# A New Computational Approach for Solving Fractional Order Telegraph Equations

**A. Devi\*, M. Jakhar**

Department of Mathematics, NIILM University, Kaithal, Haryana-136027, India

## Abstract

In this work, a modified decomposition method namely Sumudu-Adomian Decomposition Method (SADM) is implemented to find the exact and approximate solutions of fractional order telegraph equations. The derivatives of fractional-order are expressed in terms of caputo operator. Some numerical examples are illustrated to examine the efficiency of the proposed technique. Solutions of fractional order telegraph equations are obtained in the form of a series solution. It is observed that the solutions of fractional order telegraph equations converge towards the solution of an integer-order problem, which confirmed the reliability of the suggested method.

*Keywords*: Sumudu transform; Sumudu-Adomian decomposition method; Fractional order telegraph equations; Caputo fractional derivative.

## 1. Introduction

In last few years, the numbers of studies on fractional partial differential equations (FPDEs) have increased dramatically since they can be used in many areas of science and engineering. The theory of fractional partial differential equations helps to translate the real word problems in a better and systematic manner. FPDEs have been solved out by many researchers through several techniques such as iterative Laplace transform method, Adomian decomposition method, Homotopy analysis method, Homotopy perturbation method and many more.

It would be difficult to imagine a world without communication systems. The transmission of a signal from one point to another is a typical engineering problem. A transmission media is part of the circuit and represents a physical system that directly propagates the signal between two or more points. Telegraph equations is generally used to describe electrical phenomenon in a more practical approach by using short segments of an electrical system. For analysis of distributed electrical parameters of any electrical system, Telegraph equation is being used because analysis of electrical parameters in lumped form is very difficult. The Telegraph equation was developed by Oliver Heaviside [1] in the 1880s, which describes the distance and time on an electric transmission line

---

\* *Corresponding author*: anju.gill11@gmail.com

with voltage and current. The Telegraph equation has many applications in fields such as wireless signals, telephone lines, radio frequency and microwave transmission [2]. In view of the above applications, mathematicians have developed different analytic and numerical methods for solving fractional order telegraph equations. In this connection, the numerical solution of one-dimensional linear hyperbolic telegraph equation, have obtained by well-known Homotopy perturbation method and Laplace transform [3]. The numerical solution for fractional model of telegraph equation by using q-homotopy analysis transform method (q-HATM) was discussed in reference [4]. Adomain decomposition method (ADM) and Modified ADM are used to obtain the solution of nonlinear telegraph equation [5]. A new fractional Homotopy analysis transform method was successfully applied to find the solution of space-fractional telegraph equation [6]. Prakash [7] have applied Homotopy perturbation transform method to find the analytical solution of space-fractional telegraph equation. Inc *et al*. [8] have obtained the numerical solutions of the second-order one-dimensional telegraph equation based on Reproducing Kernel Hilbert Space Method. Inc *et al*. [9] have established explicit solution of telegraph equation based on Reproducing Kernel Method. Modanli *et al*. [10] have studied the numerical solution of fractional telegraph differential equations by Theta-method. Akgül *et al*. [11] have demonstrated the existence of unique solutions to the non-homogeneous telegraph equation in binary reproducing kernel Hilbert spaces. Modanli *et al*. [12] have presented the solutions of fractional order telegraph partial differential equation by Crank-Nicholson Finite Difference Method. Atangana *et al*. [13] have successfully obtained the transfer function and Bode diagram from Sumudu transform. Devi *et al*. [14] presented a reliable computational algorithm for solving fractional biological population model which is very beneficial in the present work. Manjare *et al*. [15] introduced Sumudu decomposition method for solving fractional Bratu-type differential equation.

In the present work, authors will be applying Sumudu-Adomian Decomposition Method (SADM) to solve the following type of fractional order multi-dimensional Telegraph equations [4,16].
The 1-D space-time fractional telegraph equation is defined as

$$\frac{\partial^{2\alpha}\theta}{\partial t^{2\alpha}} + 2a\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + b^2\theta = \frac{\partial^{2\beta}\theta}{\partial x^{2\beta}} + g(x,t),\ 0 < \alpha \le 1,\ 0 < \beta \le 1,$$

(1)

with ICs and BCs

$$\theta(x,0) = \phi_1(x),\ \theta_t(x,0) = \phi_2(x),$$
$$\theta(0,t) = \phi_1(t),\ \theta_x(0,t) = \phi_2(t).$$

Also 2-D time-fractional telegraph equation is

$$\frac{\partial^{2\alpha}\theta}{\partial t^{2\alpha}} + 2a\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + b^2\theta = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + g(x,y,t),\quad 0 < \alpha \le 1,$$

(2)

with ICs and BCs
$$\theta(x,y,0) = \psi_1(x,y),\ \theta_t(x,y,0) = \psi_2(x,y).$$
Similarly, 3-D time-fractional telegraph equation is given as

$$\frac{\partial^{2\alpha}\theta}{\partial t^{2\alpha}} + 2a\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + b^2\theta = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} + g(x,y,z,t), 0 < \alpha \leq 1 \qquad (3)$$

with ICs and BCs

$\theta(x,y,z,0) = \xi_1(x,y,z), \ \theta_t(x,y,z,0) = \xi_2(x,y,z).$

In (1), (2) and (3); $a$ and $b$ denote positive constants and $g, \phi_1, \phi_2, \psi_1, \psi_2, \xi_1, \ \xi_2$ are known continuous functions.

Here, the accuracy and efficiency of the proposed method are demonstrated by the five test examples.

## 2. Preliminary Results Required in the Sequel

Sumudu transform was given by Watugala [17], which is defined on the set of functions

$$A = \left\{ f(t)/\exists \ M, \tau_j > 0, \ j = 1,2 \ |f(t)| \leq Me^{\frac{|t|}{\tau_j}} \ if \ t \in (-1)^j \times [0,\infty) \right\}, \forall \ t \geq 0, \quad (4)$$

as

$$S[f(t); u] = F(u) = \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} f(t)dt, \quad f(t) \in A, u \in (-\tau_1, \tau_2) \qquad (5)$$

For more details about this transform, refer to the references [18-20].

Fractional integral (right-sided) of order $\alpha$ of Riemann-Liouville type [21], is

$$I_a^\alpha(\theta(x,t)) = {}^{RL}_a D_t^{-\alpha}(\theta(x,t)) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}\theta(x,t)d\tau, (t > a), R(\alpha) > 0. \ (6)$$

Here, $\Gamma(\cdot)$ denotes the Gamma function

$$\Gamma(\omega) = \int_0^\infty e^{-x} x^{\omega-1}dx, \quad \omega \in \mathbb{C}. \qquad (7)$$

Fractional derivative (right-sided) of order $\alpha$ of Riemann-Liouville type [21], is

$$^{RL}_a D_t^\alpha(\theta(x,t)) = \left(\frac{d}{dt}\right)^n \left(I_a^{n-\alpha}\theta(x,t)\right) \qquad (R(\alpha) > 0, \ n = [R(\alpha)] + 1), \qquad (8)$$

Here $[\cdot]$ is the integral part of $\alpha$.

