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L(2,1,1)-Labeling of Circular-Arc Graphs

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Abstract

Graph labeling problem has been broadly studied in recent past for its wide applications, in mobile communication system for frequency assignment, radar, circuit design, X-ray crystallography, coding theory, etc. An L211-labeling (L211L) of a graph G = (V, E) is a function $\gamma : V \to Z^*$ such that $|\gamma(u) - \gamma(v)| \geq 2$, if d(u,v) = 1 and $|\gamma(u)-\gamma(v)| \geq 1$, if d(u,v) = 1 or 2, where Z^* be the set of non-negative integers and d(u,v) represents the distance between the nodes u and v. The L211L numbers of a graph G, are denoted by $\lambda_{2,1,1}(G)$ which is the difference between largest and smallest labels used in L211L. In this article, for circular-arc graph (CAG) G we have proved that $\lambda_{2,1,1}(G) \leq 6\Delta - 4$, where Δ represents the degree of the graph. Beside this we have designed a polynomial time algorithm to label a CAG satisfying the conditions of L211L. The time complexity of the algorithm is $O(n\Delta^2)$, where n is the number of nodes of the graph G.

Keywords: L211-labeling; Frequency assignment; Circular-arc graph.

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1. Introduction

Graph labeling problems is one of the most important problems in discrete mathematics to solve real life problems. Different types of graph labeling problems such as node labeling, edge labeling, LrstL, graceful labeling, harmonic labeling, anti-magic labeling, magic labeling, total vertex irregular labeling, etc. are studied by many researchers. Nowadays graph domination problem is an important problem of graph theory [2].

An L211L of a graph G = (V, E) is a function γ from its node set V to Z^* such that $|\gamma(u) - \gamma(v)| \geq 2$ if d(u, v) = 1, $|\gamma(u) - \gamma(v)| \geq 1$ if d(u, v) = 1 or 2. The L211L number,

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$\lambda_{2,1,1}(G)$, of G is the least non-negative integer λ such that G has a L211L of span λ. Graph labeling problem has been extensively studied in the past [1,7,8,11-17,19-24]. Different bounds for $\lambda_{3,2,1}(G)$ and $\lambda_{4,3,2,1}(G)$ were obtained for various type of graphs. Clipperton *et al*. showed $\lambda_{3,2,1}(G) \leq \Delta^3 + \Delta^2 + 3\Delta$ for any graph [6]. Later, Chia *et al*. [5] improved this upper bound to $\lambda_{3,2,1}(G) \leq \Delta^3+2\Delta$ for any graph [5]. Lui and Shao studied the L321L of planer graph and showed that $\lambda_{3,2,1}(G) \leq 15(\Delta^2-\Delta+1)$ [9]. Also, Amanathulla *et al*. shown that $\lambda_{0,1}(G) \leq \Delta$ and $\lambda_{1,1}(G) \leq 2\Delta$ for CAGs [10]. In 2020, Rana have studied graph a new variation of graph labeling problem [3,4]. Also, in 2020, Amanathulla *et al*. have studied L(3,2,1)-labeling of trapezoid graph and obtained good result for it [18].

In this article, for CAGs G, it is shown that $\lambda_{2,1,1} \leq 6\Delta - 4$. Also an algorithm is designed to label a CAG by maintaining L211L condition. The time complexity of the algorithm is also calculated.

2. Preliminaries and Notations

A graph is a CAG if there exists a family F_A of arcs around a circle and a one-to-one correspondence between nodes of G and arcs F_A, such that two distinct nodes are adjacent in G if and only if there corresponding arcs intersect in F_A. Such a family of arcs is called an arc representation for G. It is noted that an arc F_p of F_A and a node t_p of V are one and same thing. A CAG and its corresponding circular-arc representation are shown in Fig. 1.

The degree of a node t_p is denoted by $d(t_p)$ and is defined by the maximum number of nodes which are adjacent to t_p. The degree of a CAG G, denoted by Δ, is the maximum degree of all nodes of G. Let $F = \{F_1, F_2, F_3, \ldots, F_n\}$ be a set of arcs around a circle. While going in a clockwise direction, the point at which we first encounter an arc is called the starting point of the arc. Similarly, the point at which we leave an arc is called the finishing point of the arc.

