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Applications of Composite Numerical Integrations Using Gauss-Radau and Gauss-Lobatto Quadrature Rules

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Abstract

In this paper, numerical integrals over an arbitrary triangular region are evaluated exploiting finite element method. The physical region is transformed into a standard triangular finite element using the basis functions in local space. Then the standard triangle is discretized into $4\times n^2$ right isosceles triangles, in which each of these triangles having area $1/2n^2$, and thus composite numerical integration is employed. In addition, the affine transformation over each discretized triangle and the use of linearity property of integrals are applied. Finally, each isosceles triangle is transformed into a 2-sqare finite element to generate new n^2 extended sampling points and corresponding weight coefficients, using n point's conventional Gauss-Radau and Gauss-Lobatto quadratures, which are applied again to evaluate the double integral. The performance is depicted by means of numerical examples.

Keywords: Double integral; Numerical Integration; Quadrilateral and Triangular Finite Element; Gauss-Radau and Gauss-Lobatto quadratures.

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1. Introduction

Numerical simulation in engineering science and in applied mathematics has become a powerful tool to model the physical phenomena, particularly when analytical solutions are not available and/or are very difficult to obtain. The integrals arising in practical problems are not always simple and the quadrature scheme cannot evaluate with desired accuracy, the composite numerical integration is dealt then to obtain high accuracy.

From the literature review we observe that numerical integrations over triangular regions were first introduced by Hammer and co-workers [1-3]. In the working of finite

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element method, the triangular elements are widely used in the area of numerical integration schemes [4]. The work of Hammer *et al.* [1] has been further developed by Cowper [5], and thus he provided a table of Gaussian quadrature formulae for symmetrically placed integration points. Lyness and Jespersen [6] derived symmetric quadrature rules and provided integration formulas with a precision of upto degree eleven by formulating the problem in terms of polar coordinates. Lether [7] and Hillion [8] derived formulas for triangles as product of one-dimensional Gauss quadrature rule. Lannoy [9] discussed the symmetric four-point integration rule. Lague and Baldur [10] included a technique for numerical integration over the surface of a triangle. Laursen and Gellert [11] also discussed elaborately symmetric integration formulae of precision up to degree ten. In ref. [12], double integrals over a standard triangular region are evaluated. In ref. [13], Gauss-Radau and Gauss-Lobatto quadrature rules are presented to evaluate the rational integrals of the element matrix for a general quadrilateral. In [14], double integrals over an arbitrary quadrilateral are evaluated. In [14], the physical region is transformed into a standard quadrilateral finite element using the basis functions in local space. Then the standard quadrilateral is subdivided into two triangles, and each triangle is further discretized into $4\times n^2$ right isosceles triangles, with area $1/2n^2$, and thus composite numerical integration is employed.

In this paper an arbitrary triangular region is transformed into a standard triangular finite element using the basis functions in local space. Then the standard triangle is discretized into $4\times n^2$ right isosceles triangles. Moreover, each isosceles triangle is transformed into a 2-sqare finite element to compute new n^2 extended Gauss Radau and Gauss Lobatto points and corresponding weight coefficients, using n point's, Gauss Radau and Gauss Lobatto quadratures are used instead of Gauss Lagendre quadrature [14], which are applied again to evaluate the double integral. The subsequent formulations are developed by *Mathematica*. The performance of the present formulation is excellent.

2. Numerical Integration

The most common numerical integration [13] is

$$\int_a^b w(x)\, f(x)\, dx = \sum_{k=1}^{n} w_k\; f(x_k) + E[f] \tag{1}$$

known as the Gaussian quadrature, where x_k, w_k $(k = 1, 2, \ldots.., n)$ and $E[f]$ are called nodes (abscissas), weights, and error approximation, respectively.

Assuming the exactness (i.e. $E[f] = 0$) and for our convenient (by change of variable), and set $w(x) = 1$, the equation (1) can be written as,

$$\int_1^{-1} w(x)\, f(x)\, dx = \sum_{k=1}^{m} w_k\; f(x_k) + \sum_{k=m+1}^{n-m} w_k\; f(x_k) \quad m \le n \tag{2}$$

If we put $m = 0$, then the concluded numerical integration rule is called Gauss-Legendre. If we put $m = 1$, $x_1 = -1\,(or\,1)$ and then the concluded numerical integration rule is called Gauss-Radau [13]. If we put $m = 2$, x_1 = -1 and x_n = 1 and then the concluded numerical integration rule is called Gauss-Lobatto [13].

