JSR Publications

Available Online

J. Sci. Res. **14** (1), 115-130 (2022)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

Magnetograph Transformation in Variably Inclined Two Phase MFD Flows

S. K. Singh[1*], S. Sil[2], M. Kumar[3]

[1]Department of Physics, Ranchi University, Ranchi- 834001, Jharkhand, India

[2]Department of Physics, P.K. Roy Memorial College, B.B.M.K. University, Dhanbad-826004, Jharkhand, India

[3]Department of Physics, Marwari College, Ranchi University, Ranchi, India

Received 24 May 2021, accepted in final revised form 14 October 2021

Abstract

The steady plane variably inclined two phase MFD flow is considered. Magnetograph transformation is employed to obtain the solution of governing partial differential equations of second order and also to find the solution for the vortex flow. The non-linear partial differential equations have been converted into solvable form by employing Legendre's transform function as well as the polar co-ordinates. Two different forms of Legendre transform function in terms of components of magnetic field are taken as two applications of the developed theory. The components of fluid velocity, components of magnetic field, vorticity function, current density function, the fluid pressure and the number density of dust particles are found out in both the cases. The variation of components of fluid velocity with one component of magnetic field, keeping the other component constant, is plotted for different angles. The magnetic field lines are also plotted which are found out to be concentric circles for both the cases.

Keywords: Two Phase; Variably inclined; Magnetograph transformation; Legendre transformation function; Vorticity function.

doi: http://dx.doi.org/10.3329/jsr.v14i1.53571 J. Sci. Res. **14** (1), 115-130 (2022)

1. Introduction

A phase is defined as one of the states of the matter. It can be a solid, a liquid, or a gas. Multiphase flow is the simultaneous flow of several phases. Multiphase fluid system is concerned with the motion of a liquid or gas containing immiscible inert particles. Study of multiphase phenomenon is of extreme importance as it is useful in various fields of science and technology such as nuclear engineering, chemical engineering, geophysics etc. In recent years, many researchers all over the world have been studying various aspects of multiphase fluid system in non-rotating as well as rotating frame of reference. Of all multiphase fluid system observed in nature, blood flow, flow in rocket chamber, dust in gas cooling system to enhance heat transfer process, movement of inert particles in atmosphere and sand or other suspended particles in sea beaches are the most common

* *Corresponding author:* singhsantosh2065@gmail.com

examples. Presence of particles in homogeneous fluid makes the mathematics of fluid flow complicated. By variably inclined flow in which magnetic field vector and velocity vector are coplanar and the angles between these vector fields are varying point to point in flow region. Similarly in constantly inclined flow the angel between magnetic field vector and velocity vector is a non-zero constant. Also equation of motion of such system contains partial differential equation which can be in solvable or non-solvable form. By using the techniques of various transformations, e.g., hodograph method, Magnetograph method, Inverse Method etc., one tries to find out the solution, also called exact solution, of differential equations and hence finds the variables of interests.

It was in the year 1846 that Hamilton [1] conceived the term "hodograph" for a velocity locus associated with a moving particle. If $u_1(t)$, $u_2(t)$, and $u_3(t)$ be the components of velocity, the hodograph is the locus of a point whose position coordinates in an auxiliary space are $u_i(t)$, $i = 1, 2, 3$. This means the velocity components u_i serve as the independent variables in terms of which everything else, including the original position co-ordinates x_i, is to be expressed. If the magnetic field vectors of an MHD fluid is laid off from a fixed point, the extremities of these vectors trace out a curve, called the magnetograph. Here the Magnetograph transformation is used which is analogous to the hodograph transformation and equivalent linear system is obtained by interchanging the roles of dependent and independent variables. Here the variables are transformed from physical plane to magnetograph plane. It is used to study the geometry of flow pattern and to get the exact solutions of flow variables in non-inertial frame of reference i.e. the frame which is not fixed or not moving with uniform angular velocity.

Singh *et al.* [2] obtained the solution for variably inclined MHD plane flows in porous media. Bagewadi and Siddabasappa extended the work of above authors and published some papers on MGD flow, by considering two cases, constantly inclined [3] and variably inclined [4]. They have obtained solutions for these flows by transforming the basic equations from Cartesian plane to velocity and magnetograph planes. These methods help to study the flows in a more general way by the use of Jacobian matrix. Yamamota examined the flow past porous bodies by applying the generalised law using the generalised momentum equations through a porous body [5] and porous sphere [6]. They investigated the asymptotic behaviour of the flow for small permeability of the porous medium, and they show that the flow in the porous medium is governed essentially by ordinary Darcy's law except in the boundary layer near the surface. Thakur and Singh [7] considered variably inclined viscous incompressible fluid with finite electrical conductivity and obtained the partial differential equation for the flow of fluid, using magnetograph transformation. Venkateshappa *et al.* [8] used transformation technique for variably inclined rotating MHD flow. Sil and Kumar [9] found out the solution of constantly inclined rotating two phase magnetohydrodynamic flows through porous media. They obtained a second order partial differential equation and by applying hodograph transformation found the exact solution for vortex flow. Saffman [10] has formulated the equations of motion of a dusty fluid which is represented on terms of large number density N(x,t) of very small inert particles whose volume concentration is small enough to be neglected. It is assumed that the density of the dust particles is large when compared with the fluid density so that the mass concentration of the particle is an appreciable fraction of unity. In this formulation, Saffman also assumed that the individual particles of dust are so small that stoke's law of resistance between the particles and the fluid remains valid. Using the Saffman model, Michael and Miller [11] investigated the motion of dusty gas with uniform distribution of the dust particles occupied in the semi-infinite space above a rigid plane boundary. Liu [12] has studied the flow induced by an oscillating infinite plate in a dusty gas. Thakur and Mishra [13] used the Saffman model for infinitely conducting two phase fluid flow considering constant angel between fluid velocity and magnetic field, also called constantly inclined flow, and obtained the exact solution of physical importance. Bagewadi and Bhagya [14] also studied constantly and variably MHD flows through porous media. Debnath and Basu [15], Anirban and Hari [16] studied the same problem by taking transverse magnetic field. Fenuga *et al*. [17] and Das [18] studied the MHD flow under some boundary conditions of physical importance.

This paper proceeds as follows: This paper deals with the two dimensional motion of steady variably inclined two phase MFD flow of an incompressible viscous fluid with infinite electrical conductivity. The governing equations of fluid flows are transformed into magnetograph plane. A suitable Legendre's transform function of magnetic flux function is used to recast the equations in the magnetograph plane in terms of this transformed function. The resulting partial differential equations are solved for flow problems and the exact solution of two phases variably inclined MFD flows have been determined using magnetograph transformation which has not been done earlier.