Notations: Let G be a CAG having set of arcs F, we define the following objects:
(i) $L(F_k)$: the set of labels which are used before labeling the arc F_k, $F_k \in F$.
(ii) $L^i(F_k)$: the set of labels which are used to label the nodes at distance i (i = 1, 2, 3) from the arc F_k, before labeling the arc F_k, $F_k \in F$.
(iii) $L^{ivl}(F_k)$: the set of all valid labels to label the arc F_k satisfying the condition of distance 'one', 'one and two', 'one, two and three' of L211L from the arc F_k, before labeling F_k, for (i = 1, 2, 3) respectively.
(iv) f_j: the label of the arc F_j, $F_j \in A$.
(v) L: the label set.

3. L(2,1,1)-Labeling of Circular-Arc Graphs

In this portion, some lemmas related to the proposed algorithm have been presented. Also, to label a CAG satisfying L211L an algorithm is proposed. Also we have calculated the time complexity of the proposed algorithm.

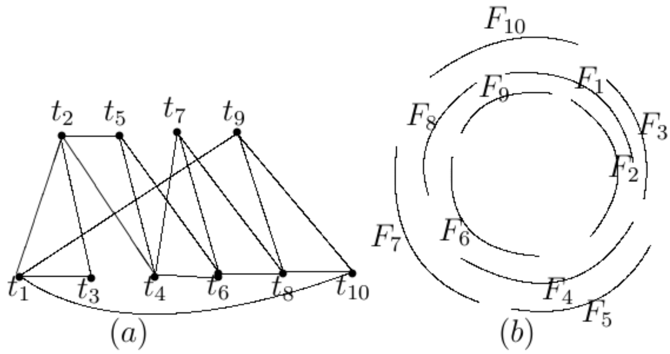

Fig. 1. A CAG and its corresponding circular-arc representation.

Lemma 1. For a CAG G, $|L^i(F_k)| \le 2\Delta - 2$, i = 2,3 for any arc $F_k \in F$.

Proof. Case 1: Let i = 2 and let G be a CAG and F_k be any arc of G. Also let $|L^2(F_k)|$ = r. This indicates that r distinct labels have been used to label the two distanced arcs from the arc F_k, before labeling the arc F_k.

Since, the degree of the graph G is Δ so, F_k is adjacent to at most Δ arcs of G. Again, since G is a CAG, among the arcs those are adjacent to F_k, there must exists at most two arcs (the arcs of maximum length) in opposite direction of the arc F_k, each of which are adjacent to at most $\Delta-1$ arcs (except F_k) of G, obviously these arcs are at distance two from F_k. In figure 2, all the two distances nodes of F_k are adjacent to either F_{k1} or F_{k2}. Except F_k, F_{k1} is adjacent at most $\Delta - 1$ arcs. Similarly, except F_k, F_{k_2} is adjacent at most $\Delta - 1$ arcs. Hence, $r \le 2(\Delta - 1)$, i.e. $|L(F_k)| \le 2\Delta - 2$.

Case 2: Let i = 3 and let G be a CAG and F_k be any arc of G and let $|L^3(F_k)|$ = s. This shows that s distinct three distanced labels are used to label the arcs from the arc F_k, before labeling the arc F_k.

Since, Δ is the degree of the graph G so, F_k is adjacent to at most Δ arcs of G. Again since G is a CAG, among the arcs those are adjacent to F_k, there must exists at most two arcs (the arcs of maximum length) in opposite direction of the arc F_k, each of which are adjacent to at most Δ arcs of G. In figure 2, F_{k1} and F_{k2} are those arcs each of which are adjacent to at most Δ arcs of G. Among the arcs those are adjacent to F_{k1} and of distance two from F_k, there exists at most one arc (the arcs of maximum length) which is adjacent to at most $\Delta-1$ arcs (expect F_{p1}) obviously these arcs are at distance three from F_k. Again among the arcs which are adjacent to F_{k2} and of distance two from F_k, there exists at most one arc (the arcs of maximum length) which is adjacent to at most $\Delta - 1$ arcs (except F_{p2}), obviously these arcs are at distance three from F_k. In Fig. 2, all the three distances arcs are adjacent to either F_{p1} or F_{p2}. Except F_{k1}, F_{p1} is adjacent at most $\Delta - 1$ arcs. Similarly, except F_{k2}, F_{p2} is adjacent at most $\Delta-1$ arcs. Hence, $s \le 2(\Delta-1)$, i.e. $|L(F_k)| \le 2\Delta-2$.

Lemma 2. For a CAG G, $L^i(F_k) \subseteq L(F_k)$, for any arc F_k of G and i = 1, 2,3

Proof. Any label used to label a CAG G belong to $L(F_k)$. So any label $l \in L^i(F_k)$ implies $l \in L(F_k)$, for i = 1, 2, 3. Hence $L^i(F_k) \subseteq L(F_k)$, for any arc F_k of G and i = 1,2,3.