Caputo [22], introduced fractional derivative of order $R(\alpha) > 0$ as

$$^C_0 D_t^\alpha(\theta(x,t)) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{\theta^m(x,\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha \leq m, R(\alpha) > 0, m \in N, \\ \frac{\partial^m}{\partial t^m}\theta(x,t), & if \ \alpha = m, \end{cases} \qquad (9)$$

The Sumudu transform of (9) is given in [23], as

$$S\left[{}^C_0 D_t^\alpha(\theta(x,t)); u\right] = u^{-\alpha}\bar{\theta}(x,u) - \sum_{k=0}^{m-1}\frac{\theta^{(k)}(x,0)}{u^{\alpha-k}}, \quad (m-1 < \alpha \leq m) \qquad (10)$$

Where $\bar{\theta}(x,u)$ is the Sumudu transform of $\theta(x,t)$.

## 3. A Computational Technique

Here, authors proposed a new computational method, called Sumudu-Adomian Decomposition Method (SADM) for solving FPDEs

$$D^\alpha\theta(x,t) + L\theta(x,t) + N\theta(x,t) = g(x,t), \quad x,t \geq 0, \ m-1 < \alpha \leq m, \qquad (11)$$

Having the initial conditions

$$\theta(x,0) = f(x). \qquad (12)$$

In eq. (11), $D^\alpha = D_t^\alpha = {}_0^C D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative operator, $m \in N$(Natural Number), L and $N$ represents linear and nonlinear terms respectively and $g(x, t)$ represents source term.

Taking Sumudu transform of (11) and using various results, as given in literature [18-20], it yields

$$S[D^\alpha \theta(x,t)] + S[L\theta(x,t)] + S[N\theta(x,t)] = S[g(x,t)], \tag{13}$$

on using (10), get as

$$u^{-\alpha}S[\theta(x,t)] - u^{-\alpha}\theta(x,0) = S[g(x,t)] - S[L\theta(x,t) + N\theta(x,t)], \tag{14}$$

again, using ICs (12), eq. (14) reduces to

$$S[\theta(x,t)] = f(x) + u^\alpha S[g(x,t)] - u^\alpha S[L\theta(x,t) + N\theta(x,t)]. \tag{15}$$

The Sumudu-Adomian Decomposition Method represents the solution as an infinite series, as

$$\theta(x,t) = \sum_{i=0}^{\infty} \theta_i(x,t), \tag{16}$$

Specific algorithms of Adomian Decomposition Method is reported in literature [24,25] and the non-linear terms (if any) in the problem are defined by the infinite series of Adomian polynomials [26,27] in the form

$$N\theta(x,t) = \sum_{i=0}^{\infty} A_i, \tag{17}$$

$$A_i = \frac{1}{i!}\left[\frac{d^i}{d\lambda^i}[N\sum_{i=0}^{\infty}(\lambda^i\theta_i)]\right]_{i=0}, \quad i = 0,1,2,3\ldots \tag{18}$$

Substituting eq. (16) and (17) into eq. (15), get as

$$S[\sum_{i=0}^{\infty}\theta_i(x,t)] = f(x) + u^\alpha S[g(x,t)] - u^\alpha S[L\sum_{i=0}^{\infty}\theta_i(x,t) + \sum_{i=0}^{\infty}A_i]. \tag{19}$$

Taking inverse Sumudu transform on (19), gives

$$\sum_{i=0}^{\infty}\theta_i(x,t) = S^{-1}[f(x) + u^\alpha S[g(x,t)]] - S^{-1}[u^\alpha S[L\sum_{i=0}^{\infty}\theta_i(x,t) + \sum_{i=0}^{\infty}A_i]]. \tag{20}$$

On using the SADM, yields the following iterative algorithm

$$\theta_0(x,t) = S^{-1}[f(x) + u^\alpha S[g(x,t)]], \tag{21}$$

$$\theta_1(x,t) = -S^{-1}[u^\alpha S[L\theta_0(x,t) + A_0]], \tag{22}$$

Generally, can be write as

$$\theta_{i+1}(x,t) = -S^{-1}[u^\alpha S[L\theta_i(x,t) + A_i]], \quad i \geq 1 \tag{23}$$

The SADM solution for eq. (11) is

$$\theta(x,t) = \sum_{i=0}^{\infty}\theta_i(x,t). \tag{24}$$

The series solution (24) converges very fast in a very few terms.

## 4. Numerical Experiments

This part of the manuscript contains the application of Sumudu-Adomian Decomposition Method to find the solution of fractional order telegraph equations to check the effectiveness and usefulness of the proposed computational method.

**Example 1.** Consider the following time-fractional linear telegraph equation

$$\frac{\partial^{2\alpha}\theta}{\partial t^{2\alpha}} + 2\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + \theta = \frac{\partial^2\theta}{\partial x^2}, \qquad 0 < \alpha \leq 1, t \geq 0, \tag{25}$$

with ICs

$$\theta(x,0) = e^x, \quad \theta_t(x,0) = -2e^x. \tag{26}$$

Applying Sumudu transform on (25) and simultaneously using ICs (26), it gives

$$S\left[\frac{\partial^{2\alpha}\theta}{\partial t^{2\alpha}}\right] = -S\left[2\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + \theta - \frac{\partial^2\theta}{\partial x^2}\right],$$

$$u^{-2\alpha}S[\theta(x,t)] - u^{-2\alpha}\theta(x,0) - u^{-2\alpha+1}\theta_t(x,0) = -S\left[2\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + \theta - \frac{\partial^2\theta}{\partial x^2}\right],$$

$$S[\theta(x,t)] = e^x - 2e^x u - \frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + \theta - \frac{\partial^2\theta}{\partial x^2}\right],$$

Applying the inverse Sumudu transform

$$\theta(x,t) = S^{-1}\left[e^x - 2e^x u - \frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + \theta - \frac{\partial^2\theta}{\partial x^2}\right]\right],$$