2. Nomenclature

$\vec{u} =$ Fluid velocity vector
$\vec{v} =$ Dust velocity vector
$\vec{H} =$ Magnetic field vector
$P =$ Fluid pressure
$\rho =$ Fluid density
$\eta =$ Kinetic Coefficient of viscosity
$\mu =$ Magnetic permeability
$N =$ Number density of dust particles
$k = 6\pi a\eta$, Stoke's resistance (drag coefficient) for the particles
$a =$ Spherical radius of dust particles
$m =$ Mass of dust particle
$\xi =$ Vorticity function
$\Omega =$ Current density function
$B =$ Bernoulli function
$\beta =$ Variable angel between $\vec{u}$ and $\vec{H}$
$J =$ Jacobian
$\phi =$ Magnetic flux function

L = Legendre transform function
f = Any constantly differentiable function

3. Basic Equations

The basic equations of motion governing the steady flow of a dusty, incompressible viscous fluid with infinite electrical conductivity in the presence of magnetic field, from reference [12], are

$\mathrm{div}(\vec{u}) = 0,$ (Continuity) (1.1)
$\rho[(\vec{u}\cdot\mathrm{grad})\vec{u}] = -\mathrm{grad}P + \mu\,\mathrm{curl}\vec{H}\times\vec{H} + KN(\vec{v}-\vec{u}) + \eta\nabla^2\vec{u},$ (Linear Momentum) (1.2)
$\mathrm{curl}(\vec{u}\times H) = 0,$ (Diffusion) (1.3)
For dust phase,
$\mathrm{div}(N\vec{v}) = 0,$ (Continuity) (1.4)
$m(\vec{v}\cdot\mathrm{grad})\vec{v} = k\,(\vec{u}-\vec{v}),$ (Linear Momentum) (1.5)
$\mathrm{div}(\vec{H}) = 0.$ (Solenoidal) (1.6)
The situation where the velocity of fluid and dust particles are everywhere parallel, is defined as

$$\vec{v} = \frac{\alpha}{N}\vec{u}, \tag{2}$$

where α is some scalar satisfying,

$$\vec{u}\cdot\mathrm{grad}\ \alpha = 0. \tag{3}$$

which implies that α is a constant on the fluid streamlines
Introducing vorticity function, current density function and Bernoulli function

$$\xi = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}, \quad \text{(Vorticity function)} \tag{4}$$

$$\Omega = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}, \quad \text{(Current density function)} \tag{5}$$

$$B = P + \frac{1}{2}\rho u^2 \text{ Here } \quad u^2 = u_1^2 + u_2^2. \quad \text{(Bernoulli function)} \tag{6}$$

System of equations from (1.1) to (1.6) can be replaced by

$$\mathrm{div}(\vec{u}) = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0, \text{ here } \vec{u} = u_1\hat{i} + u_2\hat{j}, \tag{7}$$

$$\eta\frac{\partial \xi}{\partial y} - \rho\xi u_2 + \mu\Omega H_2 - K(\alpha - N)u_1 = -\frac{\partial B}{\partial x}, \tag{8}$$

$$\eta\frac{\partial \xi}{\partial x} - \rho\xi u_1 + \mu\Omega H_1 - K(\alpha - N)u_2 = -\frac{\partial B}{\partial y}, \tag{9}$$

$$u_1 H_2 - u_2 H_1 = f, \tag{10}$$

$$m\frac{\alpha}{N}\left[\frac{\alpha}{N}\left(u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_2}{\partial y}\right) + u_1\left(u_1\frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right)\right) + u_2\frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right] = K\left(1 - \frac{\alpha}{N}\right)u_1, \tag{11}$$

$$\mathrm{m}\frac{\alpha}{\mathrm{N}}\left[\frac{\alpha}{\mathrm{N}}\left(\mathrm{u}_1\frac{\partial \mathrm{u}_2}{\partial x}+\mathrm{u}_2\frac{\partial \mathrm{u}_2}{\partial y}\right)+\mathrm{u}_2\left(\mathrm{u}_1\frac{\partial}{\partial x}\left(\frac{\alpha}{\mathrm{N}}\right)\right)+\mathrm{u}_2\frac{\partial}{\partial y}\left(\frac{\alpha}{\mathrm{N}}\right)\right] = \mathrm{K}\left(1-\frac{\alpha}{\mathrm{N}}\right)\mathrm{u}_2, \tag{12}$$

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0. \tag{13}$$

The advantage of this system over the original system is that the order of partial differential equation is decreased.

Now if $\beta=\beta(x,y)$ is the variable angle between $\vec{\mathrm{u}}$ & $\vec{\mathrm{H}}$,

then
$$\begin{aligned}\vec{\mathrm{u}}\times\vec{H} &= \mathrm{u}_1H_2-\mathrm{u}_2H_1 = \mathrm{u}H\sin\beta = \mathrm{f}\\ \&\ \vec{\mathrm{u}}\cdot\vec{H} &= \mathrm{u}_1H_1+\mathrm{u}_2H_2 = \mathrm{u}H\cos\beta = \mathrm{f}\cot\beta,\end{aligned} \tag{14}$$

Solving these two equations to find u_1 and u_2 in terms of H_1 and H_2

$$\left.\begin{aligned}\mathrm{u}_1&=\frac{f}{H^2}\left(H_2+H_1\cot\beta\right),\\ \mathrm{u}_2&=\frac{f}{H^2}\left(H_2\cot\beta-H_1\right),\end{aligned}\right] \tag{15}$$

using this equation (15), equations(7)-(13) Now converts to

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0, \tag{16}$$

$$\eta\frac{\partial\xi}{\partial y}-\rho\xi\frac{f}{H^2}\left(H_2\cot\beta-H_1\right)+\mu\Omega H_2-K(\alpha-N)\frac{f}{H^2}\left(H_2+H_1\cot\beta\right)=-\frac{\partial \mathrm{B}}{\partial x}, \tag{17}$$

$$\eta\frac{\partial\xi}{\partial x}-\rho\xi\frac{f}{H^2}\left(H_2+H_1\cot\beta\right)+\mu\Omega H_1+K(\alpha-N)\frac{f}{H^2}\left(H_2\cot\beta-H_1\right)=-\frac{\partial \mathrm{B}}{\partial y}, \tag{18}$$