Lemma 3. $L^{kvl}(F_j)$ is the non empty largest set satisfying the condition of distance 1,..., k for k = 1,2,3 of L211L, where $l \le p$ for all $l \in L^{kvl}(F_j)$ and $p = \max\{L(F_j)\} + 2$, for any $F_j \in A$ and k = 1,2, 3.

Proof. Since, $L^i(F_j) \subseteq L(F_j)$ for i = 1, 2, 3 (by Lemma 2) and $p = \max\{L(F_j)\} + 2$, so $|p - l_i| \geq 2$ for any $l_i \in L^i(F_j)$, i = 1, 2, 3. Therefore, $p \in L^{kvl}(F_j)$ for k = 1, 2, 3. Therefore, $L^{kvl}(F_j)$ is non empty set for k = 1, 2, 3.

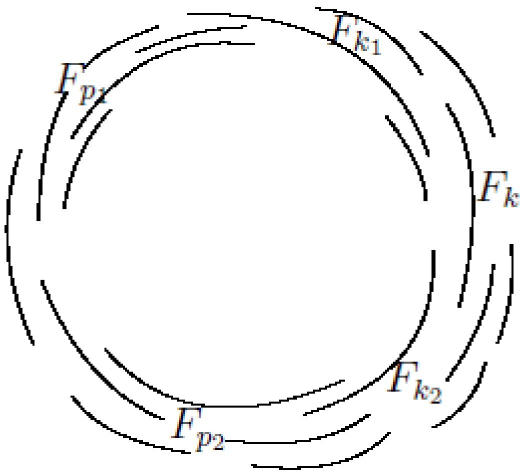

Fig. 2. A CAG.

Again, let C bean arbitrary set of labels satisfying the condition of distance one, two and three of L211L, where $l \le p$ for all $l \in C$. Also, let $c \in C$. Then $|c-l_i| \geq 2$ for any $l_i \in L^1(F_j)$ and $|c-l_i| \geq 1$ for any $l_i \in L^i(F_j)$ for i = 2,3. Thus, $c \in L^{kvl}(F_j)$, for k = 1,2,3. So $c \in C$ implies $c \in L^{kvl}(F_j)$, for k = 1, 2,3. Therefore, $C \subseteq L^{kvl}(F_j)$, for k = 1,2,3. Since, C is arbitrary, so $L^{kvl}(F_j)$ is the largest set of labels satisfying the condition of distance 1,...,k for k = 1,2,3 of L211L, where $l \le p$ for all $l \in L^{kvl}(F_j)$ for k = 1,2,3.

Now, we discuss about the upper bound of $\lambda_{2,1,1}(G)$ for a CAGs.

Theorem 1. For any CAG G, $\lambda_{2,1,1}(G) \leq 6\Delta - 4$.

Proof. Let G be a CAG having n nodes and the set of arcs $F = \{F_1, F_2, F_3, \ldots, F_n\}$. Let $L(F_k) = \{0, 1, 2, \ldots, 6\Delta - 4\}$, where $F_k \in F$. Then $|L(F_k)| = 9\Delta - 5$. Now $\lambda_{2,1,1}(G) \leq 6\Delta - 4$, if we can prove that the label in the set $L(F_k)$ is sufficient to label all the arcs of G. Suppose, we are going to label the arc F_k by L211L. We know that $|L^1(F_k)| \leq \Delta$. So, in the extreme unfavorable cases at least $(6\Delta - 3) - 2\Delta = 4\Delta - 3$ labels of the set $L(F_k)$ are available satisfying the condition of distance one of L211L. Also, since $|L^2(F_k)| \leq 2\Delta - 2$, (by Lemma 1). So in the worst cases at least $(4\Delta - 3) - (2\Delta - 2) = 2\Delta - 1$

labels of the set $L(F_k)$ are available satisfying the condition of distance one and two of L211L. Again, since $|L^3(F_k)| \leq 2\Delta - 2$, (by Lemma 1), so in the most unfavorable cases at least one (viz: $(2\Delta-1)-(2\Delta-2) = 1$) label of the set $L(F_k)$ is available satisfying L211L condition. Since F_k is arbitrary, so we can label any arc of the CAG satisfying L211L condition by using the labels from the set $L(F_k)$.

If we take $L(F_k)$ so that $L(F_k) \subseteq \{0, 1, 2, \ldots, 6\Delta - 4\}$ and we are going to label the arc F_k by L211L, then by similar arguments, it follows that the set $L(F_k)$ may or may not contain a label satisfying L211L condition. Hence, $\lambda_{2,1,1}(G) \leq 6\Delta - 4$.