3. Formulation of Integrals Over an Arbitrary Triangular Region

The integral of an arbitrary function, $f(x, y)$ over an arbitrary triangular region *AT* is given by

$$I = \iint_{AT} f(x, y)\, dy\, dx = \iint_{AT} f(x, y)\, dx\, dy \qquad (3)$$

The integral I of Eq. (3) is then transformed into an integral over the region of the standard triangle $ST = \{(u,v) : -1 \le v \le 1, -1 \le u \le -v\}$ by the liner triangular finite element basis functions $L_i(u,v)$, shown in Fig. 1:

$$L_1(u,v) = -\frac{1}{2}(u+v)\,,\; L_2(u,v) = \frac{1}{2}(1+u) \text{ and } L_3(u,v) = \frac{1}{2}(1+v)\,.$$

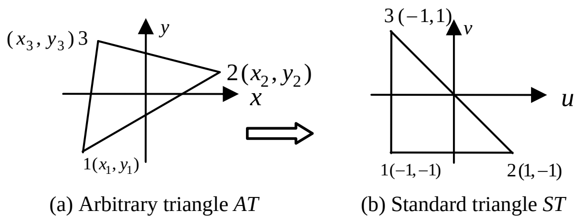

Fig. 1. Transformation of arbitrary triangle *AT* into equivalent standard triangle *ST*.

The coordinates are changed by assuming that

$$x = \sum_{i=1}^{3} x_i L_i \text{ and } y = \sum_{i=1}^{3} y_i L_i \qquad (4a)$$

and the corresponding *Jacobian*, $J_1 = \dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$ (4b)

Therefore using Eqs. (4) and (3) to obtain,

$$I = \iint_{AT} f(x, y)\, dy\, dx = \iint_{ST} f(u,v)\left|J_1\right|\, du\, dv \qquad (5)$$

The integral I of Eq. (5) can be further transformed into an integral over the standard 2-square, $\{(\xi,\eta) : -1 \le \xi, \eta \le 1\}$ using standard quadrilateral basis functions $Q_i(\xi,\eta)$, as shown in Fig. 2:

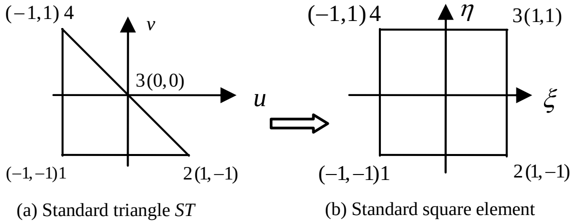

Fig. 2. Transformation of standard triangle ST into 2-square.

Assume that

$$u = \sum_{i=1}^{4} u_i Q_i = \frac{1}{4}\left(-1+3\xi-\eta\left(1+\xi\right)\right) = u\left(\xi,\eta\right) \tag{6a}$$

$$v = \sum_{i=1}^{4} v_i Q_i = \frac{1}{4}\left(-1+3\eta-\xi\left(1+\eta\right)\right) = v\left(\xi,\eta\right) \tag{6b}$$

and $J_2 = \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} = \frac{1}{4}(2-\eta-\xi)$ (6c)

Note that J_1 depends on the vertices of the given arbitrary triangular region, but J_2 is fixed.

Let $F(u,v) = f(u,v)\,|J_1|$ and using (6), Equ. (5) becomes

$$I = \int_{-1}^{1}\int_{-1}^{-u} F(u,v)\,dv\,du = \int_{-1}^{1}\int_{-1}^{1} F(u(\xi,\eta),v(\xi,\eta))\,|J_2|\,d\xi\,d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} F(\frac{1}{4}(-1+3\xi-\eta(1+\xi)),\frac{1}{4}(-1+3\eta-\xi(1+\eta)))\frac{1}{4}(2-\eta-\xi)d\xi\,d\eta \tag{7}$$

Now Eq. (5) represents an integral over the standard 2-square region:

