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# Controlling Chaos in a Newly Designed Chaotic Hamiltonian System Based on Hénon-Heiles Model using Active Controlled Hybrid Projective Synchronization

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**Abstract**

In this paper, a systematic approach is designed for investigating the hybrid projective synchronization (HPS) in identical chaotic Hamiltonian systems based on Hénon-Heiles Model by using active control method (ACM). Initially, an active control law is described to achieve asymptotic stability of state vectors of given system using Lyapunov stability theory (LST). Additionally, numerical simulations utilizing MATLAB toolbox are presented to validate the efficiency and effectiveness of the designed approach. Furthermore, the proposed strategy has numerous applications in encryption and secure communication.

*Keywords*: Active control method; Chaotic system; Hybrid projective synchronization; Lyapunov stability theory; MATLAB


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## 1. Introduction

Over the years, chaos theory has been an intriguing and a prominent field of nonlinear science that focuses on the behavioral analysis of immensely irregular or disordered dynamical systems hugely found in nature. This theory plays a significant role in various fields, for example, secure communication [1], weather models [2], neural networks [3], biomedical engineering [4], robotics [5], ecological models [6], oscillations [7], chemical reactions [8], finance models [9], jerk systems [10], encryption [11], etc. As a result, chaos control as well as synchronization have sought significant attention among several research fields.

A key characteristic of chaotic systems, mentioned as "Butterfly Effect" in available literature, is high sensitive dependency on initial conditions and it was first observed in 1963 by Lorenz [12] while studying a weather prediction model. Most importantly, Pecora and Caroll [13] firstly announced in 1990 the notion of synchronization of chaotic systems. In chaos synchronization phenomenon, the trajectories of state variables of two or more chaotic/hyperchaotic systems are regulated to follow the identical dynamics. In

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recent times, chaos synchronization in chaotic systems utilizing various control methods has become an engaging and a fascinating topic of study for the scientists and researchers. In the last three decades, several techniques of significance are initiated and studied to control [14-18] and synchronization [19-27] of chaos existing in dynamical systems.

Specifically, Bai and Lonngren [28] introduced active control method (ACM) in chaotic systems in the year 1997. Since then, numerous researches have been executed using ACM [14,19,24,29- 35]. In view of above discussions, the primary goal in this paper is to investigate hybrid projective synchronization (HPS) among identical newly designed Hamiltonian chaotic systems [36] based on Hénon-Heiles model by ACM. Primarily, Hénon and Heiles [37] in 1964 firstly modeled the Hénon-Heiles model that describes the nonlinear motion of a star around a galactic centre with the motion restricted to a plane.

## 2. Preliminaries

The master system and the corresponding slave system can be written as:

$$\dot{u}_m = g_1(u_m), \tag{1}$$

$$\dot{u}_s = g_2(u_s) + \mu, \tag{2}$$

where $u_m = (u_{m;1}, u_{m;2}, u_{m;3}, ...., u_{m;n})^T$, $u_s = (u_{s;1}, u_{s;2}, u_{s;3}, ...., u_{s;n})^T$ are the state variables of (1) and (2) respectively, $g_1, g_2 : R^n \to R^n$ are two nonlinear continuous vector functions and $\mu = (\mu_1, \mu_2, \mu_3, ......, \mu_n) \in R^n$ is the controller to be designed.

**Definition 1.** The master system (1) and the slave system (2) are said to be in hybrid projective synchronization (HPS) if

$$\lim_{t\to\infty} \| e(t) \| = \lim_{t\to\infty} \| u_s(t) - \psi u_m(t) \| = 0 \tag{3}$$

for some $\psi = diag(\psi_1, \psi_2, \psi_3, ......, \psi_n)$ and $\|.\|$ represents vector norm.

**Remark 2.1.** For $\psi_1 = \psi_2 = \psi_3 = ........ = \psi_n = 1$, complete synchronization is achieved.

**Remark 2.2.** For $\psi_1 = \psi_2 = \psi_3 = ........ = \psi_n = -1$, anti-synchronization is attained.

**Remark 2.3.** If $\psi_i's$ are not all zeros and $\psi_i \neq \psi_j$ for some $i$ and $j$, then modified projective synchronization is obtained.