Using the SADM process, get as

$$\theta_0(x,t) = S^{-1}[e^x - 2e^x u]$$

$$\theta_0(x,t) = e^x(1 - 2t)$$

and

$$\theta_{i+1}(x,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^{\alpha}\theta_i}{\partial t^{\alpha}} + \theta_i - \frac{\partial^2\theta_i}{\partial x^2}\right]\right], \quad i = 0,1,2,3\ldots \tag{27}$$

for $i = 0$, eq. (27) gives

$$\theta_1(x,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^{\alpha}\theta_0}{\partial t^{\alpha}} + \theta_0 - \frac{\partial^2\theta_0}{\partial x^2}\right]\right]$$

$$\theta_1(x,t) = -S^{-1}\left[\frac{2e^x}{u^{-2\alpha}}S\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}}(1 - 2t)\right]\right]$$

$$\theta_1(x,t) = 4e^x S^{-1}[u^{\alpha+1}]$$

$$\theta_1(x,t) = 4e^x\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

for $i = 1$, eq. (27) gives

$$\theta_2(x,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^{\alpha}\theta_1}{\partial t^{\alpha}} + \theta_1 - \frac{\partial^2\theta_1}{\partial x^2}\right]\right]$$

$$\theta_2(x,t) = -S^{-1}\left[\frac{8e^x}{u^{-2\alpha}\Gamma(\alpha+2)}S\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}}t^{\alpha+1}\right]\right]$$

$$\theta_2(x,t) = -8e^x \, S^{-1}[u^{2\alpha+1}]$$

$$\theta_2(x,t) = -8e^x \, \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

Similarly, subsequent components are

$$\theta_3(x,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}} S\left[2\frac{\partial^\alpha \theta_2}{\partial t^\alpha} + \theta_2 - \frac{\partial^2 \theta_2}{\partial x^2}\right]\right]$$

$$\theta_3(x,t) = 16e^x \, \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)},$$

$$\theta_4(x,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}} S\left[2\frac{\partial^\alpha \theta_3}{\partial t^\alpha} + \theta_3 - \frac{\partial^2 \theta_3}{\partial x^2}\right]\right]$$

$$\theta_4(x,t) = -32e^x \, \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}$$

On using eq. (24), SADM solution for example 1, is obtained in the form of series

$$\theta(x,t) = \sum_{i=0}^{\infty} \theta_i(x,t)$$

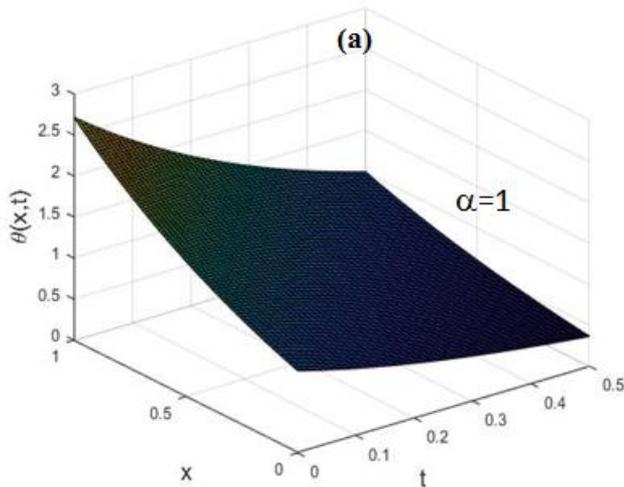$$\theta(x,t) = \theta_0(x,t) + \theta_1(x,t) + \theta_2(x,t) + \theta_3(x,t) + \theta_4(x,t) + \cdots$$

$$\theta(x,t) = e^x\left[1 - 2t + 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 8\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 16\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - 32\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} + \cdots\right]. \quad (28)$$

Putting $\alpha = 1$ in (28), get SADM solution for example 1, as

$$\theta(x,t) = e^x\left[1 - 2t + \frac{(2t)^2}{\Gamma(3)} - \frac{(2t)^3}{\Gamma(4)} + \frac{(2t)^4}{\Gamma(5)} - \frac{(2t)^5}{\Gamma(6)} + \cdots\right]. \quad (29)$$

This solution is equivalent to exact solution
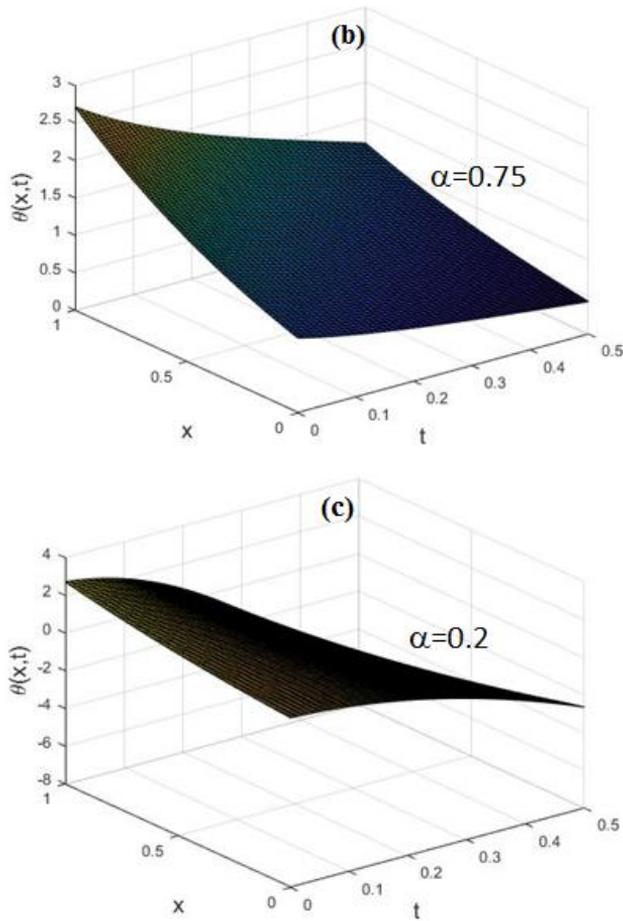
$$\theta(x,t) = e^{x-2t}. \quad (30)$$



(a)

Fig. 1. The surface shows the SADM solution $\theta(x,t)$ for example 1, when (a) $\alpha = 1$, (b) $\alpha = 0.75$, (c) $\alpha = 0.2$.