$$\begin{aligned}&\frac{\alpha}{N}\left[\frac{\partial H_2}{\partial x}\times\frac{f}{H_1^2+H_2^2}\left(H_2+H_1\cot\beta\right)\left(\frac{H_1^2-H_2^2-2H_1H_2\cot\beta}{H_1^2+H_2^2}\right)+\frac{\partial H_1}{\partial y}\frac{f}{H_1^2+H_2^2}\times\right.\\
&\left(\frac{H_2^2\cot\alpha-H_1^2\cot\beta-2H_1H_2}{H_1^2+H_2^2}\right)+\frac{\partial H_1}{\partial x}\frac{f}{H_1^2+H_2^2}\left(H_2+H_1\cot\beta\right)\\
&\left(\frac{H_2^2\cot\alpha-H_1^2\cot\beta-2H_1H_2}{H_1^2+H_2^2}\right)+\frac{\partial H_1}{\partial y}\frac{f}{H_1^2+H_2^2}\left(H_2\cot\beta-H_1\right)\left(\frac{H_1^2-H_2^2+2H_1H_2\cot\beta}{H_1^2+H_2^2}\right)\\
&+H_1\frac{f}{H_1^2+H_2^2}\left(\frac{\partial\cot\beta}{\partial x}H_2+H_1\cot\alpha\frac{\partial\cot\beta}{\partial x}+H_2\cot\alpha\frac{\partial\cot\beta}{\partial y}-H_1\frac{\partial\cot\beta}{\partial y}\right)\\
&+\frac{f}{H_1^2+H_2^2}\left(H_2+H_1\cot\beta\right)\left(H_2\frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right)+H_1\cot\beta\frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right)+H_2\cot\beta\frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right)\\
&\left.-H_1\frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right]=K\left(\frac{\alpha}{N}-1\right)\left(H_2+H_1\cot\beta\right),\qquad \text{and}\end{aligned} \tag{19}$$

$$\begin{aligned}&\frac{\alpha}{N}\left[\frac{-f}{H_1^2+H_2^2}\left(H_2+H_1\cot\beta\right)\frac{\partial H_1}{\partial x}\left(\frac{H_2^2-H_1^2+2H_1H_2\cot\beta}{H_1^2+H_2^2}\right)-\frac{f}{H_1^2+H_2^2}\left(H_2\cot\beta-H_1\right)\right.\\
&\frac{\partial H_1}{\partial y}\left(\frac{H_2^2-H_1^2+2H_1H_2\cot\beta}{H_1^2+H_2^2}\right)-\frac{f}{H_1^2+H_2^2}\left(H_2\cot\beta-H_1\right)\frac{\partial H_2}{\partial y}\left(\frac{H_1^2\cot\alpha-H_2^2\cot\beta-2H_1H_2}{H_1^2+H_2^2}\right)\end{aligned}$$

$$+\frac{f}{H_1^2+H_2^2}\left(H_2+H_1\cot\beta\right)\frac{\partial H_1}{\partial x}\left(\frac{H_1^2\cot\alpha-H_2^2\cot\beta-2H_1H_2}{H_1^2+H_2^2}\right)+\frac{f}{H_1^2+H_2^2}\left(H_2+H_1\cot\beta\right)H_2\frac{\partial\cot\beta}{\partial x}$$

$$+\frac{f}{H_1^2+H_2^2}\left(H_2\cot\beta-H_1\right)H_2\frac{\partial\cot\beta}{\partial y}+\frac{f}{H_1^2+H_2^2}\left(H_2\cot\beta-H_1\right)\times$$

$$\left\{\left(H_2+H_1\cot\beta\right)\frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right)+\left(H_2\cot\beta-H_1\right)\frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right)\right\}\Bigg]=K\left(\frac{\alpha}{N}-1\right)\left(H_2\cot\beta-H_1\right), \tag{20}$$

$$\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=0, \tag{21}$$

$$\xi=\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\Rightarrow\frac{\partial}{\partial x}\left[\frac{H_2\cot\beta-H_1}{H_1^2+H_2^2}\right]-\frac{\partial}{\partial y}\left[\frac{H_2+H_1\cot\beta}{H_1^2+H_2^2}\right]-\frac{\xi}{f}, \tag{22}$$

And,

$$\eta\frac{\partial\xi}{\partial y}-\rho\xi\frac{f}{H_1^2+H_2^2}\left(H_2\cot\beta-H_1\right)+\mu\Omega H_2-k\left(\alpha-N\right)\frac{f}{H_1^2+H_2^2}\left(H_2+H_1\cot\beta\right)=-\frac{\partial B}{\partial x}, \tag{23}$$

$$\eta\frac{\partial\xi}{\partial x}-\rho\xi\frac{f}{H_1^2+H_2^2}\left(H_2+H_1\cot\beta\right)+\mu\Omega H_2+k\left(\alpha-N\right)\frac{f}{H_1^2+H_2^2}\left(H_2\cot\beta-H_1\right)=\frac{\partial B}{\partial y}. \tag{24}$$

4. Magnetograph Transformation

As mentioned in the flow equations, $H_1=H_1(x,y),\ H_2=H_2(x,y)$, the Jacobian is $J(x,y)=\frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial y}-\frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x},\ 0<|J|<\infty$. We consider x and y as a function of H_1 and H_2 **.** So by means of $x=x(H_1,H_2)$ & $y=y(H_1,H_2)$,
We have the following relations

$$\frac{\partial H_1}{\partial x}=J\frac{\partial y}{\partial H_2},\ \frac{\partial H_2}{\partial x}=-J\frac{\partial y}{\partial H_1},\ \& \tag{25}$$

$$\frac{\partial H_1}{\partial y}=-J\frac{\partial x}{\partial H_2},\ \frac{\partial H_2}{\partial y}=J\frac{\partial x}{\partial H_1}.$$

Further,

$$J(x,y)=\frac{\partial(H_1,H_2)}{\partial(x,y)}=\left[\frac{\partial(x,y)}{\partial(H_1,H_2)}\right]^{-1}=j(H_1,H_2)\ \text{and}$$

$$\frac{\partial f}{\partial x}=j\frac{\partial(f,y)}{\partial(H_1,H_2)},\ \frac{\partial f}{\partial y}=j\frac{\partial(f,x)}{\partial(H_1,H_2)}. \tag{26}$$

where, f is any constantly differentiable function and $f(H_1,H_2)$ is its transformed function in the H_1,H_2 plane.