## 3. System Description

Proposed by Vaidyanathan *et al*. [36], the considering chaotic system can be described as

$$\begin{cases} \dot{u}_{m;1} = & u_{m;2} \\ \dot{u}_{m;2} = & -u_{m;1} - 2u_{m;1}u_{m;3} + lu_{m;1}^2 \\ \dot{u}_{m;3} = & u_{m;4} \\ \dot{u}_{m;4} = & -u_{m;3} - u_{m;1}^2 + u_{m;3}^2 + mu_{m;3}^4, \end{cases} \tag{4}$$

where $(u_{m;1}, u_{m;2}, u_{m;3}, u_{m;4})^T \in R^4$ is state vector and $l$ and $m$ are positive parameters. For $l = 0.98$ and $m = 0.99$, the system (4) exhibits chaos phenomenon. Further, the

Lyapunov exponents of system (4) are $LE_1 = 0.0015$, $LE_2 = 0$, $LE_3 = 0$, $LE_1 = -0.0015$ Also, Fig. 1 (a-c) display the phase plots of (4). However, the analytic study and numerical results in detail for the system (4) can be found in literature [34].

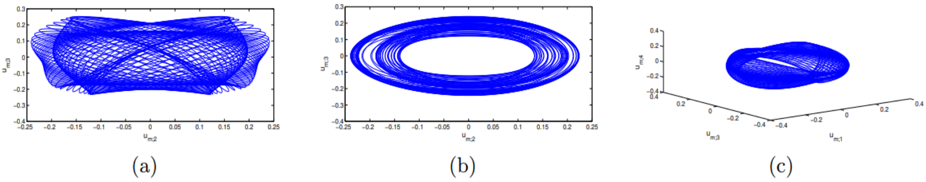

Fig. 1. Phase plots of considered Hamiltonian chaotic system in (a) $u_{m;2} - u_{m;3}$ plane, (b) $u_{m;1} - u_{m;2}$ plane, (c) $u_{m;1} - u_{m;3} - u_{m;4}$ space.

## 4. Illustrative Example

In this section, hybrid prijective synchronization scheme is discussed to design the nonlinear active control law in such a manner that the state variables $u_{m1}, u_{m2}, u_{m3}$ and $u_{m4}$ approach to equilibrium points as $t$ tends to infinity.

Conveniently, the system (4) has been selected as the master system and the corresponding slave system may be defined as:

$$\begin{cases} \dot{u}_{s;1} = & u_{s;2} + \mu_1 \\ \dot{u}_{s;2} = & -u_{s;1} - 2u_{s;1}u_{s;3} + lu_{s;1}^2 + \mu_2 \\ \dot{u}_{s;3} = & u_{s;4} + \mu_3 \\ \dot{u}_{s;4} = & -u_{s;3} - u_{s;1}^2 + u_{s;3}^2 + mu_{s;3}^4 + \mu_4, \end{cases} \tag{5}$$

where $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$ are active nonlinear controllers to be designed in a way that HPS of two identical chaotic Hamiltonian systems will be achieved. Also, Fig. 2(a-c) show the phase plots of the slave system .

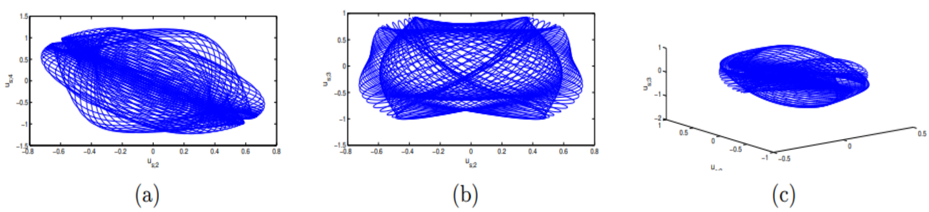

Fig. 2. Phase plots of Hamiltonian chaotic system chosen as the slave system in (a) $u_{s;2} - u_{s;4}$ plane, (b) $u_{s;2} - u_{s;3}$ plane, (c) $u_{s;1} - u_{s;2} - u_{s;4}$ space.

