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On Certain Quadruple Fixed Points of Integral Type Contraction Mappings in G-Metric Spaces and its Applications

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Abstract

In this paper, the existence of common quadruple fixed point results for integral type contraction of two mappings in *G*-metric spaces is established. Some interesting consequences of our results are achieved. Moreover, we give an illustration that presents the applicability of the achieved results.

Keywords: Integral type contraction; Completeness; Weakly compatible; Coincidence point; Quadruple fixed point.

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1. Introduction

The fixed-point theory is an influential branch of nonlinear analysis. It demonstrated the basic significance of studying various equations arising in physical, biological, social, engineering, and other branches of science and technology. It is universally used to investigate the conditions under which single-valued or multivalued mappings have solutions.

In 2002, Branciari [1] investigated the idea of using Lebesgue integrals in metric fixed point theory and proved the existence and uniqueness of fixed points for integrally contractions whenever the metric space (X; d) is complete. After that, many authors considered various versions of integral contractions and obtained fixed point results with respect to these contractions in various metric spaces in [2-5] and references contained therein.

In 2006, Mustafa and Sims [6] initiated the concept of *G*-metric spaces and gave variant related fixed point results. Afterward, many authors have developed different fixed-point results on the setting of *G*-metric spaces [7-13].

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Very recently, Karapinar [14] used quadruple fixed points and proved some quadruple fixed results in partially ordered metric spaces. Subsequently, many investigators [15-22] developed quadruple fixed theorems in various metric spaces.

This manuscript aims to provide some common quadruple fixed point results in *G*-metric spaces by using integral type contraction. Also, we give examples of applications to Homotopy theory and integral equations.

First, let's review the important concepts of *G*-metric spaces.

Preliminaries from existing literature

A mapping $G: \mathcal{P}^3 \to [0, \infty)$ satisfying the following properties is called *G*-metric on $\mathcal{P}$:

(i) $G(p_1, p_2, p_3) = 0$ if $p_1 = p_2 = p_3$

(ii) $0 < G(p_1, p_1, p_2) = 0$ for any $p_1, p_2 \in \mathcal{P}$ with $p_1 \neq p_2$

(iii) If $G(p_1, p_1, p_2) = G(p_1, p_2, p_3)$ for all $p_1, p_2, p_3 \in \mathcal{P}$ with $p_2 \neq p_3$

(iv) $G(p_1, p_2, p_3) = G(P[p_1, p_2, p_3])$, where P is a permutation of p_1, p_2, p_3 (symmetry)

(v) $G(p_1, p_2, p_3) \leq G(p_1, x, x) + G(x, p_2, p_3)$ for all $p_1, p_2, p_3, x \in \mathcal{P}$ (rectangle inequality).

If $G(p_1, p_1, p_2) = G(p_2, p_1, p_1)$ for all $p_1, p_2 \in \mathcal{P}$, then the *G*-metric is symmetric. If G is a G-metric on $\mathcal{P}$, then $(\mathcal{P}, G)$ is called G-metric space. Sequence $\{p_n\}$ in $(\mathcal{P}, G)$ is a G-Cauchy sequence if for each $\epsilon > 0$ there is an integer $n_0 \in Z^+$ such that for all $n, m, l \geq n_0$, $G(p_n, p_m, p_l) < \epsilon$ and it is G-convergent to a point $p \in \mathcal{P}$ if, for each $\epsilon > 0$, there is an integer $n_0 \in Z^+$ such that for all $n, m \geq n_0, G(p_n, p_m, p) < \epsilon$. $(\mathcal{P}, G)$ is G-complete if every G-Cauchy sequence in $\mathcal{P}$ is G-convergent in $\mathcal{P}$. Let $\mathrm{F}: \mathcal{P}^4 \to \mathcal{P}$. A point (p_1, p_2, p_3, p_4) is a quadruple fixed point of F if $\mathrm{F}(p_1, p_2, p_3, p_4) = p_1, \mathrm{F}(p_2, p_3, p_4, p_1) = p_2, \mathrm{F}(p_3, p_4, p_1, p_2) = p_3$ and $\mathrm{F}(p_4, p_1, p_2, p_3) = p_4$ for $p_1, p_2, p_3, p_4 \in \mathcal{P}$. Let $f: \mathcal{P} \to \mathcal{P}$ be a mapping. A point (p_1, p_2, p_3, p_4) is the quadruple coincident point of $F$and f if $\mathrm{F}(p_1, p_2, p_3, p_4) = fp_1, \mathrm{F}(p_2, p_3, p_4, p_1) = fp_2, \mathrm{F}(p_3, p_4, p_1, p_2) = fp_3$ and $\mathrm{F}(p_4, p_1, p_2, p_3) = fp_4$. It is a quadruple common point of F and f if $\mathrm{F}(p_1, p_2, p_3, p_4) = fp_1 = p_1, \mathrm{F}(p_2, p_3, p_4, p_1) = fp_2 = p_2, \mathrm{F}(p_3, p_4, p_1, p_2,) = fp_3 =$ and $\mathrm{F}(p_4, p_1, p_2, p_3) = fp_4 = p_4$.. A pair (F, f) is weakly compatible if $f(\mathrm{F}(p_1, p_2, p_3, p_4) = \mathrm{F}(fp_1, fp_2, fp_3, fp_4)$ for any quadruple coincident point (p_1, p_2, p_3, p_4) for all $p_1, p_2, p_3, p_4 \in \mathcal{P}$.

2. Main Results

Let $\phi, \psi: [0, \infty) \to [0, \infty)$ be two functions with the following properties:

(i) ϕ is non increasing on $[0, \infty)$, Lebesgue integrable, continuous, and for any $\mathrm{t} > 0, \int_0^{\mathrm{t}} \phi(\mathrm{s})\mathrm{ds} > 0$.

(ii) ψ is non decreasing on $[0, \infty)$, $\psi(\mathrm{s}) \leq \mathrm{s}$ for all $s > 0$, additive function and $\sum_{\mathrm{i}=0}^{\infty} 2^{\mathrm{i}} \psi^{\mathrm{i}+2}(\mathrm{s}) < \infty$ for all $s > 0$.

Theorem 2.1. Let $(\mathcal{P}, G)$ be a G-metric space. Let $T: \mathcal{P}^4 \to \mathcal{P}$ and $f: \mathcal{P} \to \mathcal{P}$ be two mappings satisfying

$$\int_0^{G\left(T(p_1,p_2,p_3,p_4),T(p_5,p_6,p_7,p_8),T(p_9,p_{10},p_{11},p_{12})\right)} \phi(s)ds$$

$$\leq \frac{\psi}{2}\left(\int_0^{G(fp_1,fp_5,fp_9)+G(fp_2,fp_6,fp_{10})+G(fp_3,fp_7,fp_{11})+G(fp_4,fp_8,fp_{12})} \phi(s)ds\right) \qquad (2.1)$$

For all $p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8,p_9,p_{10},p_{11},p_{12} \in \mathcal{P}$ and

(a) $T(\mathcal{P}^4) \subseteq f(\mathcal{P})$ and $f(\mathcal{P})$ is compatible.