5. Flow Equations in Magnetograph Plane

Employing the above transformation relations for the first order partial derivatives in the system of equations (19) - (24), the transformed system of partial differential equations in the magnetograph plane i.e. (H_1, H_2) plane is,

$$\frac{\partial x}{\partial H_1}+\frac{\partial y}{\partial H_2}=0, \tag{27}$$

$$\eta j\frac{\partial(x,\zeta)}{\partial(H_1,H_2)}-\rho\zeta\frac{f}{H_1^2+H_2^2}(H_2+H_1\cot\beta)+\mu\Omega H_2+k(\alpha-N)\frac{f}{H_1^2+H_2^2}(H_2\cot\beta-H_1)=-J\frac{\partial(B,y)}{\partial(H_1,H_2)}, \tag{28}$$

$$\eta j\frac{\partial(\zeta,Y)}{\partial(H_1,H_2)}-\rho\zeta\frac{f}{H_1^2+H_2^2}(H_2\cot\beta-H_1)+\mu\Omega H_2-k(\alpha-N)\frac{f}{H_1^2+H_2^2}(H_2+H_1\cot\beta)=-J\frac{\partial(x,B)}{\partial(H_1,H_2)}, \tag{29}$$

from equation (20) and using equation (25)

$$\frac{\alpha}{N}\left[\begin{array}{l}
-\frac{f}{H_1^2+H_2^2}(H_2+H_1\cot\beta)J\frac{\partial y}{\partial H_2}\left(\begin{array}{l}\frac{H_2^2-H_1^2+2H_1H_2\cot\beta}{H_1^2+H_2^2}-\\ H_2\frac{N}{\alpha}\frac{\partial\cot\beta}{\partial H_1}+(H_2\cot\beta-H_1)\frac{N}{\alpha}\frac{\partial}{\partial H_1}\left(\frac{\alpha}{N}\right)\end{array}\right)+\\
\frac{f}{H_1^2+H_2^2}(H_2\cot\beta-H_1)J\frac{\partial x}{\partial H_2}\left(\frac{H_2^2-H_1^2+2H_1H_2\cot\beta}{H_1^2+H_2^2}\right)-\\
\frac{f}{H_1^2+H_2^2}(H_2\cot\beta-H_1)J\frac{\partial x}{\partial H_1}\left(\begin{array}{l}\frac{H_1^2\cot\beta-H_2^2\cot\beta-2H_1H_2}{H_1^2+H_2^2}+H_2\frac{N}{\alpha}\frac{\partial\cot\beta}{\partial H_2}+\\ (H_2\cot\beta-H_1)\frac{N}{\alpha}\frac{\partial}{\partial H_2}\left(\frac{\alpha}{N}\right)\end{array}\right)\\
+\frac{f}{H_1^2+H_2^2}(H_2+H_1\cot\beta)J\frac{\partial y}{\partial H_1}\left(\frac{H_1^2\cot\beta-H_2^2\cot\beta-2H_1H_2}{H_1^2+H_2^2}\right)
\end{array}\right]$$

$$=k\left(\frac{\alpha}{N}-1\right)(H_2\cot\beta-H_1). \tag{30}$$

6. Legendre Transformation of Magnetic Flux Function

The solenoidal equation $\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0$ implies the existence of magnetic flux function $\phi(x,y)$ such that

$$d\phi=-H_2dx+H_1dy \;\; or \;\; \frac{\partial\phi}{\partial x}=-H_2,\; \frac{\partial\phi}{\partial y}=H_1.$$

Likewise equation $\frac{\partial x}{\partial H_1}+\frac{\partial y}{\partial H_2}=0$ implies the existence of a function $L(x,y)$ called Legendre Transform function of the magnetic flux function $\phi(x,y)$ such that

$$dL = ydH_1 + xdH_2 \text{ , which implies that } x=\frac{\partial L}{\partial H_1},\ y=\frac{\partial L}{\partial H_2}. \tag{31}$$

And these two are related by $L(H_1,H_2)=H_2x-H_1y+\phi(x,y)$. Introducing $L(H_1,H_2)$ into the system of equations (27) to (30) with ' j ' given by (26), it follows that equation (27) is identically satisfied and the system may be replaced by

$$\eta j\frac{\partial\left(\frac{\partial L}{\partial H_2},\zeta\right)}{\partial(H_1,H_2)}-\rho\zeta\frac{f}{H_1^2+H_2^2}(H_2+H_1\cot\beta)+\mu\Omega H_2+k(\alpha-N)\frac{f}{H_1^2+H_2^2}(H_2\cot\beta-H_1) = -J\frac{\partial\left(B,\frac{\partial L}{\partial H_1}\right)}{\partial(H_1,H_2)}, \tag{32}$$

$$\eta j\frac{\partial\left(\frac{\partial L}{\partial H_1},\zeta\right)}{\partial(H_1,H_2)}-\rho\zeta\frac{f}{H_1^2+H_2^2}(H_2\cot\beta-H_1)+\mu\Omega H_2-k(\alpha-N)\frac{f}{H_1^2+H_2^2}(H_2+H_1\cot\beta) = -J\frac{\partial\left(B,\frac{\partial L}{\partial H_2}\right)}{\partial(H_1,H_2)}, \tag{33}$$

$$\frac{\alpha}{N}\left[\begin{array}{l}
-\frac{f}{H_1^2+H_2^2}(H_2+H_1\cot\beta)J\frac{\partial^2 l}{\partial H_1\partial H_2}\left(\begin{array}{l}\frac{H_2^2-H_1^2+2H_1H_2\cot\beta}{H_1^2+H_2^2}-\\ H_2\frac{N}{\alpha}\frac{\partial\cot\beta}{\partial H_1}+(H_2\cot\beta-H_1)\frac{N}{\alpha}\frac{\partial}{\partial H_1}\left(\frac{\alpha}{N}\right)\end{array}\right)+\\
\frac{f}{H_1^2+H_2^2}(H_2\cot\beta-H_1)J\frac{\partial^2 L}{\partial H_2^2}\left(\frac{H_2^2-H_1^2+2H_1H_2\cot\beta}{H_1^2+H_2^2}\right)-\\
\frac{f}{H_1^2+H_2^2}(H_2\cot\beta-H_1)J\frac{\partial^2 L}{\partial H_1\partial H_2}\left(\begin{array}{l}\frac{H_1^2\cot\beta-H_2^2\cot\beta-2H_1H_2}{H_1^2+H_2^2}+H_2\frac{N}{\alpha}\frac{\partial\cot\beta}{\partial H_2}+\\ (H_2\cot\beta-H_1)\frac{N}{\alpha}\frac{\partial}{\partial H_2}\left(\frac{\alpha}{N}\right)\end{array}\right)\\
+\frac{f}{H_1^2+H_2^2}(H_2+H_1\cot\beta)J\frac{\partial^2 L}{\partial H_{1_1}^2}\left(\frac{H_1^2\cot\beta-H_2^2\cot\beta-2H_1H_2}{H_1^2+H_2^2}\right)
\end{array}\right] = k\left(\frac{\alpha}{N}-1\right)(H_2\cot\beta-H_1). \tag{34}$$

