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A Unified Approach to the Sandor-Smarandache Function

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Abstract

The Sandor-Smarandache function, $SS(n)$, is a recently introduced Smarandache-type arithmetic function, which involves binomial coefficients. It is known that $SS(n)$ does not possess many of the common properties of the classical arithmetic functions of the theory of numbers. Sandor gave the expression of $SS(n)$ when n (≥ 3) is an odd integer. It is found that $SS(n)$ has a simple form when n is even and not divisible by 3. In the previous papers, some closed-form expressions of $SS(n)$ have been derived for some particular cases of n. This paper continues to find more forms of $SS(n)$, starting from the function $SS(24m)$. Particular attention is given to finding necessary and sufficient conditions such that $SS(n) = n-5$ and $SS(n) = n-6$. Based on the properties of $SS(n)$, some interesting Diophantine equations have been studied. The study reveals that the form of $SS(n)$ depends on the prime factors of the integer n in the natural order of the primes.

Keywords: Sandor-Smarandache function; Binomial coefficient; Diophantine equation.

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1. Introduction

In the late 1970s, the celebrated Romanian-American number theorist, Florentin Smarandache, proposed a new arithmetic function called the Smarandache function after him. Since then, more Smarandache-type arithmetic functions have been introduced in the mathematical literature. These functions are different from the traditional arithmetic functions in many respects. Because of their special features, these functions drew the attention of different researchers. Sandor [1] introduced a new Smarandache-type function. The function, called the Sandor-Smarandache function, is denoted by $SS(n)$, and is defined as follows:

$$SS(n) = max\left\{k : 1 \leq k \leq n-2, n \text{ divides } \binom{n}{k}\right\}, n \geq 5, \tag{1.1}$$

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where by convention,

$$SS(1)=1,\ SS(2)=1,\ SS(3)=1,\ SS(4)=1,\ SS(6)=1. \quad (1.2)$$

In the defining equation (1.1), $C(n, k) \equiv \binom{n}{k} = \frac{n!}{k!\,(n-k)!}, 0 \leq k \leq n,$ are the binomial coefficients, which are all integers (Hardy and Wright [2, Theorem 73]). Throughout this paper, the following simplified form of $C(n, k)$ is used:

$$C(n,k) = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}, 0 \leq k \leq n. \quad (1.3)$$

The problem may now be reformulated as follows: Given any integer n (≥ 7), find the minimum integer k such that $k!$ divides $(n-1)(n-2)\ldots(n-k+1)$, where $1 \leq k \leq n-2$. With this minimum k, $SS(n)$ is given by $SS(n)=n-k$.

Islam et al. [3] proved the results below.

Lemma 1.1: $SS(n)=n-2$ if and only if n (≥ 7) is an odd integer.

Lemma 1.2: $SS(n)=n-3$ if and only if n is even and is not divisible by 3.

Later, Islam and Majumdar [4] established the following result.

Lemma 1.3: $SS(n)=n-4$ if and only if n is of the form $n=6(4m+3)$ for any integer $m \geq 0$.

Corollary 1.1: Let $SS(n)=n-4$ for some (positive) integer n. Then, $SS(2n) \neq 2n-4$.

Proof: If $SS(n)=n-4$, then by Lemma 1.3, $n=6(4a+3)$ for some integer a. But then, $2n$ cannot be of the form $6(4m+3)$.

Lemma 1.1 and Lemma 1.2 show that $SS(n)$ has a simple form when n is odd or when n is even but not divisible by 3. Lemma 1.3 finds the necessary and sufficient conditions such that $SS(n)=n-4$. Thus, the problem of finding $SS(n)$ in the remaining cases remains a challenging problem. The $SS(n)$ forms may be demonstrated schematically with the help of Fig. 1.1 below.