$\{(\xi,\eta): -1 \le \xi,\eta \le 1\}$. Hence using Gaussian quadrature rule for the integral *I* of Eq. (7), we have

$$I = \sum_{i=1}^{s}\sum_{j=1}^{s}\frac{1}{4}(2-\eta_j-\xi_i)w_i w_j F(\frac{1}{4}(-1+3\xi_i-\eta_j(1+\xi_i)),\frac{1}{4}(-1+3\eta_j-\xi_i(1+\eta_j))) \tag{8}$$

where (ξ_i,η_j) are Gauss- Radau points and Gauss- Lobatto points in the ξ,η directions of order *s* and w_i, w_j are the corresponding weight coefficients. We can write Eq. (8) as:

$$I = \sum_{k=1}^{N=s\times s} c_k\,F(x_k,y_k) \tag{9}$$

where c_k, x_k and y_k can be written in the form:

$$c_k = \frac{1}{4}(2-\eta_j-\xi_i)w_i w_j$$
$$x_k = \frac{1}{4}(-1+3\xi_i-\eta_j(1+\xi_i)) \ , (k=1,2,\dots \ \dots\dots\dots, N), \ (i, j=1,2,\dots \ \dots\dots\dots, s) \qquad (10)$$
$$y_k = \frac{1}{4}(-1+3\eta_j-\xi_i(1+\eta_j))$$

The weighting coefficients c_k and sampling points (x_k, y_k) of various order can be now easily computed from Eq. (10). In Table 1 and Table 2, values of, c_k , x_k and y_k for s = 3, 4, 5, 6 are calculated by using Gauss- Radau points and Gauss- Lobatto points respectively. In both table sampling points are calculated by using *Mathematica*.

Table 1. Output of c_k , x_k and y_k of Eq. (10) using Gauss-Radau points.

k	c_k	x_k	y_k
Order of Gauss-Radau Quadrature rule, s = 3			
1	0.096614387479324	-1.000000000000000	0.689897948556636
2	0.308641975308642	-0.589897948556636	0.389897948556636
3	0.087870061825481	-0.024040820577346	-0.024040820577346
4	0.187336229804627	-1.000000000000000	-0.289897948556636
5	0.677562036939952	-0.415959179422655	-0.415959179422655
6	0.308641975308642	0.389897948556636	-0.589897948556636
7	0.049382716049383	-1.000000000000000	-1.000000000000000
8	0.187336229804627	-0.289897948556636	-1.000000000000000
9	0.096614387479324	0.689897948556636	-1.000000000000000
Order of Gauss-Radau Quadrature rule, s = 4			
1	0.029999063576822	-1.000000000000000	0.822824080974592
2	0.127051925813598	-0.768848646756411	0.629294357739875
3	0.085249046429649	-0.357152738936900	0.284605070919161
4	0.017222770232317	-0.007847826570624	-0.007847826570625
5	0.068393228915845	-1.000000000000000	0.181066271118531
6	0.305638842571568	-0.700713047374403	0.055672147265822
7	0.246817075893281	-0.167663113074927	-0.167663113074927
8	0.085249046429649	0.284605070919161	-0.357152738936900
9	0.073482707507333	-1.000000000000000	-0.575318923521694
10	0.340705524244772	-0.620407427701387	-0.620407427701387
11	0.305638842571568	0.055672147265822	-0.700713047374403
12	0.127051925813598	0.629294357739875	-0.768848646756411
13	0.015625000000000	-1.000000000000000	-1.000000000000000
14	0.073482707507333	-0.575318923521694	-1.000000000000000
15	0.068393228915845	0.181066271118531	-1.000000000000000
16	0.029999063576822	0.822824080974592	-1.000000000000000