7. Applications

Application:1

Let $L(H_1,H_2)=A_1\left(H_1^2+H_2^2\right)+A_2$

And $\beta(H_1,H_2)=\cot^{-1}\left(CH_1^2+CH_2^2\right)$ form a solution set of the partial differential equation (34).

When $A_1 \neq 0, A_2$ and C are arbitrary constants, we may consider two separate cases of the solution

(i) If, $C \neq 0$ the flows will be variably inclined, and
(ii) If, $C = 0$ the flows will be crossed

So if $C \neq 0$, using above i.e. $\beta(H_1, H_2) = \cot^{-1}(CH_1^2 + CH_2^2)$ in $\frac{\partial L}{\partial H_1} = -y$ & $\frac{\partial L}{\partial H_2} = x$, we have

$2A_1H_1 = -y$ and $2A_1H_2 = x$

So, $H_1 = \frac{-y}{2A_1}$ and $H_2 = \frac{x}{2A_1}$ or $H_1 = \frac{-y}{A}$ and $H_2 = \frac{x}{A}$ (where $A = 2A_1$) (35)

This represents linear flow and magnetic field lines are concentric circles.
Using above for $H_1(x, y)$ and $H_2(x, y)$, we have

$$\mathrm{u}_1 = \frac{f}{H_1^2 + H_2^2}(H_2 + H_1 \cot \beta) \Rightarrow \mathrm{u}_1 = \frac{f}{r^2}\left(Ax - \frac{C}{A}yr^2\right), \tag{36a}$$

and
$$\mathrm{u}_2 = \frac{f}{H_1^2 + H_2^2}(H_2 \cot \beta - H_1) \Rightarrow \mathrm{u}_2 = \frac{f}{r^2}\left(Ay + \frac{C}{A}xr^2\right). \tag{36b}$$

Converting this into polar form, $x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2,\ \frac{y}{x} = \tan\theta,$ (37)

Also, vorticity function, $\xi = \frac{\partial \mathrm{u}_2}{\partial x} - \frac{\partial \mathrm{u}_1}{\partial y} = \frac{2fC}{A}.$ (38)

And current density, $\Omega = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = 2.$ (39)

Hence from equation (8) and (9), we have

$$\left(\frac{fAy}{x^2 + y^2} + \frac{fCx}{A}\right)\frac{\partial}{\partial x}(\alpha - N) - \left(\frac{fAx}{x^2 + y^2} - \frac{fCy}{A}\right)\frac{\partial}{\partial y}(\alpha - N) = -(\alpha - N)\xi. \tag{40}$$

Also, from (37), $\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$ & $\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$.

We have, equation (40) now converts to

$$\frac{-fA}{r^2}\frac{\partial t}{\partial\theta} + \frac{fCr}{A}\frac{\partial t}{\partial r} = -t\xi, \qquad Here\ t = \alpha - N \tag{41}$$

(i) (ii) (iii)

So, using (ii) and (iii), we have

$$\frac{Adr}{fCr} = \frac{-dt}{t\xi},$$

Integrating both side , we have

$$\frac{A}{fC}\ln r = -\frac{1}{\xi}\ln t + \ln c_1 \Rightarrow \frac{A}{fC}\ln r = -\frac{A}{2fC}\ln t + \ln c_1,$$ (here c_1 = arbitrary constant)

$$\ln r = \frac{-1}{2}\ln t + \frac{fC}{A}\ln c_1 \ ,$$

or $2\ln r = -\ln t + \ln (c_1)^{\frac{2fC}{A}} = \ln\left[\frac{(c_1)^{\frac{2fC}{A}}}{t}\right]$,

or $r^2 = \frac{(c_1)^{\frac{2fC}{A}}}{t} \Rightarrow t = \frac{c_2}{x^2+y^2}$, where $c_2 = (c_1)^{\frac{2fC}{A}} =$ another constant

Hence, from $t = \alpha - N$,

Number density of dust particle is, $\mathrm{N} = \alpha - \frac{c_2}{x^2+y^2}$. (42)

Now from equation (3), $\vec{u}\cdot \text{grad} \ \ \alpha = 0 \Rightarrow \left(u_1\hat{i} + u_2\hat{j}\right)\cdot\left(\frac{\partial\alpha}{\partial x}\hat{i} + \frac{\partial\alpha}{\partial y}\hat{j}\right) = 0$,

$$u_1\frac{\partial\alpha}{\partial x} + u_2\frac{\partial y}{\partial y} = 0 \ \Rightarrow u_1\frac{\partial\alpha}{\partial x} = -u_2\frac{\partial\alpha}{\partial y} \quad or \quad \frac{\partial\alpha}{u_2\partial x} = -\frac{\partial\alpha}{u_1\partial y} = D \ ,$$

$$\therefore \frac{\partial\alpha}{u_2\partial x} = D \Rightarrow \partial\alpha = Du_2\partial x \ \Rightarrow \int \partial\alpha = \int \ Df\left(\frac{Ay}{x^2+y^2} + \frac{Cx}{A}\right)\partial x \ ,$$

$$\alpha = Df\left[Ay\times\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) + \frac{Cx^2}{2A}\right]. \quad (43)$$

And also from $\int \partial\alpha = \int -Du_1\partial y$, we have ,

$$\int \partial\alpha = \int -Df\left(\frac{Ax}{x^2+y^2} - \frac{Cy}{A}\right)\partial y, \ \alpha = -Df\left[Ax\times\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right) - \frac{Cx^2}{2A}\right], \quad (44)$$

Hence from (42),

$$N = Df\left[A\tan^{-1}\left(\frac{x}{y}\right) + \frac{Cx^2}{2A}\right] - \frac{c_2}{x^2+y^2} \quad or \quad N = -Df\left[A\tan^{-1}\left(\frac{y}{x}\right) - \frac{Cy^2}{2A}\right] - \frac{c_2}{x^2+y^2}. \quad (45)$$

