JSR Publications

Available Online

J. Sci. Res. **13** (3), 845-849 (2021)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

On the Exponential Diophantine Equation $\left(13^{2m}\right) + (6r + 1)^n = z^2$

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Received 25 March 2021, accepted in final revised form 15 June 2021

Abstract

Nowadays, mathematicians are very interested in discovering new and advanced methods for determining the solution of Diophantine equations. Diophantine equations are those equations that have more unknowns than equations. Diophantine equations appear in astronomy, cryptography, abstract algebra, coordinate geometry and trigonometry. Congruence theory plays an important role in finding the solution of some special type Diophantine equations. The absence of any generalized method, which can handle each Diophantine equation, is challenging for researchers. In the present paper, the authors have discussed the existence of the solution of exponential Diophantine equation $(13^{2m}) + (6r + 1)^n = z^2$, where m, n, r, z are whole numbers. Results of the present paper show that the exponential Diophantine equation $(13^{2m}) + (6r + 1)^n = z^2$, where m, n, r, z are whole numbers, has no solution in the whole number.

Keywords: Positive integer; Diophantine equation; Solution; Congruence; Modulo system.

doi: http://dx.doi.org/10.3329/jsr.v13i3.52611 J. Sci. Res. **13** (3), 845-849 (2021)

1. Introduction

Many of the puzzles and ancient riddles depend on their solution to consider Diophantine equations of the first and second degree. Mahavira's puzzle, Monkey and coconuts puzzle have their solution by defining these puzzles in mathematical form using Diophantine equations [1]. Fermat's method of descent and method of congruence are mostly used methods for determining the positive integer solutions of Diophantine equations [2]. With the help of the Diophantine equation, we can easily prove the irrationality of a given number [3,4]. Aggarwal *et al.* [5] discussed the existence of the solution of the Diophantine equation $181^x + 199^y = z^2$.

Aggarwal *et al.* [6] discussed the Diophantine equation $223^x + 241^y = z^2$ for a solution. Gupta and Kumar [7] gave the solutions of the exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$. Kumar *et al.* [8] studied exponential Diophantine equation $601^p + 619^q = r^2$ and proved that this equation has no solution in the whole number.

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Mishra *et al.* [9] studied the existence of solution of Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ and proved that the Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ has no solution in the whole number.

Sroysang [10] discussed the Diophantine equation $3^x + 5^y = z^2$. Kumar *et al.* [11] considered the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. They showed that these equations have no non-negative integer solution. Kumar *et al.* [12] studied the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. They determined that these equations have no non-negative integer solution. Bhatnagar and Aggarwal [13] proved that the exponential Diophantine equation $421^p + 439^q = r^2$ has no solution in the whole number. Goel *et al.* [14] discussed the exponential Diophantine equation ${M_5}^p + {M_7}^q = r^2$ and proved that this equation has no solution in the whole number.

Kumar *et al.* [15] showed no solution of the exponential Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$ in the set of non-negative integers. Kumar *et al.* [16] determined that the Diophantine equation $[(7^{2m}) + (6r + 1)^n = z^2]$ has no solution in non-negative integers. The non-linear Diophantine equation $379^x + 397^y = z^2$ was examined by Aggarwal and Sharma [17]. Aggarwal and Kumar [18] studied the non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$and showed no solution to this equation in the set of whole numbers. The exponential Diophantine equation $(19^{2m}) + (12\gamma + 1)^n = \rho^2$ was studied by Aggarwal and Kumar [19]. Aggarwal [20] examined the exponential Diophantine equation$(2^{2m+1} - 1) + (13)^n = z^2$ for non-negative integer solution. Aggarwal and Kumar [21] studied the exponential Diophantine equation$(19^{2m}) + (6^{\gamma+1} + 1)^n = \rho^2$ and determined that this equation is not solvable in non-negative integers.

Aggarwal and others [22-24] studied the Diophantine equations $193^x + 211^y = z^2, 313^x + 331^y = z^2$ and $331^x + 349^y = z^2$. They proved that these equations have no solution in the set of whole numbers. Aggarwal and Kumar [25] studied the exponential Diophantine equation ${M_3}^p + {M_5}^q = r^2$, where M_3, M_5 are Mersenne primes. The exponential Diophantine equation $(19^{2m}) + (6\gamma + 1)^n = \rho^2$ was studied by Aggarwal and Kumar [26]. Kumar and Aggarwal [27] showed that the exponential Diophantine equation $439^p + 457^q = r^2$ has no solution in the whole number.

The main aim of this article is to discuss the existence of the solution of the exponential Diophantine equation $(13^{2m}) + (6r + 1)^n = z^2$, where m, n, r, z are whole numbers.