(b) (T,f) is weakly compatible. Then there exists a unique common quadruple fixed point of T and f in $\mathcal{P}$.

Proof. For any random $p_1,p_2,p_3,p_4 \in \mathcal{P}$ construct the sequences $\{p_{1_n}\}$, $\{p_{2_n}\}$, $\{p_{3_n}\}$, $\{p_{4_n}\}$, $\{q_{1_n}\}$, $\{q_{2_n}\}$, $\{q_{3_n}\}$, $\{q_{4_n}\}$ in $\mathcal{P}$ as $T(p_{1_n},p_{2_n},p_{3_n},p_{4_n}) = fp_{1_{n+1}} = q_{1_n}$, $T(p_{2_n},p_{3_n},p_{4_n},p_{1_n}) = fp_{2_{n+1}} = q_{2_n}$,
$T(p_{3_n},p_{4_n},p_{1_n},p_{2_n}) = fp_{3_{n+1}} = q_{3_n}$ and $T(p_{4_n},p_{1_n},p_{2_n},p_{3_n}) = fp_{4_{n+1}} = q_{4_n}$,
where $n = 0,1,2$ ....

Then from (2.1), we get that

$$\int_0^{G(q_{1_n},q_{1_{n+1}},q_{1_{n+1}})} \phi(s)$$

$$= \int_0^{G(T(p_{1_n},p_{2_n},p_{3_n},p_{4_n}),T(p_{1_{n+1}},p_{2_{n+1}},p_{3_{n+1}},p_{4_{n+1}}),T(p_{1_{n+1}},p_{2_{n+1}},p_{3_{n+1}},p_{4_{n+1}}))} \phi(s)ds$$

$$\leq \frac{\psi}{2}\left(\int_0^{G(fp_{1_n},fp_{1_{n+1}},fp_{1_{n+1}})+G(fp_{2_n},fp_{2_{n+1}},fp_{2_{n+1}})+G(fp_{3_n},fp_{3_{n+1}},fp_{3_{n+1}})+G(fp_{4_n},fp_{4_{n+1}},fp_{4_{n+1}})} \phi(s)ds\right)$$

$$\leq \frac{\psi}{2}\left(\int_0^{G(q_{1_{n-1}},q_{1_n},q_{1_n})+G(q_{2_{n-1}},q_{2_n},q_{2_n})+G(q_{3_{n-1}},q_{3_n},q_{3_n})+G(q_{4_{n-1}},q_{4_n},q_{4_n})} \phi(s)ds\right) \quad (2.2)$$

Similarly, it can be proved

$$\int_0^{G(q_{2_n},q_{2_{n+1}},q_{2_{n+1}})} \phi(s)ds$$

$$\leq \frac{\psi}{2}\left(\int_0^{G(q_{1_{n-1}},q_{1_n},q_{1_n})+G(q_{2_{n-1}},q_{2_n},q_{2_n})+G(q_{3_{n-1}},q_{3_n},q_{3_n})+G(q_{4_{n-1}},q_{4_n},q_{4_n})} \phi(s)ds\right) \quad (2.3)$$

$$\int_0^{G(q_{3_n},q_{3_{n+1}},q_{3_{n+1}})} \phi(s)ds$$

$$\leq \frac{\psi}{2}\left(\int_0^{G(q_{1_{n-1}},q_{1_n},q_{1_n})+G(q_{2_{n-1}},q_{2_n},q_{2_n})+G(q_{3_{n-1}},q_{3_n},q_{3_n})+G(q_{4_{n-1}},q_{4_n},q_{4_n})} \phi(s)ds\right) \quad (2.4)$$

$$\int_0^{G(q_{4_n},q_{4_{n+1}},q_{4_{n+1}})} \phi(s)ds$$

$$\leq \frac{\psi}{2}\left(\int_0^{G(q_{1_{n-1}},q_{1_n},q_{1_n})+G(q_{2_{n-1}},q_{2_n},q_{2_n})+G(q_{3_{n-1}},q_{3_n},q_{3_n})+G(q_{4_{n-1}},q_{4_n},q_{4_n})} \phi(s)ds\right) \quad (2.5)$$

For all $n \geq 0$.

Since ϕ is non increasing,

$$\int_0^{p_1+p_2+p_3+p_4}\phi(s)ds \le \int_0^{p_1}\phi(s)ds + \int_0^{p_2}\phi(s)ds + \int_0^{p_3}\phi(s)ds + \int_0^{p_4}\phi(s)ds \quad (2.6)$$

for all $p_1, p_2, p_3, p_4 \ge 0$.
Since ϕ is non increasing,

Since ψ is linear and non-decreasing and from (2.2) -(2.6),

$$\int_0^{G(q_{1_n},q_{1_{n+1}},q_{1_{n+1}})}\phi(s)ds$$
$$= \int_0^{G(T(p_{1_n},p_{2_n},p_{3_n},p_{4_n}),T(p_{1_{n+1}},p_{2_{n+1}},p_{3_{n+1}},p_{4_{n+1}}),T(p_{1_{n+1}},p_{2_{n+1}},p_{3_{n+1}},p_{4_{n+1}}))}\phi(s)ds$$
$$\le \frac{\psi}{2}\left(\int_0^{G(fp_{1_n},fp_{1_{n+1}},fp_{1_{n+1}})+G(fp_{2_n},fp_{2_{n+1}},fp_{2_{n+1}})+G(fp_{3_n},fp_{3_{n+1}},fp_{3_{n+1}i})+G(fp_{4_n},fp_{4_{n+1}},fp_{4_{n+1}})}\phi(s)ds\right)$$
$$\le$$
$$\frac{\psi}{2}\left(\int_0^{G(q_{1_{n-1}},q_{1_n},q_{1_n})}\phi(s)ds\right) + \frac{\psi}{2}\left(\int_0^{G(q_{2_{n-1}},q_{2_n},q_{2_n})}\phi(s)ds\right) +$$
$$\frac{\psi}{2}(\int_0^{G(q_{3_{n-1}},q_{3_n},q_{3_n})}\phi(s)ds) + \frac{\psi}{2}(\int_0^{G(q_{4_{n-1}},q_{4_n},q_{4_n})}\phi(s)ds)$$
$$\le \psi^2(\int_0^{G(q_{1_{n-2}},q_{1_{n-1}},q_{1_{n-1}})+G(q_{2_{n-2}},q_{2_{n-1}},q_{2_{n-1}})+G(q_{3_{n-2}},q_{3_{n-1}},q_{3_{n-1}})+G(q_{4_{n-2}},q_{4_{n-1}},q_{4_{n-1}})}\phi(s)ds)$$
$$\le 2^n\psi^{n+2}(\int_0^{G(q_{1_0},q_{1_1},q_{1_1})+G(q_{2_0},q_{2_1},q_{2_1})+G(q_{3_0},q_{3_1},q_{3_1})+G(q_{4_0},q_{4_1},q_{4_1})}\phi(s)ds) \quad (2.7)$$