Now magnetic flux function

$\frac{\partial\phi}{\partial x} = -H_2$ and $\frac{\partial\phi}{\partial y} = H_1$,

We have, $\frac{\partial\phi}{\partial x} = -\frac{x}{2A_1} \Rightarrow \phi = \frac{-x^2}{4A_1} + C_1$, and $\frac{\partial\phi}{\partial y} = -\frac{y}{2A_1} \Rightarrow \phi = \frac{-y^2}{4A_1} + C_1'$,

Therefore, magnetic flux function is given as $\phi = -\frac{x^2}{4A_1} - \frac{y^2}{4A_1} + C_1$. (46)

Using equations (35), (36), (38) and (39) in equation (8) and (9), and on solving, we get Bernoulli function as,

$$B = \frac{2\rho f^2 C}{A}\left[A\tan^{-1}\left(\frac{x}{y}\right) + \frac{Cx^2}{2A}\right] - \frac{\mu x^2}{A} + \frac{Kc_2 fA}{2\left(x^2+y^2\right)} + \frac{Kc_2 fC}{A}\tan^{-1}\left(\frac{x}{y}\right) + c_3 \ . \quad (47)$$

, Or

$$B=\frac{2\rho f^2C}{A}\left[A\tan^{-1}\left(\frac{y}{x}\right)-\frac{Cy^2}{2A}\right]-\frac{\mu y^2}{A}-\frac{Kc_2 fA}{2\left(x^2+y^2\right)}+\frac{Kc_2 fC}{A}\tan^{-1}\left(\frac{y}{x}\right)+c_3 .$$

Where, c_3 is an arbitrary constant.

Also from,

$B=P+\frac{1}{2}\rho U^2=P+\frac{1}{2}\rho\left(u_1{}^2+u_2{}^2\right)$, fluid pressure function is given as,

$$P=B-\frac{1}{2}\rho f^2\left[\frac{A^2}{x^2+y^2}+\frac{C^2}{A^2}\left(x^2+y^2\right)\right],$$

$$\therefore P=2\rho f^2C\tan^{-1}\left(\frac{x}{y}\right)+\frac{\rho f^2C^2}{2}\left(x^2+y^2\right)-\frac{\mu x^2}{A}+\frac{Kc_2 fA}{2\left(x^2+y^2\right)}+\frac{Kc_2 fC}{A}\tan^{-1}\left(\frac{x}{y}\right)+c_3 . \tag{48}$$

Application: 2

Let, $L=A_1\left(H_1{}^2+H_2{}^2\right)+A_2H_1+A_3H_2$,

And $\beta=\cot^{-1}\left(CH_1{}^2+CH_2{}^2\right)$,

$$\therefore \frac{\partial L}{\partial H_1}=-y\Rightarrow H_1=\frac{-\left(y+A_2\right)}{2A_1} .$$

And $\dfrac{\partial L}{\partial H_2}=x\Rightarrow H_2=\dfrac{\left(x-A_3\right)}{2A_1}$.

$$u_1=\frac{f}{H_1{}^2+H_2{}^2}\left(H_2+H_1\cot\alpha\right)=f\left[\frac{2A_1\left(x-A_3\right)}{\left(x-A_3\right)^2+\left(y+A_2\right)^2}-\frac{C}{2A_1}\left(y+A_2\right)\right]. \tag{49a}$$

$$u_2=\frac{f}{H_1{}^2+H_2{}^2}\left(H_2\cot\alpha-H_1\right)=f\left[\frac{C}{2A_1}\left(x-A_3\right)+\frac{2A_1\left(y+A_2\right)}{\left(x-A_3\right)^2+\left(y+A_2\right)^2}\right]. \tag{49b}$$

Also, vorticity function, $\xi=\dfrac{\partial u_2}{\partial x}-\dfrac{\partial u_2}{\partial y}=\dfrac{fC}{A_1}$. (50)

And, current density, $\Omega=\dfrac{\partial H_2}{\partial x}-\dfrac{\partial H_1}{\partial y}=\dfrac{1}{A_1}$. (51)

$\therefore$ from (8) and (9), we have

$$u_2\frac{\partial}{\partial x}(\alpha-N)-u_1\frac{\partial}{\partial y}(\alpha-N)=-(\alpha-N)\xi ,$$

$$f\left[\frac{C}{2A_1}\left(x-A_3\right)+\frac{2A_1\left(y+A_2\right)}{\left(y+A_2\right)^2+\left(x-A_3\right)^2}\right]\frac{\partial}{\partial x}(\alpha-N)-f\left[\frac{2A_1\left(x-A_3\right)}{\left(x-A_3\right)^2+\left(y+A_2\right)^2}-\frac{C}{2A_1}\left(y+A_2\right)\right]\frac{\partial}{\partial y}(\alpha-N)$$
$$=-(\alpha-N)\xi ,$$

Let $x-A_3=r\cos\theta$ & $y+A_2=r\sin\theta$, (52)

so that $\left(x-A_3\right)^2+\left(y+A_2\right)^2=r^2$, Also $\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$ & $\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$. (53)

$$\therefore\left(\frac{fCr\cos\theta}{2A_1}+\frac{2fA_1r\sin\theta}{r^2}\right)\left[\cos\theta\frac{\partial t}{\partial r}-\frac{\sin\theta}{r}\frac{\partial t}{\partial\theta}\right]-\left(\frac{2fA_1r\cos\theta}{r^2}-\frac{fCr\sin\theta}{2A_1}\right)\left[\sin\theta\frac{\partial t}{\partial r}+\frac{\cos\theta}{r}\frac{\partial t}{\partial\theta}\right]=-t\xi,$$

Here $t=\alpha-N$

On solving we get

$$\frac{fCr}{2A_1}\frac{\partial t}{\partial r}-\frac{2fA_1}{r^2}\frac{\partial t}{\partial\theta}=-t\xi\ ,$$

$$\therefore\frac{2A_1dr}{fCr}=\frac{r^2d\theta}{-2fA_1}=\frac{dt}{-t\xi}\ . \tag{54}$$

(i) (ii) (iii)