Order of Gauss-Radau Quadrature rule, s = 5			
1	0.012153616652077	1.000000000000000	0.885791607770965
2	0.058825725455510	0.852259260953794	0.754012618129610
3	0.057423793701357	0.559811698754950	0.493160773753848
4	0.027006093644855	0.235547715267332	0.203929919779881
5	0.004717626056903	0.003260889213885	0.003260889213885
6	0.028739796983092	1.000000000000000	0.446313972723752
7	0.142753113111973	0.821548593625632	0.345245650410559
8	0.150978996962694	0.468310352758179	0.145184484703406
9	0.087660908725231	0.076642054200238	0.076642054200238
10	0.027006093644855	0.203929919779881	0.235547715267332
11	0.039504439867460	1.000000000000000	0.167180864737833
12	0.200893758605095	0.778677615996012	0.225378209421406
13	0.226983484574152	0.340577792752539	0.340577792752539
14	0.150978996962694	0.145184484703406	0.468310352758179
15	0.057423793701357	0.493160773753848	0.559811698754950
16	0.033202146497371	1.000000000000000	0.720480271312439
17	0.171275017680751	0.740013090993831	0.740013090993831
18	0.200893758605095	0.225378209421406	0.778677615996012
19	0.142753113111973	0.345245650410559	0.821548593625632
20	0.058825725455510	0.754012618129610	0.852259260953794
21	0.006400000000000	1.000000000000000	1.000000000000000
22	0.033202146497371	0.720480271312439	1.000000000000000
23	0.039504439867460	0.167180864737833	1.000000000000000
24	0.028739796983092	0.446313972723752	1.000000000000000
25	0.012153616652077	0.885791607770965	1.000000000000000

Table 2. Output of c_k , x_k and y_k of Eq. (10) using Gauss-Lobatto points.

k	c_k	x_k	y_k
Order of Gauss-Lobatto Quadrature rule, s = 3			
1	0.055555555555556	1.000000000000000	1.000000000000000
2	0.111111111111111	0.500000000000000	0.500000000000000
3	0.000000000000000	0.000000000000000	0.000000000000000
4	0.333333333333333	1.000000000000000	0.000000000000000
5	0.888888888888889	0.250000000000000	0.250000000000000
6	0.111111111111111	0.500000000000000	0.500000000000000
7	0.111111111111111	1.000000000000000	1.000000000000000
8	0.333333333333333	0.000000000000000	1.000000000000000
9	0.055555555555556	1.000000000000000	1.000000000000000
Order of Gauss-Lobatto Quadrature rule, s = 4			
1	0.013888888888889	1.000000000000000	1.000000000000000
2	0.050250472065971	0.723606797749979	0.723606797749979
3	0.019193972378474	0.276393202250021	0.276393202250021
4	0.000000000000000	0.000000000000000	0.000000000000000

Table 2 (*Continued*)

5	0.088638416822918	1.000000000000000	0.447213595499958
6	0.347222222222222	0.647213595499958	0.247213595499958
7	0.191939723784737	0.076393202250021	0.076393202250021
8	0.019193972378474	0.276393202250021	0.276393202250021
9	0.119694916510415	1.000000000000000	0.447213595499958
10	0.502504720659707	0.523606797749979	0.523606797749979
11	0.347222222222222	0.247213595499958	0.647213595499958
12	0.050250472065971	0.723606797749979	0.723606797749979
13	0.027777777777778	1.000000000000000	1.000000000000000
14	0.119694916510415	0.447213595499958	1.000000000000000
15	0.088638416822918	0.447213595499958	1.000000000000000
16	0.013888888888889	1.000000000000000	1.000000000000000
Order of Gauss-Lobatto Quadrature rule, s = 5			
1	0.005000000000000	1.000000000000000	1.000000000000000
2	0.022521674962414	0.827326835353989	0.827326835353988
3	0.017777777777778	0.500000000000000	0.500000000000000
4	0.004700547259808	0.172673164646012	0.172673164646012
5	0.000000000000000	0.000000000000000	0.000000000000000
6	0.031922769482030	1.000000000000000	0.654653670707977
7	0.148209876543210	0.797510813565120	0.511796527850834
8	0.130216237304314	0.413663417676994	0.240990253030983
9	0.051183736829021	0.029816021788869	0.029816021788869
10	0.004700547259808	0.172673164646012	0.172673164646012
11	0.053333333333333	1.000000000000000	0.000000000000000
12	0.256944256522846	0.740990253030983	0.086336582323006
13	0.252839506172840	0.250000000000000	0.250000000000000
14	0.130216237304314	0.240990253030983	0.413663417676994
15	0.017777777777778	0.500000000000000	0.500000000000000
16	0.049743897184636	1.000000000000000	0.654653670707977
17	0.245236016257398	0.684469692496846	0.684469692496846
18	0.256944256522846	0.086336582323006	0.740990253030983
19	0.148209876543210	0.511796527850834	0.797510813565120
20	0.022521674962414	0.827326835353988	0.827326835353989
21	0.010000000000000	1.000000000000000	1.000000000000000
22	0.049743897184636	0.654653670707977	1.000000000000000
23	0.053333333333333	0.000000000000000	1.000000000000000
24	0.031922769482030	0.654653670707977	1.000000000000000
25	0.005000000000000	1.000000000000000	1.000000000000000