Algorithm L211

Input: A set of ordered arcs $F = \{F_1, F_2, F_3, \ldots, F_n\}$of a CAG.

Output: f_j, the L211-label of F_j, $j = 1, 2, 3, \ldots, n$.

Initialization: $f_1 = 0$;

$L(F_2) = \{0\}$;

for each $\mathbf{j} = 2$ to $n-1$ compute $L^1(F_\mathbf{j})$, $L^2(F_\mathbf{j})$ and $L^3(F_\mathbf{j})$ for $i = 0$ to r, where $r = \max\{L(F_\mathbf{j})\} +$
2

for $k = 1$ to $|L^1(F_\mathbf{j})|$

if $|i-l_k| \geq 2$, then $L^{1vl}(F_\mathbf{j}) = \{i\}$ //where $l_k \in L^1(F_\mathbf{j})$//

end for;

end for;

for k = 1 to 2

for $m = 1$ to $|L^{kvl}(F_\mathbf{j})|$ for $n = 1$ to $|L^{k+1}(F_\mathbf{j})|$

if $|l_m - p_n| \geq 1$, then $L^{k+1vl}(F_j) = \{l_m\}$ //where $l_m \in L^{kvl}(F_j)$ and $p_n \in$
$L^{k+1}(F_j)$ //

end for;

end for;

end for;

$f_\mathbf{j} = \min\{L^{3vl}(F_\mathbf{j})\}$;

$L(F_{\mathbf{j}+1}) = L(F_\mathbf{j}) \cup \{f_\mathbf{j}\}$;

end for;

for i = 0 to s,

where $s = \max\{L(F_n)\} + 2$ for k =1
to $L^1(F_n)|$

if $|i - l_k| \geq 2$, then $L^{1vl}(F_n) = \{i\}$ //where $l_k \in L^1(F_n)$//

end for;

end for;

for k = 1 to 2

for $m = 1$ to $|L^{kvl}(F_n)|$

for $q = 1$ to $|L^{k+1}(F_n)|$

if $|l_m - p_q| \geq 1$, then $L^{k+1vl}(F_n) = \{l_m\}$ //where $l_m \in L^{k+1vl}(F_n)$ and
$p_q \in L^{k+1}(F_n)$ //

end for;

end for;

$f_n = \min\{L^{3vl}(F_n)\}$;

$L = L(F_n) \cup \{f_n\}$;

end L211.

Theorem 2. The Algorithm L211 correctly labels the nodes of a CAG using L211L condition.

Proof. Let $F = \{F_1, F_2, F_3, \ldots, F_n\}$, also let $f_1 = 0$, $L(F_2) = \{0\}$. Suppose, we are going to label the arc $F_j \in F$. $L^{kvl}(F_j)$ is the non empty largest set satisfying the condition of distance 1, ..., k for k = 1, 2, 3, of L211L, where $l \leq p$ for all $l \in L^{kvl}(F_j)$ and $p = \max\{L(F_j)\} + 2$, for any $F_j \in A$ and $k = 1,2,3$ (by Lemma 3). Also, no label $l \in L^{3vl}(F_j)$ and $l \leq p$ satisfying the condition of L211L of graph. So the labels on the set are the only valid labels for F_j, which is less than or equal to p and satisfying L211L condition.

Our aim is to label the arc F_j by using as few labels as possible, satisfying L211L condition. So $f_j = q$, where $q = \min\{L^{3vl}(F_j)\}$. Now q is the least label for F_j, because no label less than q satisfies L211L condition. Since, F_j is arbitrary so this algorithm spent minimum number of labels to label any arc of a CAG satisfying L211L condition and $\lambda_{2,1,1}(G) = \max\{L(F_n) \cup \{f_n\}\}$.

Theorem 3. A CAG can be L211-labeled using $O(n\Delta^2)$ time.

Proof. Let L be the label set and $|L|$ be its cardinality. According to the algorithm L211, $|L^i(F_k)| \leq |L|$ for $i = 1,2,3$, for any $F_k \in A$, and also $r \leq 6\Delta - 2$, where $r = \max\{L(F_j)\} + 2$. So we can compute $L^{1vl}(F_j)$ using at most $|L|(6\Delta - 2)$ time, i.e. using at most $O(\Delta|L|)$ time. Also, $|L^{kvl}(F_j)| \leq 6\Delta - 4$ for $k = 1,2$, so for each k = 1, 2, $L^{k+1vl}(F_j)$ can be computed using at most $|L|(6\Delta - 4)$ time, i.e. using at most $O(\Delta|L|)$ time. This process is repeated for n−1 times. So the time complexity for the algorithm L211 is $O((n-1)\Delta|L|) = O(n\Delta|L|)$. Since, $|L| \leq 6\Delta - 3$, therefore the running time for the algorithm L211 is $O(n\Delta^2)$.