From (i) and (ii)

$$\frac{2A_1}{fC}\frac{dr}{r}=-\frac{dt}{t\xi}\ ,$$

$$\frac{2A_1}{fC}\ln r=-\frac{1}{\xi}\ln t+\ln c_3\ ,$$

$$\ln r=-\frac{fC}{2A_1}\times\frac{2A_1}{fC}\ln t+\frac{fC}{2A_1}\ln c_3\ ,$$

On solving, $t=\frac{c_4}{(x-A_3)^2+(y+A_2)^2}$, where $c_4=(c_3)^{\frac{fC}{2A_1}}$

Hence, number density of dust is, $N=\alpha-\frac{c_4}{(x-A_3)^2+(y+A_2)^2}$. (55)

And as in application:1,

$$\alpha=Df\left[A\tan^{-1}\left(\frac{x-A_3}{y+A_2}\right)+\frac{C(x-A_3)^2}{A}\right].$$

Or $\alpha=Df\left[A\tan^{-1}\left(\frac{y+A_2}{x-A_3}\right)-\frac{C(y+A_2)^2}{A}\right].$

$$\therefore N=Df\left[A\tan^{-1}\left(\frac{y+A_2}{x-A_3}\right)-\frac{C(y+A_2)^2}{A}\right]-\frac{c_4}{(x-A_3)^2+(y+A_2)^2}\ . \tag{56}$$

Bernoulli function is now,

$$B=\frac{2\rho f^2C}{A}\left[A\tan^{-1}\left(\frac{x-A_3}{y+A_2}\right)+\frac{C(x-A_3)^2}{2A}\right]+\frac{\mu(x-A_3)^2}{A}+\frac{Kc_2fA}{2\left[(x-A_3)^2+(y+A_2)^2\right]} \tag{57}$$

$$+\frac{Kc_2fC}{A}\tan^{-1}\left[\frac{x-A_3}{y+A_2}\right]+C_5\ .$$

Pressure function is now,

$$P = 2\rho f^2 C \tan^{-1}\left(\frac{x-A_3}{y+A_2}\right) + \frac{\rho f^2 C^2}{2}\left[\left(x-A_3\right)^2 + \left(y+A_2\right)^2\right] - \frac{\mu\left(x-A_3\right)^2}{A} \tag{58}$$

$$+ \frac{Kc_2 fA}{2\left[\left(x-A_3\right)^2 + \left(y+A_2\right)^2\right]} + \frac{Kc_2 fC}{A}\tan^{-1}\left(\frac{x-A_3}{y+A_2}\right) + c_5 \ .$$

where c_5 is an arbitrary constant,
and magnetic flux function is given by

$$\phi = -\frac{\left(x-A_3\right)^2}{4A_1} - \frac{\left(y+A_2\right)^2}{4A_1} + C' \ . \tag{59}$$

Hence, by using two different forms of Legendre transformation function, we have determined the exact solutions of the governing equations by Magnetograph transformation and have found out the expression for magnetic flux function, pressure function, Bernoulli function number density and also found the expressions for each of the two velocity components u_1 and u_2 .

8. Results and Discussion

In present work magnetograph transformation has been used for variable angle between fluid velocity and magnetic field and we have determined the exact solutions for the variables of interests. Magnetic field lines are found to be concentric circles for both the applications (Fig. 3) which are independent of the angle between fluid velocity and magnetic field i.e. β . Value of vorticity function is zero for constantly inclined (C = 0), equation (38) and (50), and has some finite value for variably inclined. Variations of the components of fluid velocity u_1 and u_2 with respect to H_1 are plotted. These variations are due to the fact that the velocity vector of fluid and the magnetic field vector are variable inclined. $\cot\beta$ plays the most vital role for the observed trend. In Fig. (1a) the variations are plotted for small angles between 1° to 10°. The nature of variation is almost same for all the angles. It can be seen that for the angle 1° the curve shoots high as the value of cot 1° is larger. Next in Fig. (1b) the variation of u_1 is plotted for the angles between 15° to 90°. The variations for different angles in this case too are almost having the same nature with the 15° curve getting high due to the relatively larger value of cot 15°. In each case, except for the angle 90° the curves rise to a maximum and then falls down. For the angle 90°, however, as the value of cot 90° is zero, the curve falls down from the beginning. In Fig. (2), variation of u_2 is plotted. In this case too, the nature of the curves for different angles is almost similar with the curves starting with maximum value, attaining a minimum value and then seems to converge at a point.

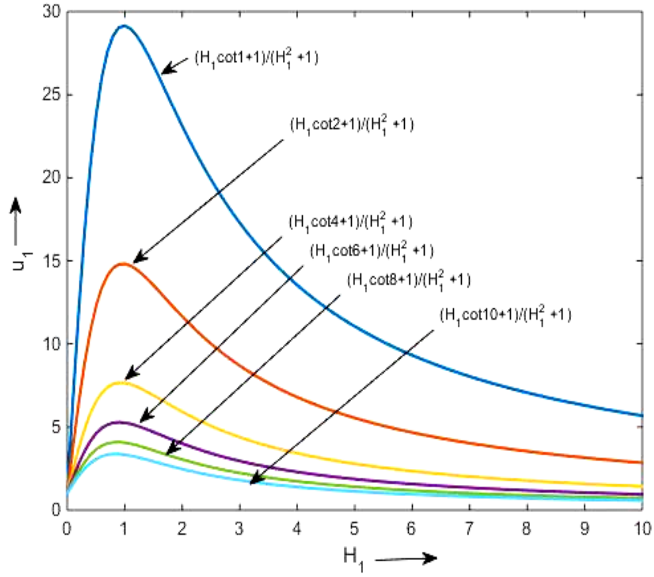

Fig.1a. Variation of u_1 with respect to H_1. $u_1 = \frac{f}{H^2}\left(H_1 \cot\beta + H_2\right),\quad f = 1, H_2 = 1$

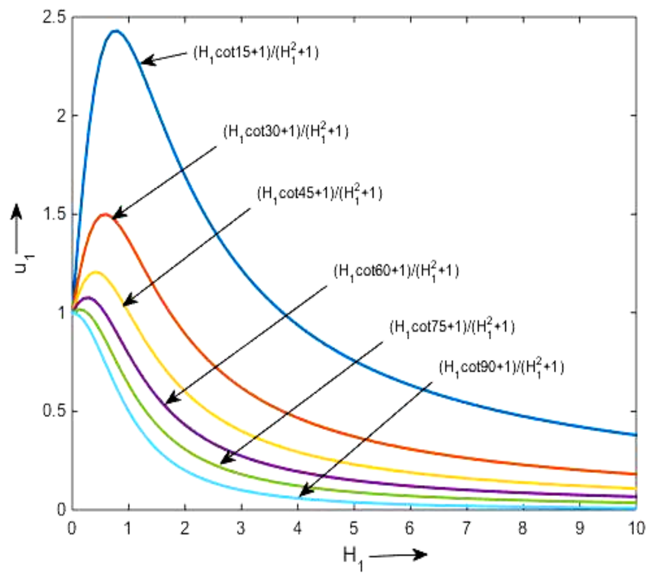

Fig. 1b. Variation of u_1 with respect to H_1. $u_1 = \frac{f}{H^2}\left(H_1 \cot\beta + H_2\right),\quad f = 1, H_2 = 1$