In a similar process,

$$\int_0^{G(q_{2_n},q_{2_{n+1}},q_{2_{n+1}})}\phi(s)ds$$
$$\le 2^n\psi^{n+2}\left(\int_0^{G(q_{2_0},q_{2_1},q_{2_1})+G(q_{3_0},q_{3_1},q_{3_1})+G(q_{4_0},q_{4_1},q_{4_1})+G(q_{1_0},q_{1_1},q_{1_1})}\phi(s)ds\right) \quad (2.8)$$
$$\int_0^{G(q_{3_n},q_{3_{n+1}},q_{3_{n+1}})}\phi(s)ds$$
$$\le 2^n\psi^{n+2}(\int_0^{G(q_{3_0},q_{3_1},q_{3_1})+G(q_{4_0},q_{4_1},q_{4_1})+G(q_{1_0},q_{1_1},q_{1_1})+G(q_{2_0},q_{2_1},q_{2_1})}\phi(s)ds) \quad (2.9)$$
$$\int_0^{G(q_{4_n},q_{4_{n+1}},q_{4_{n+1}})}\phi(s)ds$$
$$\le 2^n\psi^{n+2}(\int_0^{G(q_{4_0},q_{4_1},q_{4_1})+G(q_{1_0},q_{1_1},q_{1_1})+G(q_{2_0},q_{2_1},q_{2_1})+G(q_{3_0},q_{3_1},q_{3_1})}\phi(s)ds) \quad (2.10)$$

Now, let $m, n \in N$ such that $m > n$. Then

$$\int_0^{G(q_{1_n},q_{1_m},q_{1_m})}\phi(s)ds$$
$$\le \int_0^{G(q_{1_n},q_{1_{n+1}},q_{1_{n+1}})}\phi(s)ds + \int_0^{G(q_{1_{n+1}},q_{1_{n+2}},q_{1_{n+2}})}\phi(s)ds + \ldots\ldots + \int_0^{G(q_{1_{m-1}},q_{1_m},q_{1_m})}\phi(s)ds$$

From (2.7)

$$\int_0^{G(q_{1_n},q_{1_m},q_{1_m})}\phi(s)ds$$

$$\leq \sum_{i=n}^{m-1} 2^i \psi^{i+2} (\int_0^{G(q_{1_0}, q_{1_1}, q_{1_1}) + G(q_{2_0}, q_{2_1}, q_{2_1}) + G(q_{3_0}, q_{3_1}, q_{3_1}) + G(q_{4_0}, q_{4_1}, q_{4_1})} \phi(s)ds)$$

$$\leq \sum_{i=0}^{\infty} 2^i \psi^{i+2} (\int_0^{G(q_{1_0}, q_{1_1}, q_{1_1}) + G(q_{2_0}, q_{2_1}, q_{2_1}) + G(q_{3_0}, q_{3_1}, q_{3_1}) + G(q_{4_0}, q_{4_1}, q_{4_1})} \phi(s)ds)$$

Since $\sum_{i=0}^{\infty} 2^i \psi^{i+2} < \infty$ for all $s \in [0, \infty)$ implies that $\lim_{n,m\to\infty} G(q_{1_n}, q_{1_m}, q_{1_m}) = 0$ and the sequence $\{q_{1_n}\}$ is a Cauchy sequence in $\mathcal{P}$.

Similarly, sequences $\{q_{2_n}\}, \{q_{3_n}\}, \{q_{4_n}\}$ are Cauchy sequences in the G-metric space $(\mathcal{P}, G)$.

Sequences $\{q_{1_n}\}, \{q_{2_n}\}, \{q_{3_n}\}, \{q_{4_n}\}$ converges to q_1, q_2, q_3 and q_4 respectively in $f(\mathcal{P})$. When $f(\mathcal{P})$ is a complete subspace of $(\mathcal{P}, G)$.

This shows that there exist $p_1, p_2, p_3, p_4 \in f(\mathcal{P})$ such that

$$\lim_{n\to\infty} q_{1_n} = q_1 = fp_1, \lim_{n\to\infty} q_{2_n} = q_2 = fp_2$$

$$\lim_{n\to\infty} q_{3_n} = q_3 = fp_3, \lim_{n\to\infty} q_{4_n} = q_4 = fp_4 \tag{2.11}$$

From (2.1)

$$\int_0^{G(T(p_1, p_2, p_3, p_4), q_{1_{n+1}}, q_{1_{n+1}})} \phi(s)ds$$

$$= \int_0^{G(T(p_1,p_2,p_3,p_4), T(p_{1_{n+1}}, p_{2_{n+1}}, p_{3_{n+1}}, p_{4_{n+1}}), T(p_{1_{n+1}}, p_{2_{n+1}}, p_{3_{n+1}}, p_{4_{n+1}}))} \phi(s)ds$$

$$\leq \frac{\psi}{2} (\int_0^{G(fp_1, fp_{1_{n+1}}, fp_{1_{n+1}}) + G(fp_2, fp_{2_{n+1}}, fp_{2_{n+1}}) + G(fp_3, fp_{3_{n+1}}, fp_{3_{n+1}}) + G(fp_4, fp_{4_{n+1}}, fp_{4_{n+1}})} \phi(s)ds)$$

$$\leq \frac{\psi}{2} (\int_0^{G(fp_1, q_{1_n}, q_{1_n}) + G(fp_2, q_{2_n}, q_{2_n}) + G(fp_3, q_{3_n}, q_{3_n}) + G(fp_4, q_{4_n}, q_{4_n})} \phi(s)ds)$$

As $n \to \infty$

$G(T(p_1, p_2, p_3, p_4), q_1, q_1) = 0 \Rightarrow T(p_1, p_2, p_3, p_4) = q_1$

Similarly, $T(p_2, p_3, p_4, p_1) = q_2, T(p_3, p_4, p_1, p_2) = q_3, T(p_4, p_1, p_2, p_3) = q_4$.