2. Preliminaries

2.1. *Lemma 1*

The exponential Diophantine equation $(13^{2m}) + 1 = z^2$, where m, z are the whole numbers, has no solution in the whole number.

Proof: Since (13^{2m}) is an odd number for all whole number m.

$\Rightarrow (13^{2m}) + 1 = z^2$ is an even number for all whole number m.

$\Rightarrow z$ is an even number.

$\Rightarrow z^2 \equiv 0(mod3)$ or $z^2 \equiv 1(mod3)$ (1)

Now, $13 \equiv 1(mod3)$

$\Rightarrow (13^{2m}) \equiv 1(mod3)$, for all whole number m.

$\Rightarrow (13^{2m}) + 1 \equiv 2(mod3)$, for all whole number m.

$\Rightarrow z^2 \equiv 2(mod3)$ (2)

Equation (2) contradicts equation (1).

Hence the exponential Diophantine equation $(13^{2m}) + 1 = z^2$, where m, z are the whole numbers, has no solution in the whole number.

2.2. *Lemma 2*

The exponential Diophantine equation $1 + (6r + 1)^n = z^2$, where r, n, z are whole numbers, has no solution in the whole number.

Proof: Since $(6r + 1)$ is an odd number for all whole numbers r so $(6r + 1)^n$ is an odd number for all numbers r and n.

$\Rightarrow 1 + (6r + 1)^n = z^2$ is an even number for all whole numbers r and n.

$\Rightarrow z$ is an even number

$\Rightarrow z^2 \equiv 0(mod3)$ or $z^2 \equiv 1(mod3)$ (3)

Now, $(6r + 1) \equiv 1(mod3)$, for all whole number r.

$\Rightarrow (6r + 1)^n \equiv 1(mod3)$, for all whole numbers r and n.

$\Rightarrow 1 + (6r + 1)^n \equiv 2(mod3)$, for all whole numbers r and n.

$\Rightarrow z^2 \equiv 2(mod3)$ (4)

Equation (4) contradicts equation (3).

Hence the exponential Diophantine equation $1 + (6r + 1)^n = z^2$, where r, n, z are whole numbers, has no solution in the whole number.

2.3. *Main theorem*

The exponential Diophantine equation $(13^{2m}) + (6r + 1)^n = z^2$, where m, n, r, z are whole numbers, has no solution in the whole number.

Proof: There are four cases:

Case: 1 If $m = 0$ then the exponential Diophantine equation $(13^{2m}) + (6r + 1)^n = z^2$ becomes $1 + (6r + 1)^n = z^2$, which has no whole number solution by lemma 2.

Case: 2 If $n = 0$ then the exponential Diophantine equation $(13^{2m}) + (6r + 1)^n = z^2$ becomes $(13^{2m}) + 1 = z^2$, which has no whole number solution by lemma 1.

Case: 3 If m, n are positive integers, then $(13^{2m}), (6r + 1)^n$ are odd numbers.

$\Rightarrow (13^{2m}) + (6r + 1)^n = z^2$ is an even number

$\Rightarrow z$ is an even number

$\Rightarrow z^2 \equiv 0(mod3)\ or\ z^2 \equiv 1(mod3)$ (5)

Now, $13 \equiv 1(mod3)$

$\Rightarrow (13^{2m}) \equiv 1(mod3)\ and\ (6r + 1) \equiv 1(mod3)$

$\Rightarrow (13^{2m}) \equiv 1(mod3)\ and\ (6r + 1)^n \equiv 1(mod3)$

$$\Rightarrow (13^{2m}) + (6r+1)^n \equiv 2(mod3)$$
$$\Rightarrow z^2 \equiv 2(mod3) \quad (6)$$

Equation (6) contradicts equation (5).

Hence the exponential Diophantine equation $(13^{2m}) + (6r+1)^n = z^2$, where m, n are positive integers and r, z are whole numbers, has no solution in whole number.

Case: 4 If $m, n = 0$, then $(13^{2m}) + (6r+1)^n = 1 + 1 = 2 = z^2$, which is impossible because z is a whole number. Hence exponential Diophantine equation $(13^{2m}) + (6r+1)^n = z^2$, where $m, n = 0$ and r, z are whole numbers, has no solution in the whole number.

3. Conclusion

In this article, the authors successfully discussed the existence of the solution of exponential Diophantine equation $(13^{2m}) + (6r+1)^n = z^2$, where m, n, r, z are whole numbers. They determined that the exponential Diophantine equation $(13^{2m}) + (6r+1)^n = z^2$, where m, n, r, z are whole numbers, has no solution in the whole number.

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