4. Composite Integration over Standard Triangle ST

Let us discretize ST in (u,v) space into $4(n\times n)=4n^2$ right isosceles triangle T_i each of area $1/(2n^2)$ [14]. This is depicted in Fig. 3. Let $F(u,v)= f(u,v)\,|J_1|$. Then

$$I = \iint_{AT} f(x,y)\ dy\,dx = \iint_{ST} f(u,v)\left|J_1\right|\ du\,dv\ = \sum_{i=1}^{4(n\times n)} \iint_{T_i} F(u,v)\ du\,dv \tag{11}$$

Now each T_i is to be transformed again into a standard triangle in (X,Y) space, say. Observe that each T_i is either Type I (Fig. 4a) whose vertices are $1(-1+\frac{i}{n},-1+\frac{j}{n})$, $2(-1+\frac{i+1}{n},-1+\frac{j}{n})$ and $3(-1+\frac{i}{n},-1+\frac{j+1}{n})$, or Type II (Fig.4b) whose vertices are $1(-1+\frac{i+1}{n},-1+\frac{j+1}{n})$, $2(-1+\frac{i}{n},-1+\frac{j+1}{n})$ and $3(-1+\frac{i+1}{n},-1+\frac{j}{n})$.

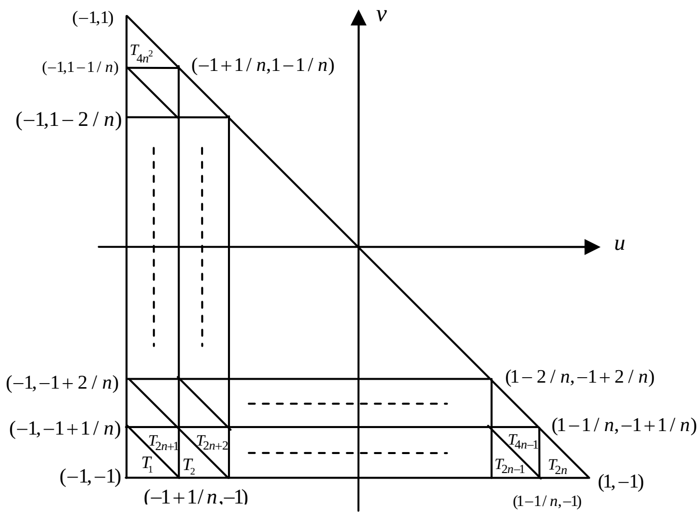

Fig. 3. Discretization of T into $4\,n^2$ sub triangles T_i .

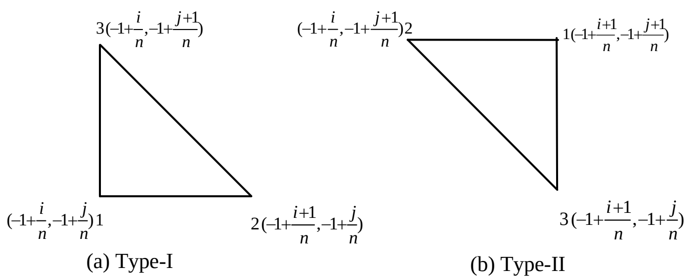

Fig. 4. Types of triangles of T_i of Fig. 3 in (u,v) space.