**Example 2.** Consider the following space-fractional linear telegraph equation

$$\frac{\partial^\alpha \theta}{\partial x^\alpha} = \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} + \theta , \qquad 1 < \alpha \le 2, t \ge 0, \tag{31}$$

with initial conditions

$$\theta(0,t) = e^{-t}, \quad \theta_x(0,t) = e^{-t}. \tag{32}$$

Taking the Sumudu transform on (31) and simultaneously using ICs (32), it yields

$$S\left[\frac{\partial^\alpha \theta}{\partial x^\alpha}\right] = S\left[\frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} + \theta\right],$$

after little simplification, get as

$$u^{-\alpha} S[\,\theta(x,t)] - u^{-\alpha}\theta(0,t) - u^{-\alpha+1}\theta_x(0,t) = S\left[\frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} + \theta\right],$$

$$\Rightarrow S[\,\theta(x,t)] = e^{-t} + ue^{-t} + \frac{1}{u^{-\alpha}} S\left[\frac{\partial^2\theta}{\partial t^2} + \frac{\partial\theta}{\partial t} + \theta\right],$$

again, applying the inverse Sumudu transform, it gives

$$\theta(x,t) = S^{-1}[e^{-t} + ue^{-t}] + S^{-1}\left[\frac{1}{u^{-\alpha}} S\left[\frac{\partial^2\theta}{\partial t^2} + \frac{\partial\theta}{\partial t} + \theta\right]\right],$$

using the SADM process, it gives

$$\theta_0(x,t) = S^{-1}[e^{-t} + ue^{-t}]$$

$$\theta_0(x,t) = e^{-t}(1+x)$$

and

$$\theta_{i+1}(x,t) = S^{-1}\left[\frac{1}{u^{-\alpha}} S\left[\frac{\partial^2\theta_i}{\partial t^2} + \frac{\partial\theta_i}{\partial t} + \theta_i\right]\right], \quad i = 0,1,2,3\ldots \tag{33}$$

for $i = 0$, eq. (33) reduces

$$\theta_1(x,t) = S^{-1}\left[\frac{1}{u^{-\alpha}} S\left[\frac{\partial^2\theta_0}{\partial t^2} + \frac{\partial\theta_0}{\partial t} + \theta_0\right]\right]$$

$$\theta_1(x,t) = S^{-1}\left[\frac{1}{u^{-\alpha}} S[e^{-t}(1+x)]\right]$$

$$\theta_1(x,t) = e^{-t} S^{-1}[u^\alpha + u^{\alpha+1}]$$

$$\theta_1(x,t) = e^{-t}\frac{x^\alpha}{\Gamma(\alpha+1)} + e^{-t}\frac{x^{\alpha+1}}{\Gamma(\alpha+2)},$$

The subsequent components are

$$\theta_2(x,t) = S^{-1}\left[\frac{1}{u^{-\alpha}} S\left[\frac{\partial^2\theta_1}{\partial t^2} + \frac{\partial\theta_1}{\partial t} + \theta_1\right]\right]$$

$$\theta_2(x,t) = e^{-t}\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + e^{-t}\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

$$\theta_3(x,t) = S^{-1}\left[\frac{1}{u^{-\alpha}} S\left[\frac{\partial^2\theta_2}{\partial t^2} + \frac{\partial\theta_2}{\partial t} + \theta_2\right]\right]$$

$$\theta_3(x,t) = e^{-t}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + e^{-t}\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)},$$

On using eq. (24), SADM solution for example 2, is obtained in the form of series

$$\theta(x,t) = \theta_0(x,t) + \theta_1(x,t) + \theta_2(x,t) + \theta_3(x,t) + \theta_4(x,t) + \cdots$$

$$\theta(x,t) = e^{-t}\left[1 + x + \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots\right].$$
(34)

Putting $\alpha = 2$ in (34), get SADM solution for example 2, as

$$\theta(x,t) = e^{-t}\left[1 + x + \frac{x^2}{\Gamma(3)} + \frac{x^3}{\Gamma(4)} + \frac{x^4}{\Gamma(5)} + \frac{x^5}{\Gamma(6)} + \frac{x^6}{\Gamma(7)} + \frac{x^7}{\Gamma(8)} + \cdots\right]. \tag{35}$$

This solution is equivalent to exact solution
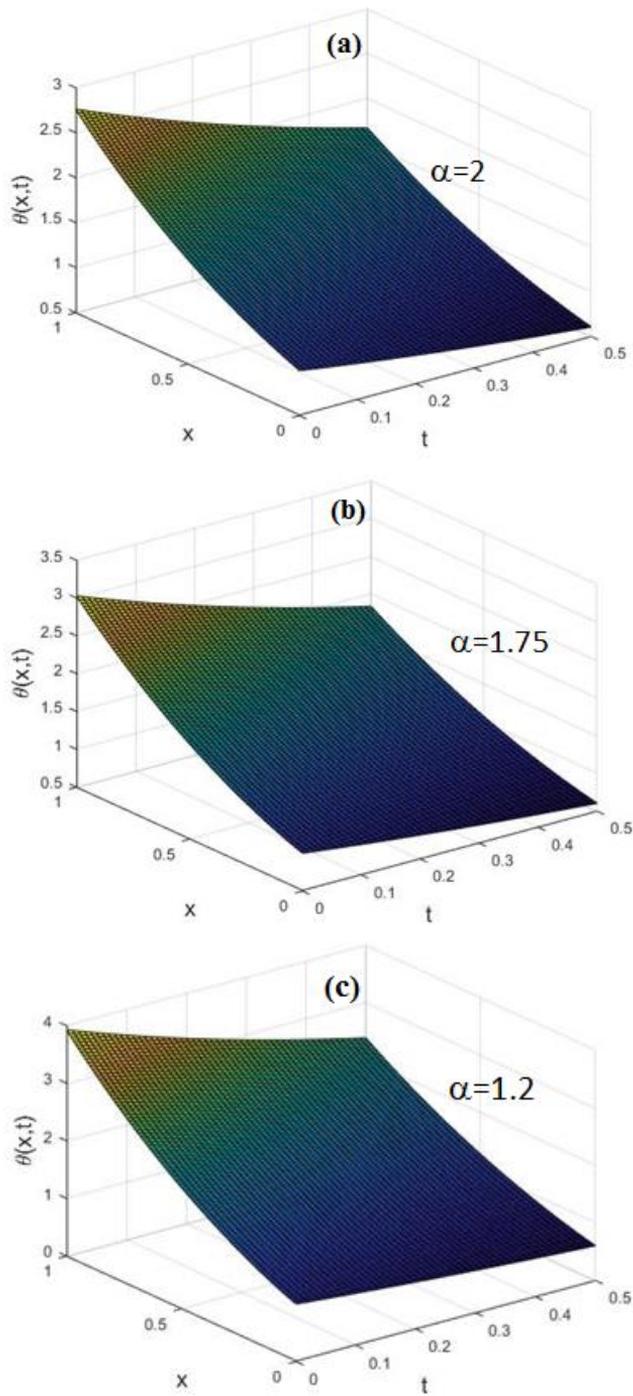
$$\theta(x,t) = e^{x-t}. \tag{36}$$

Fig. 2. The surface shows the SADM solution $\theta(x, t)$ for example 2, when (a) $\alpha = 2$, (b) $\alpha = 1.75$, (c) $\alpha = 1.2$.