$\Rightarrow \quad T(p_1, p_2, p_3, p_4) = q_1 = fq_1, T(p_2, p_3, p_4, p_1) = q_2 = fq_2$

$T(p_3, p_4, p_1, p_2) = q_3 = fq_3, T(p_4, p_1, p_2, p_3) = q_4 = fq_4$.

Since (T, f) is weakly compatible pair,

$T(q_1, q_2, q_3, q_4) = fq_1, T(q_2, q_3, q_4, q_1) = fq_2$

$T(q_3, q_4, q_1, q_2) = fq_3, T(q_4, q_1, q_2, q_4) = fq_4$. Since

$$\int_0^{G(fq_1, q_{1_{n+1}}, q_{1_{n+1}})} \phi(s)ds$$

$$= \int_0^{G(T(q_1,q_2,q_3,q_4), q_{1_{n+1}}, q_{1_{n+1}})} \phi(s)ds$$

$$= \int_0^{G(T(q_1,q_2,q_3,q_4), T(p_{1_{n+1}}, p_{2_{n+1}}, p_{3_{n+1}}, p_{4_{n+1}}), T(p_{1_{n+1}}, p_{2_{n+1}}, p_{3_{n+1}}, p_{4_{n+1}}))} \phi(s)ds$$

$$\leq \frac{\psi}{2} (\int_0^{G(fq_1, fp_{1_{n+1}}, fp_{1_{n+1}}) + G(fq_2, fp_{2_{n+1}}, fp_{2_{n+1}}) + G(fq_3, fp_{3_{n+1}}, fp_{3_{n+1}}) + G(fq_4, fp_{4_{n+1}}, fp_{4_{n+1}})} \phi(s)ds)$$

$$\leq \frac{\psi}{2}(\int_0^{G(fq_1,q_{1_{n+1}},q_{1_{n+1}})+G(fq_2,q_{2_{n+1}},q_{2_{n+1}})+G(fq_3,q_{3_{n+1}},q_{3_{n+1}})+G(fq_4,q_{4_{n+1}},q_{4_{n+1}})} \phi(s)ds)$$

Therefore

$$\int_0^{G(fq_1,q_{1_{n+1}},q_{1_{n+1}})+G(fq_2,q_{2_{n+1}},q_{2_{n+1}})+G(fq_3,\, q_{3_{n+1}},q_{3_{n+1}})+G(fq_4,q_{4_{n+1}},q_{4_{n+1}})} \phi(s)ds$$

$$\leq 2\psi(\int_0^{G(fq_1,q_{1_{n+1}},q_{1_{n+1}})+G(fq_2,q_{2_{n+1}},q_{2_{n+1}})+G(fq_3,\, q_{3_{n+1}},q_{3_{n+1}})+G(fq_4,q_{4_{n+1}},q_{4_{n+1}})} \phi(s)ds)$$

As $n \to \infty$, $G(fq_1, q_1, q_1) = 0, G(fq_2, q_2, q_2) = 0, G(fq_3, q_3, q_3) = 0, G(fq_4, q_4, q_4) = 0$
$\Rightarrow fq_1 = q_1, fq_2 = q_2, fq_3 = q_3$ and $fq_4 = q_4$.
Therefore
$T(q_1, q_2, q_3, q_4) = fq_1 = q_1, T(q_2, q_3, q_4, q_1) = fq_2 = q_2$
$T(q_3, q_4, q_1, q_2) = fq_3 = q_3, T(q_4, q_1, q_2, q_4) = fq_4 = q_4$
Thus (q_1, q_2, q_3, q_4) is a quadruple fixed point of T and f. Suppose that (q_1', q_2', q_3', q_4') is another quadruple fixed point of T, f. Then

$$\int_0^{G(q_1',q_{1_{n+1}},q_{1_{n+1}})} \phi(s)ds \quad = \quad \int_0^{G(T(q_1',q_2',q_3',q_4'),q_{1_{n+1}},q_{1_{n+1}})} \phi(s)ds$$

$$= \int_0^{G\left(T(q_1',q_2',q_3',q_4'),T(p_{1_{n+1}},p_{2_{n+1}},p_{3_{n+1}},p_{4_{n+1}}),T(p_{1_{n+1}},p_{2_{n+1}},p_{3_{n+1}},p_{4_{n+1}})\right)} \phi(s)ds$$

$$\leq \frac{\psi}{2}(\int_0^{G(fq_1',fp_{1_{n+1}},fp_{1_{n+1}})+G(fq_2',fp_{2_{n+1}},fp_{2_{n+1}})+G(fq_3',fp_{3_{n+1}},fp_{3_{n+1}})+G(fq_4',fp_{4_{n+1}},fp_{4_{n+1}})} \phi(s)ds)$$

$$\leq \frac{\psi}{2}(\int_0^{G(fq_1',q_{1_n},q_{1_n})+G(fq_2',q_{2_n},q_{2_n})+G(fq_3',q_{3_n},q_{3_n})+G(fq_4',q_{4_n},q_{4_n})} \phi(s)ds)$$

Therefore

$$\int_0^{G(fq_1',q_{1_{n+1}},q_{1_{n+1}})+G(fq_2',q_{2_{n+1}},q_{2_{n+1}})+G(fq_3',q_{3_{n+1}},q_{3_{n+1}})+G(fq_4',q_{4_{n+1}},q_{4_{n+1}})} \phi(s)ds$$

$$\leq 2\psi(\int_0^{G(q_1',q_{1_n},q_{1_n})+G(q_2',q_{2_n},q_{2_n})+G(q_3',q_{3_n},q_{3_n})+G(q_4',q_{4_n},q_{4_n})} \phi(s)ds)$$

Taking the limit as $n \to \infty$ in the above inequality and which possibility holds only $G(q_1', q_1, q_1) = 0$, $G(q_2', q_2, q_2) = 0, G(q_3', q_3, q_3) = 0$ and $G(q_4', q_4, q_4) = 0$ implies that $q_1' = q_1$, $q_2' = q_2$, $q_3' = q_3$, $q_4' = q_4$. Therefore (q_1, q_2, q_3, q_4) is a unique quadruple common fixed point of T and f.