Illustration of Algorithm L211

Let us consider the CAG of Fig. 3 to illustrate Algorithm L211.

For this graph, $F = \{F_1, F_2, F_3, \ldots, F_{13}\}$ and $\Delta = 3$. f_j = the label of the arc F_j, for j = 1,2,3,...,13. $f_1 = 0$, $L(F_2) = \{0\}$.

Iteration 1: j = 2.

$L^1(F_2) = \{0\}$, $L^2(F_2) = \varphi$, $L^3(F_2) = \varphi$.

$L^{1vl}(F_2) = \{2\}$, $L^{2vl}(F_2) = \{2\}$, $L^{3vl}(F_2) = \{2\}$.

Therefore, $f_2 = \min\{L^{3vl}(F_2)\} = 2$ and $L(F_3) = L(F_2) \cup \{f_2\} = \{0\} \cup \{2\} = \{0, 2\}$.

Iteration 2: j = 3.

$L^1(F_3) = \{0\}$, $L^2(F_3) = \{2\}$, $L^3(F_3) = \varphi$.

$L^{1vl}(F_3) = \{2, 3, 4\}$, $L^{2vl}(F_3) = \{3, 4\}$, $L^{3vl}(F_3) = \{3, 4\}$.

So $f_3 = \min\{L^{3vl}(F_3)\} = 3$ and $L(F_4) = L(F_3) \cup \{f_3\} = \{0, 2, \} \cup \{5\} = \{0, 2, 3\}$.

Iteration 3: j = 4.

$L^1(F_4) = \{3\}$, $L^2(F_4) = \{0\}$, $L^3(F_4) = \{2\}$.

$L^{1vl}(F_4) = \{0, 1, 5\}$, $L^{2vl}(F_4) = \{1, 5\}$, $L^{3vl}(F_4) = \{1, 5\}$.

Therefore, $f_4 = \min\{L^{3vl}(F_4)\} = 1$ and $L(F_5) = L(F_4) \cup \{f_4\} = \{0, 2, 3\} \cup \{1\} = \{0, 1, 2, 3\}$.

Iteration 4: j = 5.
$L^1(F_5) = \{1, 3\}$, $L^2(F_5) = \{0, 2\}$, $L^3(F_5) = \varphi$.
$L^{1vl}(F_5) = \{5\}$, $L^{2vl}(F_5) = \{5\}$, $L^{3vl}(F_5) = \{5\}$.
Therefore, $f_5 = \min\{L^{3vl}(F_5)\} = 5$ and $L(F_6) = L(F_5) \cup \{f_5\} = \{0, 1, 2, 3\} \cup \{5\} =\{0, 1, 2, 3, 5\}$.

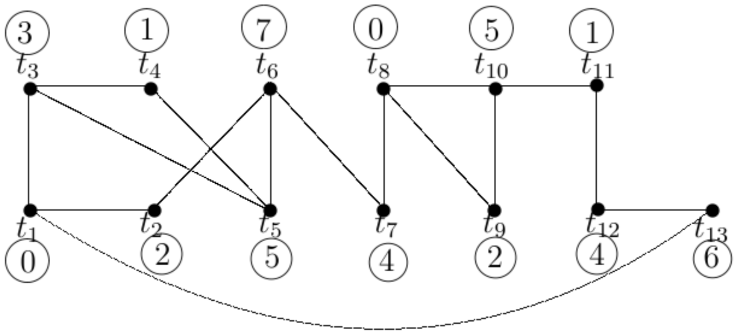

Fig. 3. A CAG labeled by L211L, the number within the circle represents the label of the corresponding nodes.

In this way $f_6 = 7$, $f_7 = 4$ $f_8 = 0$, $f_9 = 2$, $f_{10} = 5$, $f_{11} = 1$, $f_{12} = 4$ and $f_{13} = 6$.

The nodes and the label of the corresponding nodes are shown below:

Nodes	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}
L211-labels	0	2	3	1	5	7	4	0	2	5	1	3	6

4. Conclusion

In this article, we have computed the upper bound of L211L for CAG, and have proved that $\lambda_{2,1,1}(G) \leq 6\Delta - 4$. This upper bound is very closed to the exact value of L211L number of CAG and this is the first upper bound for CAG. Also, an algorithms is designed to L211-label for CAGs. The time complexity for this algorithm is $O(n\Delta^2)$.

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