**Example 3.** Consider the space-fractional linear telegraph equation of non-homogeneous type

$$\frac{\partial^\alpha \theta}{\partial x^\alpha} = \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} + \theta - x^2 - t + 1, \qquad 1 < \alpha \le 2, t \ge 0 \tag{37}$$

with initial condition

$$\theta(0, t) = t, \quad \theta_x(0, t) = 0. \tag{38}$$

Taking the Sumudu transform on (37) and simultaneously using ICs (38), it yields

$$S\left[\frac{\partial^\alpha \theta}{\partial x^\alpha}\right] = S\left[\frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} + \theta - x^2 - t + 1\right],$$

$$\Rightarrow u^{-\alpha} S[\theta(x, t)] - u^{-\alpha} \theta(0, t) - u^{-\alpha+1} \theta_x(0, t) = -2u^2 - t + 1 + S\left[\frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} + \theta\right],$$

$$\Rightarrow S[\theta(x, t)] = t - 2u^{\alpha+2} - tu^\alpha + u^\alpha + \frac{1}{u^{-\alpha}} S\left[\frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} + \theta\right],$$

again, applying the inverse Sumudu transform, it gives

$$\theta(x, t) = S^{-1}[t - 2u^{\alpha+2} - tu^\alpha + u^\alpha] + S^{-1}\left[\frac{1}{u^{-\alpha}} S\left[\frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} + \theta\right]\right],$$

using the SADM process, it gives

$$\theta_0(x, t) = S^{-1}[t - 2u^{\alpha+2} - tu^\alpha + u^\alpha]$$

$$\theta_0(x, t) = t - \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{tx^\alpha}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)},$$

and

$$\theta_{i+1}(x, t) = S^{-1}\left[\frac{1}{u^{-\alpha}} S\left[\frac{\partial^2 \theta_i}{\partial t^2} + \frac{\partial \theta_i}{\partial t} + \theta_i\right]\right], \quad i = 0, 1, 2, 3 \ldots \tag{39}$$

for $i = 0$, eq. (39) gives

$$\theta_1(x, t) = S^{-1}\left[\frac{1}{u^{-\alpha}} S\left[\frac{\partial^2 \theta_0}{\partial t^2} + \frac{\partial \theta_0}{\partial t} + \theta_0\right]\right]$$

$$\theta_1(x, t) = \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{tx^\alpha}{\Gamma(\alpha+1)} - \frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{tx^{2\alpha}}{\Gamma(2\alpha+1)},$$

for $i = 1$, eq. (39) gives

$$\theta_2(x, t) = S^{-1}\left[\frac{1}{u^{-\alpha}} S\left[\frac{\partial^2 \theta_1}{\partial t^2} + \frac{\partial \theta_1}{\partial t} + \theta_1\right]\right]$$

$$\theta_2(x, t) = \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{tx^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{2x^{3\alpha+2}}{\Gamma(3\alpha+3)} - \frac{tx^{3\alpha}}{\Gamma(3\alpha+1)},$$

Similarly, will find out the components ahead as authors have found out the previous components. On using eq. (24), SADM solution for example 3 is obtained in the form of series

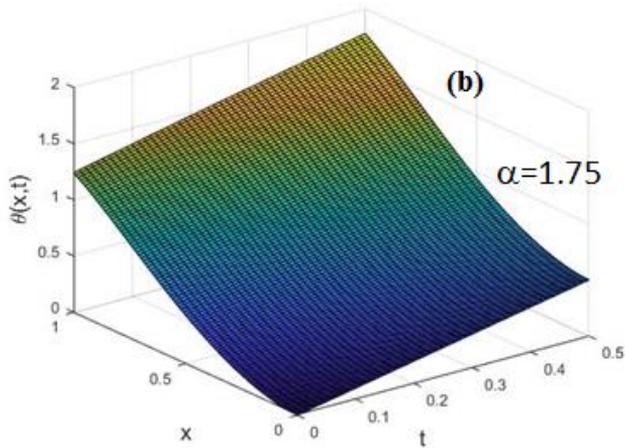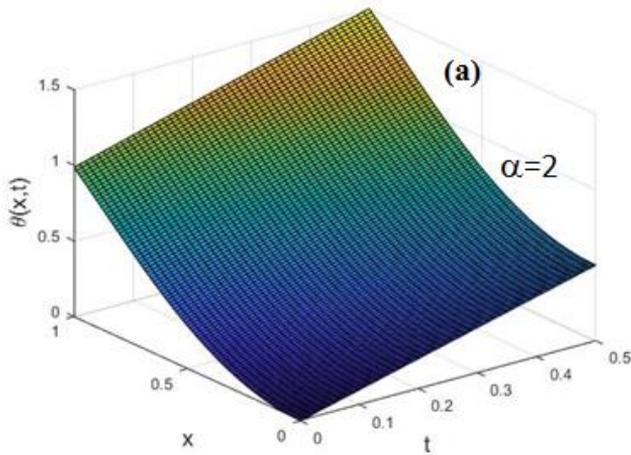$$\theta(x, t) = \theta_0(x, t) + \theta_1(x, t) + \theta_2(x, t) + \theta_3(x, t) + \cdots$$

$$\theta(x,t) = t - \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{tx^\alpha}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{tx^\alpha}{\Gamma(\alpha+1)} - \frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{tx^{2\alpha}}{\Gamma(2\alpha+1)} +$$

$$\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{tx^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{2x^{3\alpha+2}}{\Gamma(3\alpha+3)} - \frac{tx^{3\alpha}}{\Gamma(3\alpha+1)} \dots, \tag{40}$$

Putting $\alpha = 2$ in (40), we get SADM solution for example 3, as

$$\theta(x,t) = t - \frac{2x^4}{\Gamma(5)} - \frac{tx^2}{\Gamma(3)} + \frac{x^2}{\Gamma(3)} + \frac{x^2}{\Gamma(3)} + \frac{tx^2}{\Gamma(3)} - \frac{2x^6}{\Gamma(7)} - \frac{tx^4}{\Gamma(5)} + \frac{2x^4}{\Gamma(5)} - \frac{x^6}{\Gamma(7)} + \frac{tx^4}{\Gamma(5)} - \frac{2x^8}{\Gamma(9)} -$$

$$\frac{tx^6}{\Gamma(7)} \dots \ , \tag{41}$$

this solution is equivalent to exact solution
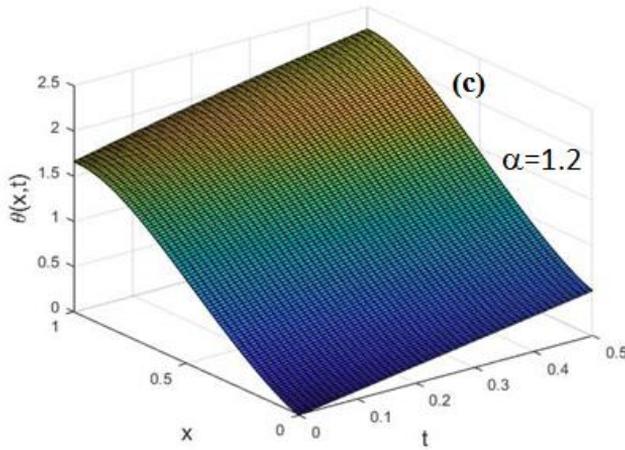
$$\theta(x,t) = t + x^2. \tag{42}$$

Fig. 3. The surface shows the SADM solution $\theta(x,t)$ for example 3, when (a) $\alpha = 2$, (b) $\alpha = 1.75$, (c) $\alpha = 1.2$.