Now $\int_0^{G(q_{1_{n+1}},q_{1_{n+1}},q_{2_{n+1}})} \phi(s)ds$

$$=\int_0^{G\left(T(p_{1_{n+1}},p_{2_{n+1}},p_{3_{n+1}},p_{4_{n+1}}),T(p_{1_{n+1}},p_{2_{n+1}},p_{3_{n+1}},p_{4_{n+1}}),T(p_{2_{n+1}},p_{3_{n+1}},p_{4_{n+1}},p_{1_{n+1}})\right)} \phi(s)ds$$

$$\leq \frac{\psi}{2}(\int_0^{G(fp_{1_{n+1}},fp_{1_{n+1}},fp_{2_{n+1}})+G(fp_{2_{n+1}},fp_{2_{n+1}},fp_{3_{n+1}})+G(fp_{3_{n+1}},fp_{3_{n+1}},fp_{4_{n+1}})+G(fp_{4_{n+1}},fp_{4_{n+1}},fp_{1_{n+1}})} \phi(s)ds)$$

$$\leq \frac{\psi}{2}\left(\int_0^{G(q_{1_n},q_{1_n},q_{2_n})+G(q_{2_n},q_{2_n},q_{3_n})+G(q_{3_n},q_{3_n},q_{4_n})+G(q_{4_n},q_{4_n},q_{1_n})} \phi(s)ds\right)$$

Therefore

$$\int_0^{G(q_{1_{n+1}},q_{1_{n+1}},q_{2_{n+1}})+G(q_{2_{n+1}},q_{2_{n+1}},q_{3_{n+1}})+G(q_{3_{n+1}},q_{3_{n+1}},q_{4_{n+1}})+G(q_{4_{n+1}},q_{4_{n+1}},q_{1_{n+1}})} \phi(s)ds$$

$$\leq 2\psi\left(\int_0^{G(q_{1_n},q_{1_n},q_{2_n})+G(q_{2_n},q_{2_n},q_{3_n})+G(q_{3_n},q_{3_n},q_{4_n})+G(q_{4_n},q_{4_n},q_{1_n})} \phi(s)ds\right).$$

Taking the limit as $n \to \infty$ in the above inequality, we get. Hence, we get $q_1 = q_2 = q_3 = q_4$, which means that T and f have a unique common fixed point.

Corollary 2.2. Let $(\mathcal{P}, G)$ be a G-metric space. Suppose that $T: \mathcal{P}^4 \to \mathcal{P}$ be a mapping satisfying

$$\int_0^{G\left(T(p_1,p_2,p_3,p_4),T(p_5,p_6,p_7,p_8),T(p_9,p_{10},p_{11},p_{12})\right)} \phi(s)ds$$

$$\leq \frac{\psi}{2}\left(\int_0^{\max\{G(p_1,p_5,p_9),G(p_2,p_6,p_{10}),G(p_3,p_7,p_{11}),G(p_4,p_8,p_{12})} \phi(s)ds\right) \tag{2.12}$$

For all $p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8,p_9,p_{10},p_{11},p_{12} \in \mathcal{P}$. Then there is a unique quadruple fixed point of T in $\mathcal{P}$.

Example 2.3. Define a *G*-mapping on a complete *G*-metric space $(\mathcal{P}, G)$ as $G(p_1,p_2,p_3) = |p_1 - p_2| + |p_1 - p_3| + |p_2 - p_3|$, where $\mathcal{P} = [0,1]$.
Define $T(p_1,p_2,p_3,p_4) = \frac{p_1+p_2+p_3+p_4}{16}$ and $f(x) = 8x$, also $\varphi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$.
Then $T(\mathcal{P}^4) \subseteq f(\mathcal{P})$ and the pair (T,f) is weakly compatible.

Infect, $\int_0^{G\left(T(p_1,p_2,p_3,p_4),T(p_5,p_6,p_7,p_8),T(p_9,p_{10},p_{11},p_{12})\right)} \phi(s)ds$

$$= \int_0^{|T(p_1,p_2,p_3,p_4)-T(p_5,p_6,p_7,p_8)|+|T(p_1,p_2,p_3,p_4)-T(p_9,p_{10},p_{11},p_{12})|+|T(p_5,p_6,p_7,p_8)-T(p_9,p_{10},p_{11},p_{12})|} \phi(s)ds$$

$$= \int_0^{\left|\frac{p_1+p_2+p_3+p_4}{16}-\frac{p_5+p_6+p_7+p_8}{16}\right|+\left|\frac{p_1+p_2+p_3+p_4}{16}-\frac{p_9+p_{10}+p_{11}+p_{12}}{16}\right|+\left|\frac{p_5+p_6+p_7+p_8}{16}-\frac{p_9+p_{10}+p_{11}+p_{12}}{16}\right|} \phi(s)ds$$

$$\leq \frac{1}{32}\left(\int_0^{G(fp_1,fp_5,fp_9)+G(fp_2,fp_6,fp_{10})+G(fp_3,fp_7,fp_{11})+G(fp_4,fp_8,fp_{12})} \phi(s)ds\right)$$

$$\leq \frac{\psi}{2}\left(\int_0^{G(fp_1,fp_5,fp_9)+G(fp_2,fp_6,fp_{10})+G(fp_3,fp_7,fp_{11})+G(fp_4,fp_8,fp_{12})} \phi(s)ds\right).$$

In this case, (0,0,0,0) is the unique quadruple fixed point satiating all the environments of theorem 2.1.

Theorem 2.4. Let $T: I \times \mathbb{R}^4 \to \mathbb{R}$ and $r_{1_0}, r_{2_0}, r_{3_0}, r_{4_0} \in \mathbb{R}$. Consider the initial value problem defined as $r'(t) = T\left(t,(r_1,r_2,r_3,r_4)(t)\right)$, $t \in I = [0,1]$, $(r_1,r_2,r_3,r_4)(0) = (r_{1_0}, r_{2_0}, r_{3_0}, r_{4_0})$. Then there exists a unique solution in $C(I,\mathbb{R})$.

Proof. The integral equation corresponding to the initial value problem $r'(t) = T\big(t,(r_1,r_2,r_3,r_4)(t)\big), t \in I = [0,1]$, $(r_1,r_2,r_3,r_4)(0) = \big(r_{1_0}, r_{2_0}, r_{3_0}, r_{4_0}\big)$ is $r'(t) = r_{1_0} + 4\int_0^t \mathrm{T}(\mathrm{s},(r_1,r_2,r_3,r_4)(\mathrm{s}))\,ds$.