State errors are defined by the following rule:

$$\begin{cases} E_1 = & u_{s;1} - \psi_1 u_{m;1} \\ E_2 = & u_{s;2} - \psi_2 u_{m;2} \\ E_3 = & u_{s;3} - \psi_3 u_{m;3} \\ E_4 = & u_{s;4} - \psi_4 u_{m;4} \end{cases} \tag{6}$$

The ultimate aim here is to construct controllers $\mu_i, (i = 1, 2, 3, 4)$ using LST [38,39] so that the synchronization errors defined in (6) satisfy
$\lim_{t\to\infty} E_i(t) = 0 \quad \text{for } (i = 1, 2, 3, 4).$

The resulting error dynamics has been given by

$$\begin{cases} \dot{E}_1 = & E_2 + (\psi_2 - \psi_1)u_{m;2} + \mu_1 \\ \dot{E}_2 = & -E_1 + (\psi_2 - \psi_1)u_{m;1} - 2(u_{s;1}u_{s;3} - \psi_2 u_{m;1}u_{m;3}) + l(u_{s;1}^2 - \psi_2 u_{m;1}^2) + \mu_2 \\ \dot{E}_3 = & E_4 + (\psi_4 - \psi_3)u_{m;4} + \mu_3 \\ \dot{E}_4 = & -E_3 + (\psi_4 - \psi_3)u_{m;4} - (u_{s;1}^2 - \psi_4 u_{m;1}^2) + (u_{s;3}^2 - \psi_4 u_{m;3}^2) + m(u_{s;3}^4 - \psi_4 u_{m;3}^4) + \mu_4. \end{cases} \tag{7}$$

Now, the active controllers are described as:

$$\begin{cases} \mu_1 = & -E_2 - (\psi_2 - \psi_1)u_{m;2} - L_1 E_1 \\ \mu_2 = & E_1 - (\psi_2 - \psi_1)u_{m;1} + 2(u_{s;1}u_{s;3} - \psi_2 u_{m;1}u_{m;3}) - l(u_{s;1}^2 - \psi_2 u_{m;1}^2) - L_2 E_2 \\ \mu_3 = & -E_4 - (\psi_4 - \psi_3)u_{m;4} - L_3 E_3 \\ \mu_4 = & E_3 - (\psi_4 - \psi_3)u_{m;4} + (u_{s;1}^2 - \psi_4 u_{m;1}^2) - (u_{s;3}^2 - \psi_4 u_{m;3}^2) - m(u_{s;3}^4 - \psi_4 u_{m3;}^4) - L_4 E_4, \end{cases} \tag{8}$$

where $L_1 > 0$, $L_2 > 0$, $L_3 > 0$, $L_4 > 0$ are gain constants

By putting the active controllers (8) in error dynamics (7), we find that

$$\begin{cases} \dot{E}_1 = & -L_1 E_1 \\ \dot{E}_2 = & -L_2 E_2 \\ \dot{E}_3 = & -L_3 E_3 \\ \dot{E}_4 = & -L_4 E_4 \end{cases} \tag{9}$$

The classic Lyapunov function is described by the rule:

$$V = \frac{1}{2}[E_1^2 + E_2^2 + E_3^2 + E_4^2], \tag{10}$$

which confirms that $V$ is positive definite.
Derivative for Lyapunov function $V$ takes the form:

$$\dot{V} = E_1\dot{E}_1 + E_2\dot{E}_2 + E_3\dot{E}_3 + E_4\dot{E}_4. \tag{11}$$

**Theorem 1.** The chaotic systems (4)-(5) are asymptotically hybrid projective synchronized globally for all initial states $(u_{m;1}(0), u_{m;2}(0), u_{m;3}(0), u_{m;4}(0)) \in R^4$ by the designed active controller.

**Proof**. The Lyapunov function $V$ as given in (10) is a positive definite function. On solving, equations (9) and (11) give rise to

$$\begin{aligned} \dot{V} &= -L_1E_1^2 - L_2E_2^2 - L_3E_3^2 - L_4E_4^2 \\ &< 0, \end{aligned}$$

ensuring that $\dot{V}$ is negative definite .