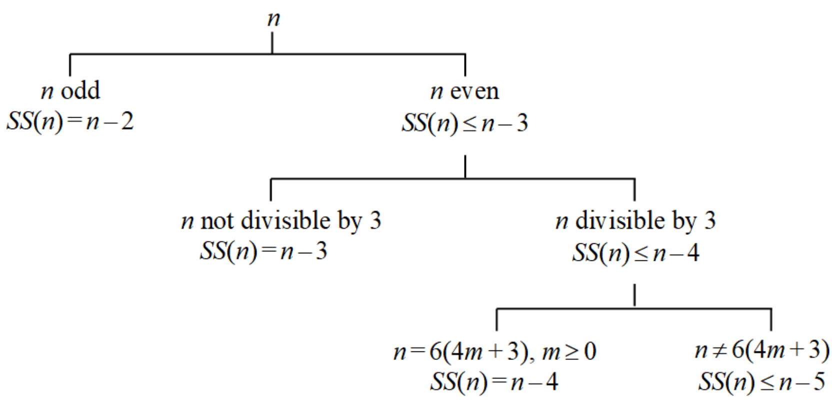

Fig. 1.1. A tree diagram of $SS(n)$.

From the tree above, it is clear that the problems of interest are the ones given in the final branch. Majumdar [5] concentrated solely on the form $SS(p+1)$ functions, where p is an odd prime. Later, the problem was studied to some extent by Islam, and Majumdar

[6], who derived the expressions of $SS(2mp)$, $SS(6mp)$, $SS(60mp)$, and $SS(420mp)$, where p is an odd prime and m is any (positive) integer. Islam *et al*. [3] subsequently found explicit forms of $SS(6t)$, $SS(12t)$, $SS(18t)$, $SS(42t)$, $SS(30t)$, and $SS(210t)$ for some particular cases of t. Later, Majumdar and Ahmed [7] extended the results of Islam *et al.* [3] by considering all the possible cases involved in $SS(210t)$. Recently, Islam and Majumdar [4] derived the expressions of $SS(120m)$, $SS(840m)$, $SS(9240m)$, and $SS(120120m)$ for some particular cases of m.

This paper first derives the necessary and sufficient conditions such that $SS(n) = n - 5$ and $SS(n) = n - 6$. This is done in Section 3 in Theorem 3.1 and 3.2, respectively. Theorem 3.1 shows that one needs to consider the function $SS(12m)$, m (≥ 1) being an integer. And Theorem 3.2 shows that attention needs to be given to the study of the function $SS(60(6m+5))$, $m \geq 0$ being an integer. Thus, starting from $SS(12m)$, one has to consider the functions $SS(60m)$, and then $SS(420m)$, $SS(4620m)$ in succession. Some remarks are made in Section 4, based on the results. Some interesting equations involving $SS(n)$ have been derived. Section 2 summarizes the relevant background materials. The paper concludes with some concluding remarks in Section 5. This paper's unified and detailed analyses suggest that the form of $SS(n)$ depends on the prime factors 2, 3, 5, … (in this order) of the integer n. Another objective is to study how the form of $SS(n)$ changes if some prime factor of n is repeated. At the end of the paper, four tables are appended, which give respectively the values of $SS(60m)$, $SS(420m)$, $SS(4620m)$, and $SS(60060m)$, calculated on a computer, using equation (1.3).

2. Background Material

This section gives the necessary background material that would be needed later. These are given in the following lemmas. For proof, the readers are referred to Islam *et al*. [3].

Lemma 2.1: (*Fundamental Theorem of Arithmetic*) Let a and b be two (positive) integers with $(a, b) = 1$. Let the integer N be such that both a and b divide N. Then, ab divides N.

An alternative proof of Lemma 2.1 may be found in Olds, Lax, and Davidoff [8].

Lemma 2.2: Let A and B be two (positive) integers such that A is divisible by the integer a and B is divisible by the integer b. Then, AB is divisible by ab.

Lemma 2.3: For any integer $a \geq 1$ fixed, $a(a-1)\ldots(a-s+1)$ is divisible by $s!$, where s is an integer with $1 \leq s \leq a$.

Lemma 2.3 states that the product of s consecutive (positive) integers is divisible by s!; for proof, the reader refers to Hardy and Wright [2]. The result below follows readily from Lemma 2.3.

Corollary 2.1: For any integer $a \geq 1$ fixed, let $P(a, s) \equiv a(a-1)\ldots(a-s+1)$ for any integer $1 \leq s \leq a$. Then, s divides $(a-1)(a-2)\ldots(a-s+1)$ if and only if s does not divide a.

The paper's main results are derived in Section 3, where the following result would be required frequently.