Let $\mathcal{P} = C(I,\mathbb{R})$ and $G(p_1,p_2,p_3) = |p_1 - p_2| + |p_1 - p_3| + |p_2 - p_3|$ for all $p_1,p_2,p_3 \in P$ and $\psi(t) = t$ for all t$\in [0,\infty)$. Define $R\colon \mathcal{P}^4 \to \mathcal{P}$ by

$$R(r_1,r_2,r_3,r_4) = \frac{r_{1_0}}{4} + \int_0^t \mathrm{T}(\mathrm{s},(r_1,r_2,r_3,r_4)(\mathrm{s}))\,ds \tag{2.13}$$

Now

$$\int_0^{G(R(r_1,r_2,r_3,r_4)(t),R(r_5,r_6,r_7,r_8)(t),R(r_9,r_{10},r_{11},r_{12})(t))} \phi(s)ds$$

$$\int_0^{|R(r_1,r_2,r_3,r_4)(t)-R(r_5,r_6,r_7,r_8)(t)|+|R(r_1,r_2,r_3,r_4)(t)-R(r_9,r_{10},r_{11},r_{12})(t)|+|R(r_5,r_6,r_7,r_8)(t)-R(r_9,r_{10},r_{11},r_{12})(t)|} \phi(s)ds$$

$$=\int_0^{\left\{\begin{matrix} \frac{r_{1_0}}{4}+\int_0^t \mathrm{T}(\mathrm{s},(r_1,r_2,r_3,r_4)(\mathrm{s}))ds-\frac{r_{5_0}}{4}+\int_0^t \mathrm{T}(\mathrm{s},(r_5,r_6,r_7,r_8)(\mathrm{s}))ds+ \\ \frac{r_{1_0}}{4}+\int_0^t \mathrm{T}(\mathrm{s},(r_1,r_2,r_3,r_4)(\mathrm{s}))ds-\frac{r_{9_0}}{4}+\int_0^t \mathrm{T}(\mathrm{s},(r_9,r_{10},r_{11},r_{12})(\mathrm{s}))ds+ \\ \frac{r_{5_0}}{4}+\int_0^t \mathrm{T}(\mathrm{s},(r_5,r_6,r_7,r_8)(\mathrm{s}))ds-\frac{r_{9_0}}{4}+\int_0^t \mathrm{T}(\mathrm{s},(r_9,r_{10},r_{11},r_{12})(\mathrm{s}))ds \end{matrix}\right\}} \phi(s)ds$$

$$=\int_0^{|r_1(t)-r_5(t)|+|r_1(t)-r_9(t)|+|r_5(t)-r_9(t)|} \phi(s)ds$$

$$\le \frac{1}{4}\int_0^{G(r_1,r_5,r_9)} \phi(s)ds$$

$$\le \frac{\psi}{2}\Big(\int_0^{\max\{\mathrm{G}(r_1,r_5,r_9),\mathrm{G}(r_2,r_6,r_{10}),\mathrm{G}(r_3,r_7,r_{11}),\mathrm{G}(r_4,r_8,r_{12})\}} \phi(s)ds\Big).$$

It implies that R has a unique fixed point in $\mathcal{P}$ [from corollary 2.2]. It is one of the applications to integral equations.

Theorem 2.5 is an application to Homotopy theory.

Theorem 2.5. For an open and closed subset U and $\overline{U}$ of $\mathcal{P}$ such that $U \subseteq \overline{U}$, assume that $H\colon \overline{U}^4 \times [0,1] \to \mathcal{P}$satisfying the conditions

(τ_0) $u_1 \neq H(u_1,u_2,u_3,u_4,k), u_2 \neq H(u_2,u_3,u_4,u_1,k), u_3 \neq H(u_3,u_4,u_1,u_2,k)$ and $u_4 \neq H(u_4,u_1,u_2,u_3,k)$ for

each $u_1,u_2,u_3,u_4 \in \partial U$ and $k \in [0,1]$.

(τ_1) $$\int_0^{G\left(H(u_1,u_2,u_3,u_4,k),H(u_1,u_2,u_3,u_4,k)H(u_5,u_6,u_7,u_8,k)\right)} \phi(s)ds$$

$$\le \frac{\psi}{2}\Big(\int_0^{G(u_1,u_1,u_5),G(u_2,u_2,u_6),G(u_3,u_3,u_7),G(u_4,u_4,u_8)} \phi(s)ds\Big)$$

(τ_2) There exists $M \ge 0$ such that

$$\int_0^{G\left(H(u_1,u_2,u_3,u_4,k),H(u_1,u_2,u_3,u_4,k)H(u_5,u_6,u_7,u_8,\varkappa)\right)} \phi(s)ds \le \Big(\int_0^{M|k-\varkappa|} \phi(s)ds\Big)$$

for all $u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8 \in \overline{U}$ and $k,\varkappa \in [0,1]$.

Then $H(.,0)$ has a quadruple fixed point $\Leftrightarrow$ $H(.,1)$ has a quadruple fixed point.

Proof. Let

$$X = \left\{\begin{matrix} k \in [0,1]\colon H(u_1,u_2,u_3,u_4,k) = u_1, H(u_2,u_3,u_4,u_1,k) = u_2, \\ H(u_3,u_4,u_1,u_2,k) = u_3, H(u_4,u_1,u_2,u_3,k) = u_4 \; for\ some\ u_1,u_2,u_3,u_4 \in U \end{matrix}\right\}.$$

Since $(0,0,0,0) \in X^4$ and $H(.,0)$ has a quadruple fixed point in U^4 implies that $X \neq \emptyset$.

Let $H(\{k_n\}_{n=1}^{\infty}) \subseteq X$ with $k_n \to k$ as $n \to \infty$. Since $k_n \in X$ with $n = 0, 1, 2 \ldots$ there exist sequences $\{u_{1_n}\}, \{u_{2_n}\}, \{u_{3_n}\}, \{u_{4_n}\}$ such that $u_{1_{n+1}} = H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n)$, $u_{2_{n+1}} = H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n)$,

$u_{3_{n+1}} = H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n)$ and $u_{4_{n+1}} = H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n)$.

Consider

$$\int_0^{G(u_{1_{n+1}}, u_{1_{n+1}}, u_{1_{n+2}})} \phi(s)ds$$

$$= \int_0^{G(H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n), H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n), H(u_{1_{n+1}}, u_{2_{n+1}}, u_{3_{n+1}}, u_{4_{n+1}}, k_{n+1}))} \phi(s)ds)$$

$$\leq \int_0^{\left\{\begin{array}{c} G\left(H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n), H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n), H(u_{1_{n+1}}, u_{2_{n+1}}, u_{3_{n+1}}, u_{4_{n+1}}, k_n)\right) + \\ G(H(u_{1_{n+1}}, u_{2_{n+1}}, u_{3_{n+1}}, u_{4_{n+1}}, k_n), H(u_{1_{n+1}}, u_{2_{n+1}}, u_{3_{n+1}}, u_{4_{n+1}}, k_n), H(u_{1_{n+1}}, u_{2_{n+1}}, u_{3_{n+1}}, u_{4_{n+1}}, k_{n+1})) \end{array}\right\}} \phi(s)ds$$

$$\leq \int_0^{G\left(H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n), H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n), H(u_{1_{n+1}}, u_{2_{n+1}}, u_{3_{n+1}}, u_{4_{n+1}}, k_n)\right)} \phi(s)ds)$$

