma_t^e(H) = \gamma_t^e(P_2) + \gamma_t^e(L_{n-1}) = \gamma_t^e(L_n) + 2$. Hence, $b_t(L_n) = b_t^e(L_n)$.
Case II: For $n \equiv 1(\text{mod } 3)$
The graph H obtained by removal of three edge namely e_{n-1}, e_n and $e_n{}'$ for all $i \in \{2,3,\dots,n-1\}$ from L_n is a graph with two components namely $L_{n-1} - e_{n-1}$ and P_2. Then $\gamma_t^e(H) = \gamma_t^e(L_{n-1} - e_{n-1}) + \gamma_t^e(P_2) = \gamma_t^e(L_{n-1}) + 1 + 2 = \gamma_t^e(L_n) + 1$. Hence, $b_t(L_n) = b_t^e(L_n)$.

Theorem 2.4: $b_t\big(DS(P_n)\big) = \begin{cases} 1\,;\text{if } n = 4 \\ 4\,;\text{if } n > 4 \end{cases}$

Proof: Let $V(DS(P_n)) = \{v_i \ / \ 1 \le i \le n\} \cup S_1 \cup S_2$, where $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i / \ 1 \le i \le n-1\}$. To obtained $DS(P_n)$ from P_n, add two new vertices x and y corresponding to S_1 and S_2 respectively. Thus $V\big(DS(P_n)\big) = V(P_n) \cup \{x, y\}$ and $E\big(DS(P_n)\big) = E(P_n) \cup \{xv_i \ / \ v_i \in S_1\} \cup \{yv_j \ / \ v_j \in S_2\}$.

To prove the result we consider following cases:
Case I: For $n = 4$
If the graph H is obtained by removal of an edge v_2v_3 from $DS(P_n)$ then H is isomorphic to C_6. Therefore, $\gamma_t(H) = \gamma_t(C_6) = 4 = \gamma_t\big(DS(P_4)\big) + 1$. Hence $b_t\big(DS(P_4)\big) = 1$.
Case II: For $n > 4$
The graph H is obtained by removal of four edges namely $e_1 = v_2v_3$, $e_2 = v_4v_5$, $e_3 = yv_3$ and $e_4 = yv_4$ from $DS(P_n)$ then H is a graph with two components namely P_2 and H_1. Further the minimal total dominating set of H_1 is $\{v_1, v_2, v_{n-1}, y\}$. Then $\gamma_t(H) = \gamma_t(P_2) + \gamma_t(H_1) = 2 + 4 = \gamma_t\big(DS(P_n)\big) + 2$. Hence $b_t\big(DS(P_n)\big) = 2$ for $n > 4$.

We introduce a concept of total equitable edge critical graph.

Definition 2.5: A graph G is called total equitable domination edge critical or γ_t^e −critical graph if the removal of any edge in the graph changes the total equitable domination number. *i.e.* $\gamma_t^e(G-e) \neq \gamma_t^e(G)$ for every edge $e \in E(G)$.

We note that removing an edge from a graph G cannot decrease the total equitable domination number. Hence if G is γ_t^e −critical, then $\gamma_t^e(G-e) > \gamma_t^e(G)$ for every edge $e \in E(G)$. An edge $e \in E(G)$ is a critical edge of G if $\gamma_t^e(G-e) > \gamma_t^e(G)$.

Theorem 2.6: Let G be a γ_t^e − critical graph and D be total equitable dominating set, then the induce subgraph $< D >$ is a family of stars.

Proof: Let D be a total equitable dominating set in the γ_t^e − critical graph G. Let e be any edge from $< D >$. We assume that $< D >$ is not a family of stars. Then there exist an edge $e \in D$ such that both terminals of an edge e have degree at least two in $< D >$ then D is a total dominating set in $G-e$. Thus $\gamma_t^e(G-e) \leq |D| = \gamma_t^e(G)$. This contradicts the graph G is γ_t^e − critical graph (*i.e.* $\gamma_t^e(G-e) > \gamma_t^e(G)$). Therefore at least one terminal of an edge e is pendant vertex in $< D >$. Hence $< D >$ is a family of stars.