For type II triangles (Fig. 4b)

$$u(X,Y) = (-1+\frac{i+1}{n})(-\frac{1}{2}(X+Y)) + (-1+\frac{i}{n})\ (\frac{1}{2}(1+X)) + (-1+\frac{i+1}{n})(\frac{1}{2}(1+Y))$$
$$= \frac{-X+2(i-n)+1}{2n}$$
$$v(X,Y) = (-1+\frac{j+1}{n})(-\frac{1}{2}(X+Y)) + (-1+\frac{j+1}{n})\ (\frac{1}{2}(1+X)) + (-1+\frac{j}{n})(\frac{1}{2}(1+Y))$$
$$= \frac{-Y+2(j-n)+1}{2n}$$

Therefore from Eq. (11)

$$I = \sum_{i=1}^{4(n\times n)} \iint_{T_i} F(u,v)\ du\,dv = \frac{1}{4n^2}\int_{-1}^{1}\int_{-1}^{-y} H(X,Y)\ dX\ dY = \frac{1}{4n^2}\int_{-1}^{1}\int_{-1}^{-x} H(X,Y)\ dY\,dX \qquad (12)$$

where

$$H(X,Y) = \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1-i} F\left(\frac{X+2(i-n)+1}{2n}, \frac{Y+2(j-n)+1}{2n}\right)$$
$$+ \sum_{i=0}^{2n-2}\sum_{j=0}^{2n-2-i} F\left(\frac{-X+2(i-n)+1}{2n}, \frac{-Y+2(j-n)+1}{2n}\right) \qquad (13)$$

We can now apply Gauss Legendre quadrature rules on the integral, in a manner similar to the procedure we already developed for integral $I = \iint_{AT} f(x,y)\ dy\,dx$ in previous section. Following the method already developed in previous section, we have now on using the transformation:

$$X = \frac{1}{4}(-1+3\xi-\eta(1+\xi)) \qquad Y = \frac{1}{4}(-1+3\eta-\xi(1+\eta)) \qquad (14)$$

Therefore (12) becomes

$$I = \iint_{AT} f(x,y)\ dy\,dx = \frac{1}{4n^2}\iint_{ST} H(X,Y)\,dX\ dY$$
$$= \frac{1}{4n^2}\int_{-1}^{1}\int_{-1}^{1}\frac{1}{4}(2-\xi-\eta)\,H(X(\xi,\eta),Y(\xi,\eta))\,d\xi\,d\eta$$
$$= \frac{1}{4n^2}\sum_{p=1}^{s}\sum_{q=1}^{s}\frac{(2-\xi-\eta)}{8}\ W_p W_q\ H(X(\xi_p,\eta_q),Y(\xi_p,\eta_q)) \qquad (15)$$

where

$$H(X,Y)=\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1-i}F\left(\frac{X+2(i-n)+1}{2n},\frac{Y+2(j-n)+1}{2n}\right)$$

$$+\sum_{i=0}^{2n-2}\sum_{j=0}^{2n-2-i}F\left(\frac{-X+2(i-n)+1}{2n},\frac{-Y+2(j-n)+1}{2n}\right) \qquad (16a)$$

$$X = X(\xi_p,\eta_q)=\frac{1}{4}(-1+3\xi_p-\eta_q(1+\xi_p)) \qquad (16b)$$

$$Y=Y(\xi_p,\eta_q)=\frac{1}{4}(-1+3\eta_q-\xi_p(1+\eta_q)) \;\; (p,q=1,2,3,\ldots\ldots\ldots,s) \qquad (16c)$$

From Eqs. (10)–(14), it is clear that we can obtain the following composite integration rule:

$$I=\frac{1}{4n^2}\sum_{k=1}^{N=s\times s}c_k\,H(x_k,y_k) \qquad (17)$$

where

$$H(x_k,y_k)=\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1-i}F\left(\frac{x_k+2(i-n)+1}{2n},\frac{y_k+2(j-n)+1}{2n}\right)$$

$$+\sum_{i=0}^{2n-2}\sum_{j=0}^{2n-2-i}F\left(\frac{-x_k+2(i-n)+1}{2n},\frac{-y_k+2(j-n)+1}{2n}\right)\;, \qquad (18a)$$

and

$$c_k=\frac{1}{4}(2-\xi_p-\eta_q)\,w_p\,w_q$$

$$x_k=\frac{1}{4}(-1+3\xi_p-\eta_q(1+\xi_p)) \;\; (k=1,2,\ldots\ldots\ldots\ldots,N),(i,\,j=1,2,\ldots\ldots\ldots\ldots,s) \qquad (18b)$$

$$y_k=\frac{1}{4}(-1+3\eta_q-\xi_p(1+\eta_q))$$

5. Numerical Examples

In this section we estimate errors for three simple examples to show that the present formulation converges to exact solution with great accuracy than that of the conventional Gauss-Radau and Gauss-Lobatto quadratures.