Lemma 2.4: Let A, B, and C be any three integers. The Diophantine equation $Ax + By = C$ has an (integer) solution if and only if C is divisible by $D \equiv (A, B)$. Moreover, if (x_0, y_0) is a solution, then there are an infinite number of solutions, given parametrically by $x = x_0 + (\frac{B}{D})t$, $y = y_0 + (\frac{-A}{D})t$ for any integer t.

Proof: See, for example, Gioia [9].

In applying Lemma 2.4, one has to find the solution of the equation $Ax + By = C$ with minimum (positive) x_0 (in the sense that there is no solution x less than x_0). Then, if, in particular, $(A, B) = 1$, then the solutions of the equation are given simply by $x = x_0 + Bt$, $y = y_0 - At$, where t is a parameter. On the other hand, if $(A, B) = D > 1$, it is sufficient to consider the simplified equation $(A/D)x + (B/D)y = C/D$, where $(A/D, B/D) = 1$.

Another interesting result is the following (see Hardy and Wright [2] for proof).

Lemma 2.5: (*Dirichlet Theorem*) If A and B are two integers with $A > 0$ and $(A, B) = 1$, then there are infinitely many primes of the form $Ax + B$, x (> 0) being an integer.

The main results of the paper are given in the next section.

3. Main Results

First, the following two general results are proved.

Theorem 3.1: Let $N \equiv N(p_1, p_2, \ldots, p_k) = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are the first k odd primes in increasing order (so that $2 < p_1 < \ldots < p_k$), $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ and $\alpha \geq 1$ are fixed integers. Then, $SS(Nm) \neq Nm - p_k$ for any integer $m \geq 1$.

Proof: Since

$$C(Nm, Nm - p_k) = Nm[\frac{(Nm - 1)(Nm - 2) \ldots (Nm - p_k + 1)}{2 \times 3 \times \ldots \times p_k}],$$

and since p_k does not divide any of $Nm - 1$, $Nm - 2$, $\ldots$, $Nm - p_k + 1$, it follows that the term inside the square bracket cannot be an integer.

Theorem 3.2: Let $N \equiv N(p_1, p_2, \ldots, p_k) = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are the first k odd primes with $2 < p_1 < \ldots < p_k$, $\alpha \geq 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are fixed integers. Let p $(> p_k)$ be any prime. Then, $SS(Npm) \neq Npm - p_k$ for any integer $m \geq 1$.

Proof: Since

$$C(Npm, Npm - p_k) = Npm[\frac{(Npm - 1)(Npm - 2) \ldots (Npm - p_k + 1)}{2 \times 3 \times \ldots \times p_k}],$$

it follows that the term inside the square bracket cannot be an integer.

To illustrate the application of the above two theorems, note that, by Theorem 3.1,

$SS(30m) \neq 30m - 5$ for any integer $m \geq 1$.

It then follows, by virtue of Theorem 3.2, that

$SS(30mp) \neq 30mp - 5$ for any integer $m \geq 1$, and for any prime p.

The following theorem gives two sets of necessary and sufficient conditions: $SS(n) = n - 5$.

Theorem 3.3: Let n (> 0) be an integer. Then,

$$SS(n) = n - 5 \tag{3.1}$$

if and only if n is one of the following two forms :

(1) $n = 12m$ for some integer $m \geq 0$, where m is not divisible by 5,

(2) $n = 6(4m + 1)$ for some integer $m \geq 0$ with $m \neq 5u + 1$, $u \geq 0$ being an integer,

Proof: By Lemma 1.1 and Lemma 1.2, any integer n satisfying (3.1) must be even and divisible 3; moreover, by Lemma 1.3, $n \neq 6(4u+3)$ for any integer $u \geq 0$.
Now, consider the expression:

$$C(n, n-5) \equiv n[\frac{(n-1)(n-2)(n-3)(n-4)}{2 \times 3 \times 4 \times 5}].$$

Here, the numerator of the term inside the square bracket is divisible by 3 (by Lemma 2.1, coupled with Lemma 2.3); also, the numerator is divisible by 5 if and only if 5 does not divide n. Hence, the term inside the square bracket is an integer if and only if one of the following three conditions is satisfied:
(1) 4 divides $(n-4)$, (2) 4 divides $(n-2)$, (3) 8 divides $(n-4)$.
Moreover, such an n must be divisible by 3.