**Example 4.** Next, time-fractional telegraph equation of linear type

$$\frac{\partial^{2\alpha}\theta}{\partial t^{2\alpha}} + 3\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + 2\theta = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}, \qquad 0 < \alpha \leq 1, t \geq 0, \tag{43}$$

with initial condition

$$\theta(x,y,0) = e^{x+y}, \quad \theta_t(x,y,0) = -3e^{x+y}. \tag{44}$$

Taking the Sumudu transform on (43) and simultaneously using ICs (44), it yields

$$S\left[\frac{\partial^{2\alpha}\theta}{\partial t^{2\alpha}}\right] = -S\left[3\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + 2\theta - \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2}\right],$$

$$\Longrightarrow u^{-2\alpha}S[\,\theta(x,y,t)] - u^{-2\alpha}\theta(x,y,0) - u^{-2\alpha+1}\theta_t(x,y,0) = -S\left[3\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + 2\theta - \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2}\right],$$

$$\Longrightarrow S[\theta(x,y,t)] = e^{x+y} - 3ue^{x+y} - \frac{1}{u^{-2\alpha}}S\left[3\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + 2\theta - \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2}\right],$$

again, applying the inverse Sumudu transform

$$\theta(x,y,t) = S^{-1}[e^{x+y} - 3ue^{x+y}] - S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[3\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + 2\theta - \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2}\right]\right],$$

using the SADM process, it gives

$$\theta_0(x,y,t) = S^{-1}[e^{x+y} - 3e^{x+y}u]$$

$$\theta_0(x,y,t) = e^{x+y}(1 - 3t),$$

and

$$\theta_{i+1}(x,y,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[3\frac{\partial^{\alpha}\theta_i}{\partial t^{\alpha}} + 2\theta_i - \frac{\partial^2\theta_i}{\partial x^2} - \frac{\partial^2\theta_i}{\partial y^2}\right]\right], \quad i = 0, 1, 2, 3 \dots \tag{45}$$

for $i = 0$, eq. (45) redues

$$\theta_1(x,y,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[3\frac{\partial^\alpha\theta_0}{\partial t^\alpha} + 2\theta_0 - \frac{\partial^2\theta_0}{\partial x^2} - \frac{\partial^2\theta_0}{\partial y^2}\right]\right],$$

$$\theta_1(x,y,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S[-9e^{x+y}\,u^{1-\alpha}]\right]$$

$$\theta_1(x,y,t) = 9e^{x+y}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

after putting different values of $i$ in eq. (45), get as

$$\theta_2(x,y,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[3\frac{\partial^\alpha\theta_1}{\partial t^\alpha} + 2\theta_1 - \frac{\partial^2\theta_1}{\partial x^2} - \frac{\partial^2\theta_1}{\partial y^2}\right]\right],$$

$$\theta_2(x,y,t) = -27e^{x+y}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

similarly

$$\theta_3(x,y,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[3\frac{\partial^\alpha\theta_2}{\partial t^\alpha} + 2\theta_2 - \frac{\partial^2\theta_2}{\partial x^2} - \frac{\partial^2\theta_2}{\partial y^2}\right]\right],$$

$$\theta_3(x,y,t) = 81e^{x+y}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)},$$

On using eq. (24), SADM solution for example 4, is obtained in the form of series

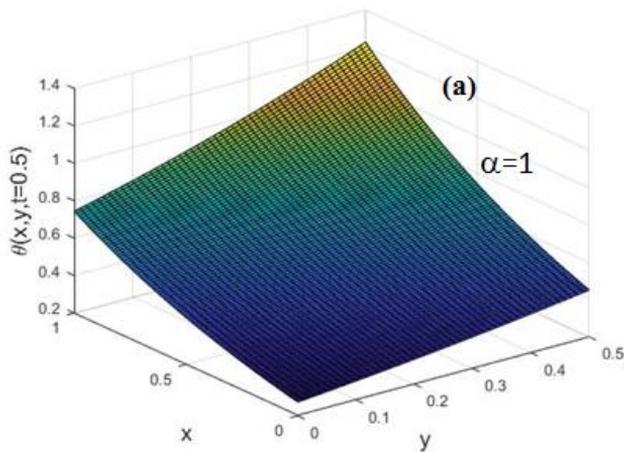$$\theta(x,y,t) = \theta_0(x,y,t) + \theta_1(x,y,t) + \theta_2(x,y,t) + \theta_3(x,y,t) + \theta_4(x,y,t) + \cdots$$

$$\theta(x,y,t) = e^{x+y}\left[1 - 3t + \frac{9t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{27t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{81t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \cdots\right]. \tag{46}$$

Putting $\alpha = 1$ in (46), we get SADM solution for example 4, as

$$\theta(x,y,t) = e^{x+y}\left[1 - 3t + \frac{(3t)^2}{\Gamma(3)} - \frac{(3t)^3}{\Gamma(4)} + \frac{(3t)^4}{\Gamma(5)} - \cdots\right], \tag{47}$$