Thus, using Lyapunov stability theory [38,39], we deduce that synchronization error $e(t) \to 0$ exponentially as $t \to \infty$ for each initial conditions $e(0) \in R^4$. This ends the proof.

## 5. Numerical Simulation and Discussion

This section presents some numerical simulations to show effectively the proposed HPS technique via ACM. The parameters of system (4) are selected as $a = 0.98$ and $b = 0.99$ to confirm the chaoticity of considered system without control inputs. Also, Fig. 1(a-c) and Fig. 2(a-c) display phase plots of the master (4) and slave system (5) respectively. The initial states of the master (4) and slave systems (5) are $(u_{m;1}(0) = 0.2, u_{m;2}(0) = 0, u_{m;3}(0) = -0.2, u_{m;4}(0) = 0)$ and $(u_{s;1}(0) = 0.2, u_{s;2}(0) = 0.2, u_{s;3}(0) = 0.2, u_{s;4}(0) = 0)$ respectively. The control gains are taken as $K_i = 6$ for $i = 1, 2, ..., 6$. Further, simulation results regarding the state hybrid projective synchronized trajectories of chaotic systems (4) and (5) are shown in Fig. 3(a-d). Moreover, Fig. 3(e) depict that the synchronization error $(E_1, E_2, E_3, E_4) = (0.6, 0.2, -0.6, 0)$ converging to zero as $t$ tending to infinity. Hence, the proposed HPS strategy in master and slave system has been achieved computationally.

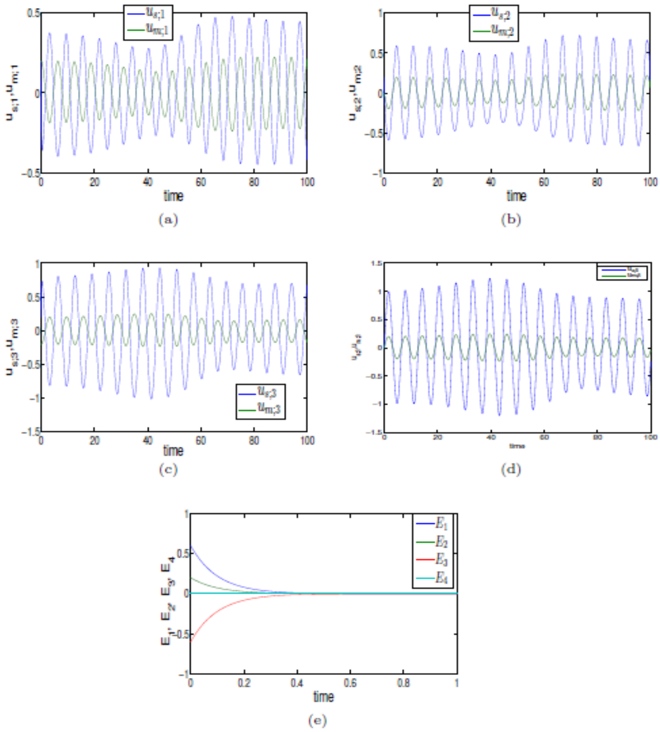


Fig. 3. HPS synchronization of Hamiltonian chaotic system (a) between $u_{m;1}(t) - u_{s;1}(t)$, (b) between $u_{m;2}(t) - u_{s;2}(t)$, (c) between $u_{m;3}(t) - u_{s;3}(t)$, (d) between $u_{m;4}(t) - u_{s;4}(t)$, (e) synchronization errors.

## 6. Conclusion

In this paper, hybrid projective synchronization of integer order identical Hamiltonian chaotic systems has been investigated using active control method keeping Lyapunov stability theory in mind. Further numerical simulations in MATLAB toolbox are presented to validate the effectiveness of the proposed technique. Remarkably, the theoretical results are in complete agreement with computational results. Such strategy can be used to control the nonlinear motion of a star around a galactic centre with motion restricted to a plane. Moreover, the proposed scheme may find applications in the field of image encryption and secure communication.

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