$$+ \int_0^{M|k_n - k_{n+1}|} \phi(s)ds)$$

Now as $n \to \infty$,

$$\lim_{n\to\infty} \int_0^{G(u_{1_{n+1}}, u_{1_{n+1}}, u_{1_{n+2}})} \phi(s)ds$$

$$\leq \lim_{n\to\infty} \int_0^{G(H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n), H(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}, k_n), H(u_{1_{n+1}}, u_{2_{n+1}}, u_{3_{n+1}}, u_{4_{n+1}}, k_n))} \phi(s)ds$$

$$\leq \lim_{n\to\infty} \frac{\psi}{2} (\int_0^{G(u_{1_n}, u_{1_n}, u_{1_{n+1}}) + G(u_{2_n}, u_{2_n}, u_{2_{n+1}}) + G(u_{3_n}, u_{3_n}, u_{3_{n+1}}) + G(u_{4_n}, u_{4_n}, u_{4_{n+1}})} \phi(s)ds)$$

$$\leq \lim_{n\to\infty} \psi^2 (\int_0^{G(u_{1_{n-1}}, u_{1_{n-1}}, u_{1_n}) + G(u_{2_{n-1}}, u_{2_{n-1}}, u_{2_n}) + G(u_{3_{n-1}}, u_{3_{n-1}}, u_{3_n}) + G(u_{4_{n-1}}, u_{4_{n-1}}, u_{4_n})} \phi(s)ds)$$

$$\leq \lim_{n\to\infty} 2^n \psi^{n+2} (\int_0^{G(u_{1_0}, u_{1_0}, u_{1_1}) + G(u_{2_0}, u_{2_0}, u_{2_1}) + G(u_{3_0}, u_{3_0}, u_{3_1}) + G(u_{4_0}, u_{4_0}, u_{4_1})} \phi(s)ds)$$

$$\leq \sum_{n=0}^{\infty} 2^n \psi^{n+2} (\int_0^{G(u_{1_0}, u_{1_0}, u_{1_1}) + G(u_{2_0}, u_{2_0}, u_{2_1}) + G(u_{3_0}, u_{3_0}, u_{3_1}) + G(u_{4_0}, u_{4_0}, u_{4_1})} \phi(s)ds) \quad (2.14)$$

Since $\sum_{n=0}^{\infty} 2^n \psi^{n+2}(s) < \infty$ for all $s \in [0, \infty) \Rightarrow \lim_{n\to\infty} G(u_{1_{n+1}}, u_{1_{n+1}}, u_{1_{n+2}}) = 0$

Let $m, n \in N$ with $m > n$. Then

$$\int_0^{G(u_{1_n}, u_{1_m}, u_{1_m})} \phi(s)ds$$

$$\leq \int_0^{G(u_{1_n}, u_{1_{n+1}}, u_{1_{n+1}})} \phi(s)ds + \int_0^{G(u_{1_{n+1}}, u_{1_{n+2}}, u_{1_{n+2}})} \phi(s)ds + \cdots + \int_0^{G(u_{1_{m-1}}, u_{1_m}, u_{1_m})} \phi(s)ds$$

It follows from (2.1) that

$$\int_0^{G(u_{1_n}, u_{1_m}, u_{1_m})} \phi(s)ds$$

$$\leq \sum_{i=1}^{m-1} 2^i \psi^{i+2} (\int_0^{G(u_{1_0}, u_{1_1}, u_{1_1}) + G(u_{2_0}, u_{2_1}, u_{2_1}) + G(u_{3_0}, u_{3_1}, u_{3_1}) + G(u_{4_0}, u_{4_1}, u_{4_1})} \phi(s)ds)$$

$$\leq \sum_{i=1}^{\infty} 2^i \psi^{i+2} (\int_0^{G(u_{1_0}, u_{1_1}, u_{1_1}) + G(u_{2_0}, u_{2_1}, u_{2_1}) + G(u_{3_0}, u_{3_1}, u_{3_1}) + G(u_{4_0}, u_{4_1}, u_{4_1})} \phi(s)ds)$$

Since $\sum_{i=0}^{\infty} 2^n \psi^{n+2}(s) < \infty$ for all $s \in [0, \infty)$ and $\lim_{n\to\infty} G(u_{1_{n+1}}, u_{1_{n+1}}, u_{1_{n+2}}) = 0$

This shows that sequence $\{u_{1_n}\}$ is a Cauchy sequence in $\mathcal{P}$.

Similarly, $\{u_{2_n}\}, \{u_{3_n}\}, \{u_{4_n}\}$ are Cauchy sequences in the G-metric space $(\mathcal{P}, G)$.

By the completeness of $(\mathcal{P}, G)$, there exist $p_1, p_2, p_3, p_4 \in \mathcal{P}$ with

$\lim_{n\to\infty} u_{1_{n+1}} = p_1$, $\lim_{n\to\infty} u_{2_{n+1}} = p_2$, $\lim_{n\to\infty} u_{3_{n+1}} = p_3$, $\lim_{n\to\infty} u_{4_{n+1}} = p_4$

By using (2.1) and property of ψ, ϕ we have

$$\leq \int_0^{\left\{\begin{matrix} G(H(p_1,p_2,p_3,p_4,k),H(p_1,p_2,p_3,p_4,k),p_1)+G(H(p_2,p_3,p_4,k,p_1),H(p_2,p_3,p_4,k),p_2))+ \\ G(H(p_3,p_4,k,p_1,p_2),H(p_3,p_4,k,p_1,p_2),p_3))+G(H(p_4,k,p_1,p_2,p_3),H(p_4,k,p_1,p_2,p_3),p_4)\end{matrix}\right\}} \phi(s)ds$$

$$\leq \int_0^{\left\{\begin{matrix} G\left(H(p_1,p_2,p_3,p_4,k),H(p_1,p_2,p_3,p_4,k),H\left(u_{1_{n+1}},u_{2_{n+1}},u_{3_{n+1}},u_{4_{n+1}},k\right)\right)+ \\ G\left(H(p_2,p_3,p_4,k,p_1),H(p_2,p_3,p_4,k,p_1),H(u_{2_{n+1}},u_{3_{n+1}},u_{4_{n+1}},u_{1_{n+1}},k)\right)+ \\ G\left(H(p_3,p_4,k,p_1,p_2),H(p_3,p_4,k,p_1,p_2),H\left(u_{3_{n+1}},u_{4_{n+1}},u_{1_{n+1}},u_{2_{n+1}},k\right)\right)+ \\ G(H(p_4,k,p_1,p_2,p_3),H(p_4,k,p_1,p_2,p_3),H(u_{4_{n+1}},u_{1_{n+1}},u_{2_{n+1}},u_{3_{n+1}},k))\end{matrix}\right\}} \phi(s)ds$$

$$\leq \lim_{n\to\infty} 2\psi \left[\int_0^{\left\{\begin{matrix} G(p_1,p_1,u_{1_{n+1}})+G(p_2,p_2,u_{2_{n+1}})+ \\ G(p_3,p_3,u_{3_{n+1}})+G(p_4,p_4,u_{4_{n+1}})\end{matrix}\right\}} \phi(s)ds \right] = 0$$

It follows that $H(p_1, p_2, p_3, p_4, k) = p_1, H(p_2, p_3, p_4, k, p_1) = p_2, H(p_3, p_4, k, p_1, p_2) = p_3$ and $H(p_4, k, p_1, p_2, p_3) = p_4$.Thus $k \in X$. Hence X is closed in [0,1].