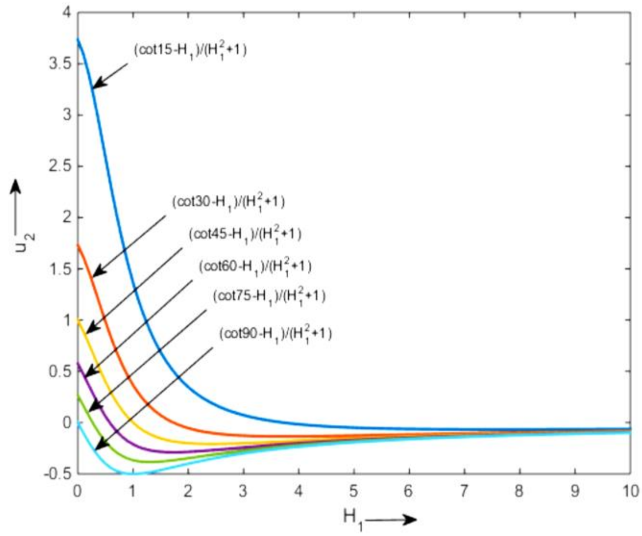

Fig. 2. Variation of u_2 with respect to H_1. $u_2 = \frac{f}{H^2}\left(H_2 \cot\beta - H_1\right),\ \ f = 1, H_2 = 1$

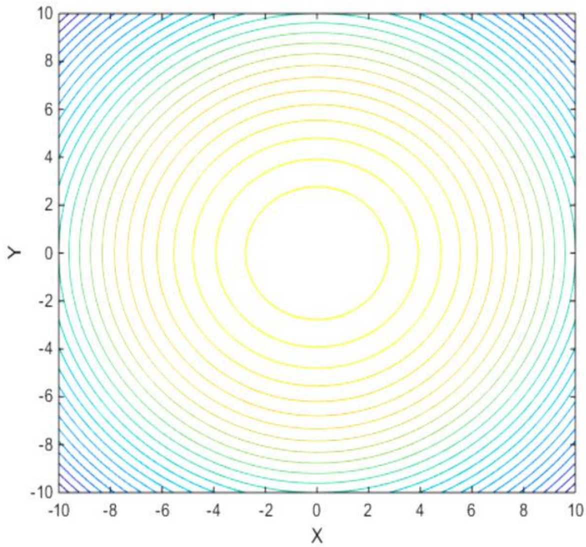

Fig. 3. Concentric circular magnetic field lines.

9. Conclusion

This paper aims to introduce the reader to the modelling of two-phase flow in general, liquid-gas flow in particular, and the prediction of fluid pressure gradient specially. Two different forms of Legendre transform function, application 1 and 2, were presented for two-phase flow. Due to the mathematical complexity of two phase MFD flow, very few exact solutions for the conducting fluid flow under the presence of electric and magnetic field are known till date. To reduce some of the complexity, it becomes necessary to make certain assumptions about the inherent properties of two phase fluid. Here, in this paper, we have found out the expressions for magnetic flux function, (46) and (59), Bernoulli function, (47) and (57), pressure function, (48) and (58), and number density of fluid particle , (45) and (56), by determining the exact solution. These results resembles exactly as obtained by C. Thakur and Ram Babu Mishra,[10] , if we assume β, i.e. angle between $\vec{u}$ and $\vec{H}$ constant. If we consider the motion of fluid particles only (one phase) i.e. $\vec{v} = 0$, $N = 0$, then we obtain the same flow equation as is obtained by C. S. Bagewadi [11].

References

1. S. N. Singh, Int. J. Theor. Phys. **27**, 1137 (1988). https://doi.org/10.1007/BF00674355
2. K. K. Singh and D. P. Singh, Bull. Cal. Math. Soc. **85**, 255 (1993).
3. C. S. Bagewadi and Siddabasappa, Ganita **42**, 13 (1991).
4. C. S. Bagewadi and Siddabasappa, Bul. Cal. Math. Soc. **85**, 513 (1993).
5. K. Yamamoto, J. Phys. Soc. **34**, 814 (1973). https://doi.org/10.1143/JPSJ.34.814
6. K. Yamamoto, J. Phys. Soc. **31**, 572 (1971). https://doi.org/10.1143/JPSJ.31.1572
7. C. Thakur and B. Singh, Bull. Cal. Math. Soc. **92**, 39 (2000).
8. Y. Venkateshappa, Siddabasappa, and B. Rudraswamy, IJAMAA **11**, 13 (2016).
9. S. Sil, M. Kumar, and S. Singh, Int. J. Math. Arch. **9**, 225 (2018).
10. P. G. Saffman, J. Fluid Mech. **13**, 120 (1962). https://doi.org/10.1017/S0022112062000555
11. D. H. Michael and D. A. Miller, Mathematika **13**, 97 (1966). https://doi.org/10.1112/S0025579300004289
12. J. T. C. Liu, Phys. Fluid **9**, 1716 (1966). https://doi.org/10.1063/1.1761926
13. C. Thakur and R. B. Mishra, Ind. J. Pure Appl. Math. **20**, 728 (1989).
14. C. S. Bagewadi and S. Bhagya, Mapana J. Sci. **2**, 1 (2003). https://doi.org/10.12723/mjs.3.1
15. L. Debnath, U. Basu, II Nuvo Cimento B Series 11. **28**, 349 (1975).
16. A. Roy and H. Baskar, Int. J. Curr. Eng. Sci. Res. **4** (2017).
17. O. J. Fenuga , S. J. Aroloye, and S. O. Salawu, J. Sci. Res. **12**, 485 (2020). https://doi.org/10.3329/jsr.v12i4.45551
18. U. J. Das, J. Sci. Res. **13**, 785 (2021). https://doi.org/10.3329/jsr.v13i3.52234