We define a family of some trees τ as: If T is a star $K_{1,n}$ or T is a subdivided star $K_{1,n}^*$ then a tree T is member of τ ($T \in \tau$).

Theorem 2.7: A tree $T \in \tau$ if and only if T is a γ_t^e − critical.

Proof: First we assume that a tree $T \in \tau$. Let e be any edge of T. If one end vertex of e is pendant vertex in T then $\gamma_t^e(T-e) = \infty$, thus an edge e of T is a critical edge. Therefore we may suppose that both end vertices of an edge e are not pendant in T. Then T is a subdivided star by adding pendant edge to each pendent vertex of T. If e is an edge from T then $\gamma_t^e(T-e) = \gamma_t^e(T) + 2$. Therefore $\gamma_t^e(T-e) = \gamma_t^e(T)$. Hence T is γ_t^e − critical.

Conversely, suppose that $T = \big(V(T), E(T)\big)$ is a γ_t^e − critical. Let D be a total equitable dominating set in T. By Theorem 2.6, $< D >$ is a family of stars and for any edge e of T, the set D is not a total equitable dominating set in $T-e$. Then exactly one vertex of D is adjacent to $V(T) - D$ (*i.e.* $V(T) - D$ is independent set). Thus T is a connected and also $< D >$ is a connected graph. Therefore $< D >$ is a star. We note that the pendant vertex of $< D >$ is adjacent to at least one vertex in $V(T) - D$. Therefore T is a star $K_{1,n}$ or T is a subdivided star $K_{1,n}^*$ as $V(T) - D$ is an independent set. Hence $T \in \tau$.

3. Conclusion

We have introduced the concept of total equitable bondage number in reference [8]. Here we have explored the same concept. Also proved some characterizations and investigate total equitable bondage number for Ladder L_n and degree splitting graph of path P_n.

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References

1. J. F. Fink, M. S. Jacobson, L. F. Kinch, and J. Roberts, Discrete Math. **86**, 47 (1990). https://doi.org/10.1016/0012-365X(90)90348-L
2. V. Swaminathan and K. M. Dharmalingam, J. Math. **35**, 191 (2011).
3. G. Deepak, N. D. Soner, and A. Alwaedi, Res. J. Pure Algebra **1**, 201 (2011).
4. E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, Networks **10**, 211 (1980). https://doi.org/10.1002/net.3230100304
5. V. R. Kulli and D. K. Patwari, Vishwa Int. Publications 227 (1991).
6. B. Basavanagoud, V. R. Kulli, and V. V. Teli, J. Computer. Math. Sci. **5**, 235 (2014).
7. S. K. Vaidya and A. D. Parmar, J. Comput. Mathematica **1**, 98 (2017).
8. S. K. Vaidya and A. D. Parmar, Int. J. Math. Soft Computing **7**, 103 (2017). https://doi.org/10.26708/IJMSC.2017.2.7.09
9. S. K. Vaidya and A. D. Parmar, Malaya J. Matematik **6**, 375 (2018). https://doi.org/10.26637/MJM0602/0012
10. S. K. Vaidya and A. D. Parmar, On Total and Total Equitable Domination Number of Some Graphs, Thai J. Math. Submitted.
11. S. K. Vaidya and A. D. Parmar, J. Sci. Res. **10**, 231 (2018). https://doi.org/10.3329/jsr.v10i3.33940
12. F. Harary, Graph Theory (Addison Wesley, Massachusetts, 1969). https://doi.org/10.21236/AD0705364
13. R. Ponraj and S. Somasundaram, Natl. Acad. Sci. Letters, **27**, 275 (2004).
14. F. Hu, Y. Lu, and J. Xu, Discrete Appl. Math. **160**, 2408 (2012). https://doi.org/10.1016/j.dam.2012.06.012
15. D. B. West, Introduction to Graph Theory (Prentice Hall, New Delhi, India, 2003).
16. T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).