this solution is equivalent to exact solution

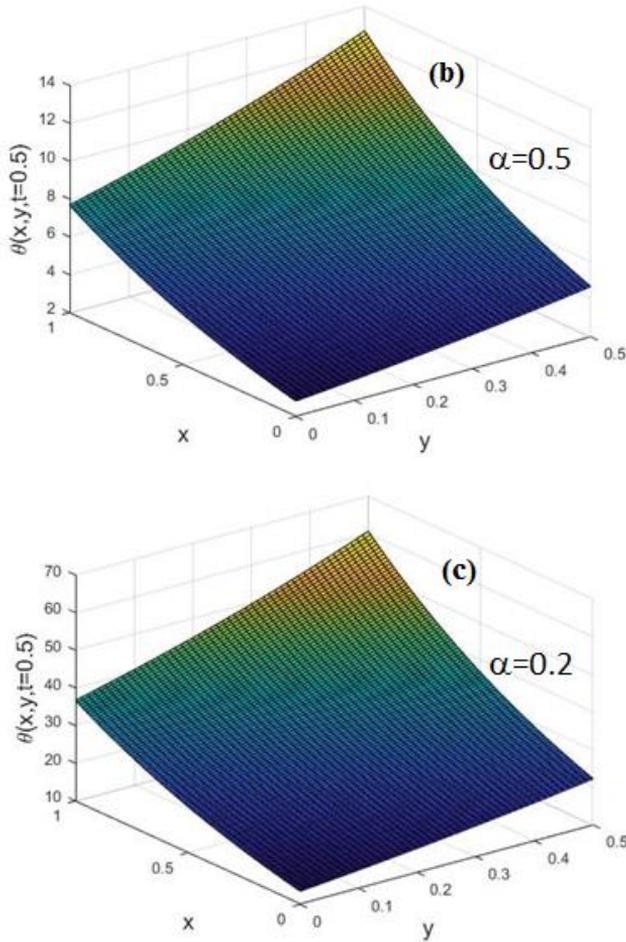$$\theta(x,y,t) = e^{x+y-3t}. \tag{48}$$

Fig. 4. The surface shows the SADM solution $\theta(x, y, t)$ for example 4, when (a) $\alpha = 1$, (b) $\alpha = 0.5$, (c) $\alpha = 0.2$.

**Example 5.** Finally, consider time-fractional telegraph equation of linear type

$$\frac{\partial^{2\alpha}\theta}{\partial t^{2\alpha}} + 2\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + 3\theta = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}, \qquad 0 < \alpha \le 1, t \ge 0 \tag{49}$$

with initial condition

$$\theta(x, y, z, 0) = \sinh(x)\sinh(y)\sinh(z), \ \theta_t(x, y, z, 0) = -\sinh(x)\sinh(y)\sinh(z) \tag{50}$$

Taking the Sumudu transform on (49) and simultaneously using ICs (50), it yields

$$S\left[\frac{\partial^{2\alpha}\theta}{\partial t^{2\alpha}}\right] = -S\left[2\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} + 3\theta - \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial^2 \theta}{\partial y^2} - \frac{\partial^2 \theta}{\partial z^2}\right],$$

$$\Rightarrow u^{-2\alpha}S[\theta(x,y,z,t)] - u^{-2\alpha}\theta(x,y,z,0) - u^{-2\alpha+1}\theta_t(x,y,z,0) = -S\left[2\frac{\partial^\alpha\theta}{\partial t^\alpha} +\right.$$

$$\left.3\theta - \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2} - \frac{\partial^2\theta}{\partial z^2}\right],$$

$$\Rightarrow S[\theta(x,y,z,t)] = \sinh(x)\sinh(y)\sinh(z) - u\sinh(x)\sinh(y)\sinh(z)$$

$$-\frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^\alpha\theta}{\partial t^\alpha} + 3\theta - \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2} - \frac{\partial^2\theta}{\partial z^2}\right],$$

again, applying the inverse Sumudu transform

$$\theta(x,y,z,t) = S^{-1}[\sinh(x)\sinh(y)\sinh(z) - u\sinh(x)\sinh(y)\sinh(z)]$$

$$-S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^\alpha\theta}{\partial t^\alpha} + 3\theta - \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2} - \frac{\partial^2\theta}{\partial z^2}\right]\right],$$

using the SADM procedure, we get

$$\theta_0(x,y,z,t) = S^{-1}[\sinh(x)\sinh(y)\sinh(z) - u\sinh(x)\sinh(y)\sinh(z)]$$

$$\theta_0(x,y,z,t) = \sinh(x)\sinh(y)\sinh(z)(1-t),$$

and

$$\theta_{i+1}(x,y,z,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^\alpha\theta_i}{\partial t^\alpha} + 3\theta_i - \frac{\partial^2\theta_i}{\partial x^2} - \frac{\partial^2\theta_i}{\partial y^2} - \frac{\partial^2\theta_i}{\partial z^2}\right]\right], \quad i = 0,1,2,3\ldots (51)$$

For $i = 0$, eq. (51) gives

$$\theta_1(x,y,z,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^\alpha\theta_0}{\partial t^\alpha} + 3\theta_0 - \frac{\partial^2\theta_0}{\partial x^2} - \frac{\partial^2\theta_0}{\partial y^2} - \frac{\partial^2\theta_0}{\partial z^2}\right]\right],$$

$$\theta_1(x,y,z,t) = 2\sinh(x)\sinh(y)\sinh(z)S^{-1}[u^{\alpha+1}],$$

$$\theta_1(x,y,z,t) = 2\sinh(x)\sinh(y)\sinh(z)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

similarly, for $i = 1$ in eq. (51), it gives

$$\theta_2(x,y,z,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^\alpha\theta_1}{\partial t^\alpha} + 3\theta_1 - \frac{\partial^2\theta_1}{\partial x^2} - \frac{\partial^2\theta_1}{\partial y^2} - \frac{\partial^2\theta_1}{\partial z^2}\right]\right],$$

$$\theta_2(x,y,z,t) = -4\sinh(x)\sinh(y)\sinh(z)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

Also, for $i = 2$, eq. (51) reduces

$$\theta_3(x,y,z,t) = -S^{-1}\left[\frac{1}{u^{-2\alpha}}S\left[2\frac{\partial^\alpha\theta_2}{\partial t^\alpha} + 3\theta_2 - \frac{\partial^2\theta_2}{\partial x^2} - \frac{\partial^2\theta_2}{\partial y^2} - \frac{\partial^2\theta_2}{\partial z^2}\right]\right],$$

$$\theta_3(x,y,z,t) = 8\sinh(x)\sinh(y)\sinh(z)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)},$$

On using eq. (24), SADM solution for example 5, is obtained in the form of series

$$\theta(x,y,z,t) = \theta_0(x,y,z,t) + \theta_1(x,y,z,t) + \theta_2(x,y,z,t) + \theta_3(x,y,z,t) + \cdots$$

$$\theta(x,y,z,t) = \sinh(x)\sinh(y)\sinh(z)\left[1 - t + \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{8t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \cdots\right]. (52)$$

When $\alpha = 1$ in (52), get SADM solution for example 5, as

$$\theta(x,y,z,t) = \sinh(x)\sinh(y)\sinh(z)\left[1 - t + 2\frac{t^2}{\Gamma(3)} - 4\frac{t^3}{\Gamma(4)} + 8\frac{t^4}{\Gamma(5)} - \cdots\right]. \quad (53)$$