Let $k_0 \in X$. Then there exist $u_{1_0}, u_{2_0}, u_{3_0}, u_{4_0} \in U$ with $u_{1_0} = H(u_{1_0}, u_{2_0}, u_{3_0}, u_{4_0}, k_0)$,
$u_{2_0} = H(u_{2_0}, u_{3_0}, u_{4_0}, u_{1_0}, k_0), u_{3_0} = H(u_{3_0}, u_{4_0}, u_{1_0}, u_{2_0}, k_0)$ and
$u_{4_0} = H(u_{4_0}, u_{1_0}, u_{2_0}, u_{3_0}, k_0)$

Since U is open, then there exist $r > 0$ such that $B_G(u_{1_0}, u_{1_0}, r) \subseteq U$.

Choose $k \in (k_0 - \epsilon, k_0 + \epsilon)$ such that $|k - k_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}$.

Then for $x \in B_G(u_{1_0}, u_{1_0}, r) = \{x \in \frac{X}{G(u_1, u_1, u_{1_0})} \leq r + G(u_{1_0}, u_{1_0}, u_{1_0})\}$

Also $\int_0^{G(H(u_1,u_2,u_3,u_4,k),H(u_1,u_2,u_3,u_4,k)u_{1_0})} \phi(s)ds$

$$= \int_0^{G\left(H(u_1,u_2,u_3,u_4,k),H(u_1,u_2,u_3,u_4,k),H(u_{1_0},u_{2_0},u_{3_0},u_{4_0},k_0)\right.} \phi(s)ds$$

$$\leq \int_0^{\left\{\substack{G(H(u_1,u_2,u_3,u_4,k),H(u_1,u_2,u_3,u_4,k),H(u_1,u_2,u_3,u_4,k_0))+ \\ G(H(u_1,u_2,u_3,u_4,k_0),H(u_1,u_2,u_3,u_4,k_0),H(u_{1_0},u_{2_0},u_{3_0},u_{4_0},k_0))}\right\}} \phi(s)ds$$

$$\leq \int_0^{M|k-k_0|} \phi(s)ds + \int_0^{G(H(u_1,u_2,u_3,u_4,k_0),H(u_1,u_2,u_3,u_4,k_0),H(u_{1_0},u_{2_0},u_{3_0},u_{4_0},k_0))} \phi(s)ds$$

$$\leq \frac{1}{M^{n-1}} \int_0^{1} \phi(s)ds + \int_0^{G(H(u_1,u_2,u_3,u_4,k_0),H(u_1,u_2,u_3,u_4,k_0),H(u_{1_0},u_{2_0},u_{3_0},u_{4_0},k_0))} \phi(s)ds$$

Letting $n \to \infty$,

$$\int_0^{G(H(u_1,u_2,u_3,u_4,k),H(u_1,u_2,u_3,u_4,k),u_{1_0})} \phi(s)ds$$

$$\leq \int_0^{G(H(u_1,u_2,u_3,u_4,k_0),H(u_1,u_2,u_3,u_4,k_0),H(u_{1_0},u_{2_0},u_{3_0},u_{4_0},k_0))} \phi(s)ds$$

$$\leq \frac{\psi}{2} (\int_0^{G(u_1,u_1,u_{1_0})+G(u_2,u_2,u_{2_0})+G(u_3,u_3,u_{3_0})+G(u_4,u_4,u_{4_0})} \phi(s)ds)$$

Therefore

$$\int_0^{\left\{\substack{G(H(u_1,u_2,u_3,u_4,k),H(u_1,u_2,u_3,u_4,k),u_{1_0})+ \\ G(H(u_2,u_3,u_4,u_1,k),H(u_2,u_3,u_4,u_1,k),u_{2_0})+ \\ G(H(u_3,u_4,u_1,u_2,k),H(u_3,u_4,u_1,u_2,k),u_{3_0})+ \\ G(H(u_4,u_1,u_2,u_3,k),H(u_4,u_1,u_2,u_3,k),u_{4_0})}\right\}} \phi(s)ds$$

$$\leq 2\psi(\int_0^{G(u_1,u_1,u_{1_0})+G(u_2,u_2,u_{2_0})+G(u_3,u_3,u_{3_0})+G(u_4,u_4,u_{4_0})} \phi(s)ds)$$

$$\leq 2\psi(\int_0^{4r+G(u_{1_0},u_{1_0},u_{1_0})+G(u_{2_0},u_{2_0},u_{2_0})+G(u_{3_0},u_{3_0},u_{3_0})+G(u_{4_0},u_{4_0},u_{4_0})} \phi(s)ds)$$

Thus for each fixed $k \in (k_0 - \epsilon, k_0 + \epsilon), H(.,k): \overline{B_G(u_{1_0}, u_{1_0}, r)} \to \overline{B_G(u_{1_0}, u_{1_0}, r)}$,
$H(.,k): \overline{B_G(u_{2_0}, u_{2_0}, r)} \to \overline{B_G(u_{2_0}, u_{2_0}, r)}$,
$H(.,k): \overline{B_G(u_{3_0}, u_{3_0}, r)} \to \overline{B_G(u_{3_0}, u_{3_0}, r)}, H(.,k): \overline{B_G(u_{4_0}, u_{4_0}, r)} \to \overline{B_G(u_{4_0}, u_{4_0}, r)}$.

All the conditions of the theorem 2.5 are satisfied, therefore $H(.,k)$ has a quadruple fixed point in $\overline{U}^4$.
But this must be in U^4. Since (τ_0) $i$holds, $k \in X$ for any $k \in (k_0 - \epsilon, k_0 + \epsilon)$.
$\Rightarrow (k_0 - \epsilon, k_0 + \epsilon) \subseteq X$ and X is open in [0,1].
The other part can follow the same argument.

3. Conclusion

We ensured the existence and uniqueness of a common fixed point for two mappings in the class of *G*-metric spaces via Integral type contraction. Two illustrated applications have been provided.

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