$$I_1=\iint_R (x+y)^{-1/2}\,dy\,dx \;=1.995191271069104,$$

where R is a triangular region connecting with the points (-1,2), (2,1) and (3,3).

$$I_2 = \int_0^1 \int_0^x e^{|x+y-1|} \, dy \, dx = 0.718281834100861, \text{ and}$$

$$I_3 = \int_{-1}^{1} \int_{-1}^{-x} \sin(x^2 + y^2) \, dy \, dx = 1.122580796643812.$$

In Table 3, errors of integrals are calculated by using Table 1, in which sampling points are generated by using Gauss-Radau points. In Table 4, errors of integrals are calculated by using Table 1, in which sampling points are generated by using Gauss-Lobatto points. Finally in Table 5, errors of integrals are calculated by conventional Gauss-Radau and Gauss-Lobatto quadratures. The errors given in Table 3 and Table 4 are very negligible compared to those of Table 5.

Table 3. Errors of integrals in which sampling points are generated by using Gauss-Radau points.

$4\times n^2$	I_1	I_2	I_3
Gauss-Radau Quadrature rule, s = 3			
4×1^2	0.00042	0.00423	0.00179
4×2^2	0.00003	0.00106	0.00011
4×3^2	5.05972×10^{-6}	0.00047	0.00002
4×4^2	1.55175×10^{-6}	0.00027	6.46490×10^{-6}
4×5^2	6.23037×10^{-7}	0.00017	2.64101×10^{-6}
4×6^2	2.96591×10^{-7}	0.00012	1.27179×10^{-6}
4×7^2	1.58702×10^{-7}	0.00009	6.85876×10^{-7}
4×8^2	9.24637×10^{-8}	0.00007	4.01818×10^{-7}
4×9^2	5.74720×10^{-8}	0.00005	2.50755×10^{-7}
4×10^2	3.75849×10^{-8}	0.00004	1.64474×10^{-7}
Gauss-Radau Quadrature rule, s = 4			
4×1^2	0.00002	0.00228	0.00002
4×2^2	4.84714×10^{-7}	0.00057	3.66263×10^{-7}
4×3^2	4.56336×10^{-8}	0.00025	3.33264×10^{-8}
4×4^2	8.18940×10^{-9}	0.00014	5.98998×10^{-9}
4×5^2	2.13738×10^{-9}	0.00009	1.57668×10^{-9}
4×6^2	7.11321×10^{-10}	0.00006	5.29143×10^{-10}
4×7^2	2.80489×10^{-10}	0.00005	2.10100×10^{-10}
4×8^2	1.25297×10^{-10}	0.00004	9.43656×10^{-11}
4×9^2	6.15765×10^{-11}	0.00003	4.65725×10^{-11}
4×10^2	3.26277×10^{-11}	0.00002	2.47597×10^{-11}
Gauss-Radau Quadrature rule, s = 5			
4×1^2	1.20974×10^{-6}	0.00143	3.82664×10^{-7}
4×2^2	1.17070×10^{-8}	0.00036	8.78387×10^{-10}
4×3^2	5.78610×10^{-10}	0.00016	2.77998×10^{-11}
4×4^2	6.27607×10^{-11}	0.00009	2.55751×10^{-12}
4×5^2	1.08593×10^{-11}	0.00006	4.12559×10^{-13}
4×6^2	2.55840×10^{-12}	0.00004	9.48130×10^{-14}
4×7^2	7.50733×10^{-13}	0.00003	2.79776×10^{-14}
4×8^2	2.59792×10^{-13}	0.00002	9.99201×10^{-15}
4×9^2	1.03251×10^{-13}	0.00002	4.66294×10^{-15}
4×10^2	4.59632×10^{-14}	0.00001	2.66454×10^{-15}

The error is calculated as follows

$$\text{Error} = \left|(\text{Exact value - Approximate value})\right|$$

Table 4. Errors of integrals in which sampling points are generated by using Gauss-Lobatto points.