This solution is approximate equivalent to exact solution in closed form

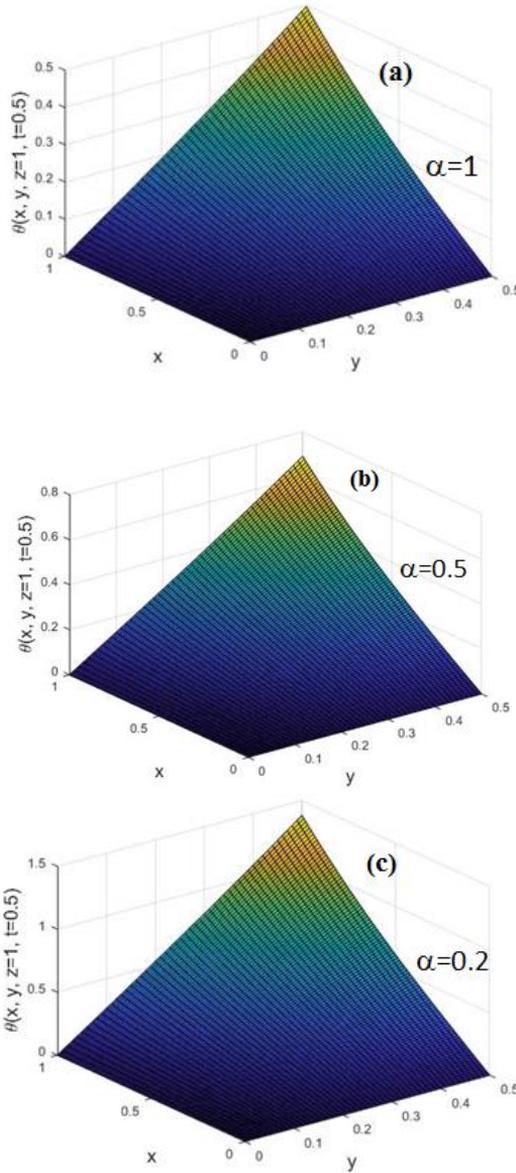$$\theta(x,y,z,t) = e^{-t}\sinh(x)\sinh(y)\sinh(z). \quad (54)$$



Fig. 5. The surface shows the SADM solution $\theta(x,y,z,t)$ for example 5, when (a) $\alpha = 1$, (b) $\alpha = 0.5$, (c) $\alpha = 0.2$.

**5. Conclusion**

In this study, a powerful computational technique, called Sumudu-Adomian Decomposition Method (SADM) is applied to find the solution of some fractional order telegraph equations. This computational technique provides the numerical solution in form of series solution, this series converge to exact solution very rapidly. Numerical experiments on test examples show that proposed computational technique is of high accuracy and support the theoretical results. The fractional order derivatives are computed in Caputo sense. Obviously, SADM is easy to implement for the multi-dimensional space and time fractional order physical problems emerging in various fields of science and engineering.

**References**

1. O. Heaviside, Electromagnetic Theory, (Chelsea Publishing Company, New York, 1899) **2.**
2. A. Ōkubo, Application of the Telegraph Equation to Oceanic Diffusion: Another Mathematic Model (Chesapeake Bay Institute, The Johns Hopking Unversity, Baltimore, MD, USA, 1971).
3. M. Javidi and N. Nyamoradi, Int. J. Adv. Math. Sci. **1**, 64 (2013). https://doi.org/10.14419/ijams.v1i2.780
4. P. Veeresha and D. G. Prakasha, arXiv (2018). arXiv:1805.03968v1
5. H. Al-badrani, S. Saleh, H. O. Bakodah, and M. Al-Mazmumy, Nonlinear Anal. Differ. Eq. **4**, 243 (2016). http://dx.doi.org/10.12988/nade.2016.6418
6. S. Kumar, Appl. Math. Mode. **38**, 3154 (2014). https://doi.org/10.1016/j.apm.2013.11.035
7. A. Prakash, Nonlinear Eng. **5**, 123 (2016). https://doi.org/10.1515/nleng-2016-0008
8. M. Inc, A. Akgül, and A. Kılıçman, Abstr. Appl. Anal. **2013**, ID 768963 (2013). https://doi.org/10.1155/2013/768963
9. M. Inc, A. Akgül, and A. Kılıçman, J. Funct. Space Appl. **2012**, ID 984682 (2012). https://doi.org/10.1155/2012/984682
10. M. Modanli and A. Akgül, Eur. Phys. J. Spec. Top. **226**, 3693 (2017). https://doi.org/10.1140/epjst/e2018-00088-6
11. A. Akgül and D. Grow, Differ. Equ. Dyn. Syst. **28**, 715 (2020). https://doi.org/10.1007/s12591-019-00453-3
12. M. Modanli and A. Akgül, Appl. Math. Nonlinear Sci. **5**, 163 (2020). https://doi.org/10.2478/amns.2020.1.00015
13. A. Atangana and A. Akgül, Alex. Eng. J. **59**, 1971 (2020). https://doi.org/10.1016/j.aej.2019.12.028
14. A. Devi and M. Jakhar, J. Sci. Res. **13**, 59 (2021). https://doi.org/10.3329/jsr.v13i1.47521
15. N. B. Manjare and H. T. Dinde, J. Sci. Res. **12**, 585 (2020). http://dx.doi.org/10.3329/jsr.v12i4.47163
16. V. K. Srivastava, M. K. Awasthi, and M. Tamsir, AIP Adv. **3**, ID 032142 (2013). https://doi.org/10.1063/1.4799548
17. G. K. Watugala, Int. J. Math. Educ. Sci. Technol. **24**, 35 (1993). https://doi.org/10.1080/0020739930240105
18. F. B. M. Belgacem, Applications of the Sumudu Transform to Indefinite Periodic Parabolic Equations - *Proc. of the 6ᵗʰ Int. Conf. on Mathematical Problems and Aerospace Sciences* (ICNPAA'06) (Cambridge Scientific, Cambridge, UK, 2007) pp. 51-60.
19. F. B. M. Belgacem, A. A. Karaballi, and S. L. Kalla, Math. Probl. Eng. **2003**, ID 439059 (2003). https://doi.org/10.1155/S1024123X03207018
20. Q. D. Katatbeh and F. B. M. Belgacem, Nonlinear Studies **18**, 99 (2011).

21. S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives- Theory and Applications (CRC Press, Linghorne, 1993).
22. M. Caputo, Elasticità e Dissipazione (Bologana, Zanichelli, 1969).
23. V. B. L. Chaurasia and J. Singh, Appl. Math. Sci. **4**, 2843 (2010).
24. P. Pue-on, Appl. Math. Sci. **7**, 6593 (2013).  http://dx.doi.org/10.12988/ams.2013.310603
25. R. Ezzati and K. Shakibi, Procedia Comput. Sci. **3**, 1043 (2011). https://doi.org/10.1016/j.procs.2010.12.171
26. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method (Kluwer Academic Publishers, Boston, 1994). https://doi.org/10.1007/978-94-015-8289-6
27. A. M. Wazwaz, Appl. Math. Comput. **111**, 33 (2000). https://doi.org/10.1016/S0096-3003(99)00063-6