$4\times n^2$	I_1	I_2	I_3
Gauss-Lobatto Quadrature rule, s = 3			
4×1^2	0.00247	0.00686	0.01439
4×2^2	0.00022	0.00173	0.00084
4×3^2	0.00005	0.00077	0.00016
4×4^2	0.00002	0.00043	0.00005
4×5^2	6.67386×10^{-6}	0.00028	0.00002
4×6^2	3.26412×10^{-6}	0.00019	9.84195×10^{-6}
4×7^2	1.77757×10^{-6}	0.00014	5.30244×10^{-6}
4×8^2	1.04816×10^{-6}	0.00011	3.10437×10^{-6}
4×9^2	6.57073×10^{-7}	0.00009	1.93641×10^{-6}
4×10^2	4.32402×10^{-7}	0.00007	1.26971×10^{-6}
Gauss-Lobatto Quadrature rule, s = 4			
4×1^2	0.00010	0.00316	0.00008
4×2^2	3.37072×10^{-6}	0.00079	2.70476×10^{-6}
4×3^2	3.89712×10^{-7}	0.00035	2.51565×10^{-7}
4×4^2	7.87776×10^{-8}	0.00020	4.54702×10^{-8}
4×5^2	2.21231×10^{-8}	0.00013	1.19959×10^{-8}
4×6^2	7.72044×10^{-9}	0.00009	4.03050×10^{-9}
4×7^2	3.14377×10^{-9}	0.00006	1.60140×10^{-9}
4×8^2	1.43645×10^{-9}	0.00005	7.19568×10^{-10}
4×9^2	7.17607×10^{-10}	0.00004	3.55230×10^{-10}
4×10^2	3.84926×10^{-10}	0.00003	1.88892×10^{-10}
Gauss-Lobatto Quadrature rule, s = 5			
4×1^2	5.18813×10^{-6}	0.00182	3.22066×10^{-6}
4×2^2	7.54983×10^{-8}	0.00046	7.36389×10^{-9}
4×3^2	4.73682×10^{-9}	0.00020	2.27718×10^{-10}
4×4^2	5.96499×10^{-10}	0.00011	2.06979×10^{-11}
4×5^2	1.13768×10^{-10}	0.00007	3.30980×10^{-12}
4×6^2	2.86215×10^{-11}	0.00005	7.48512×10^{-13}
4×7^2	8.77809×10^{-12}	0.00004	2.13607×10^{-13}
4×8^2	3.12350×10^{-12}	0.00003	7.17204×10^{-14}
4×9^2	1.24745×10^{-12}	0.00002	2.68674×10^{-14}
4×10^2	5.45786×10^{-13}	0.00002	1.11022×10^{-14}

Table 5. Errors of integrals with conventional Gauss-Radau and Gauss-Lobatto quadratures.

Order of quadrature (s)	I_1	I_2	I_3
Gauss-Radau Quadrature rule			
3	0.00521	0.01666	0.02830
4	0.00050	0.00910	0.00026
5	0.00006	0.00570	0.00007
Gauss-Lobatto Quadrature rule			
3	0.02165	0.02639	0.14217
4	0.00169	0.01259	0.00598
5	0.00018	0.00728	0.00044

6. Conclusion

We have discussed, in details, the formulation of double integrals over an arbitrary triangular region. For this, we have transformed an arbitrary triangular region into a standard triangle $\{(u,v): -1 \le u \le 1, -1 \le v \le -u\}$ by the use of triangular basis functions. This standard triangle is then discretized into $4 \times n^2$ right isosceles triangles, in which the area of each of these triangles is $1/(2n^2)$. We then compute new n^2 sampling points and coefficients through the affine transformations via 2-sqaure, $\{(\xi,\eta): -1 \le \xi, \eta \le 1\}$ finite element by using the *n*-points Gauss-Radau, and Gauss-Lobatto quadratures. Numerical errors show that the present formulation converges to the exact solutions with a large significant digits compare to the conventional Gauss Radau and Gauss Lobatto quadratures.

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