In case (1), 4 divides $(n-4)$ if and only if n is a multiple of 4. Since n must also be divisible by 3, it follows by Lemma 2.1 that n must be of the form $n = 12m$ for some integer $m \geq 1$. Note that, if $n = 12m$ then $n \neq 6(4u+3)$, for otherwise, $12m = 6(4u+3)$, which, by virtue of Lemma 2.4, has no solution.
In Case (2), 4 divides $(n-2)$; moreover, n is divisible by 3. This leads to the following combined Diophantine equation:

$n = 4\alpha + 2 = 3\beta$ for some integers $\alpha \geq 1$, $\beta \geq 2$,

whose solution is $\alpha = 3m+1$ for any integer $m \geq 0$. Thus,

$n = 4(3m+1)+2 = 6(2m+1)$.

Now, considering the Diophantine equation $6(2m+1) = 6(4u+3)$, the solution is found to be $m = 2u+1$. Thus, m must be even, so that

$n = 6(4m+1)$, $m \geq 0$ being an integer.

Since 5 does not divide n, the Diophantine equation to be considered is

$6(4m+1) = 5x$ for some integer x (>1),

whose solution is $m = 5u+1$ for any integer $u \geq 0$.
In Case (3), 8 divides $(n-4)$; also, n is divisible by 3. Thus,

$n = 8y+4 = 3z$ for some integers $y \geq 1$, $z \geq 4$,

whose solution is $y = 3m+1$, $m \geq 0$ being any integer. Hence,

$n = 8(3m+1)+4 = 12(2m+1)$.

Thus, case (3) is a particular case of the case (1).
Using Theorem 3.1, the following values are found:

$SS(12) = 7$, $SS(24) = 19$, $SS(36) = 31$, $SS(48) = 43$, $SS(72) = 66$, $SS(96) = 91$,
$SS(54) = 49$, $SS(78) = 73$, $SS(102) = 97$, $SS(126) = 121$, $SS(174) = 169$,

Note that, by Theorem 3.3, $SS(6) = 1$, which is consistent with the conventional value. Also, Lemma 3.7 in Islam *et al*. [3] follows directly from part (1) of Theorem 3.3. Moreover, parts (2) and (3) of Theorem 3.3 prove more than those proved in Proposition 3.1 and Proposition 3.2 in Islam Majumdar [4] by different approaches.
The following results are the trivial consequences of Theorem 3.3.
Corollary 3.1: For any prime $p \neq 5$, $SS(12p) = 12p-5$.
Corollary 3.2: Let $SS(n) = n-5$ for some (positive) integer n. Then, $SS(2n) = 2n-5$.
Corollary 3.3: For any prime $p \neq 5$, $SS(24p) = 24p-5$.

After having the expression of $SS(12m)$ (m (≥ 1) being an integer not divisible by 5), the expressions of $SS(12m+i)$ for $1 \leq i \leq 11$) are given in the corollary below.

Corollary 3.4: For any integer $m \geq 1$,

(1) $SS(12m+1) = 12m-1$, (2) $SS(12m+2) = 12m-1$,

(3) $SS(12m+3) = 12m+1$, (4) $SS(12m+4) = 12m+1$,

(5) $SS(12m+5) = 12m+3$,

(6) (a) $SS(12m+6) = 12m+2$, if m is odd,

(b) $SS(12m+6) = \begin{cases} 12m+1, if\ m \text{ is even with } m \neq 10s+2, s \geq 0 \\ 12m-1, \quad if\ m = 10s+2, s \geq 0 \end{cases}$

(7) $SS(12m+7) = 12m+5$, (8) $SS(12m+8) = 12m+5$,

(9) $SS(12m+9) = 12m+7$, (10) $SS(12m+10) = 12m+7$,

(11) $SS(120m+11) = 12m+9$.

Proof: Parts (1), (3), (5), (7), (9), and (11) follow readily from Lemma 1.1, while parts (2), (4), (8), and (10) follow from Lemma 1.2. It thus remains to prove part (6).

Consider the expression:

$$C(12m+6, 12m+2) \equiv (12m+6)[\frac{(12m\ +\ 5)(12m\ +\ 4)(12m\ +\ 3)}{2 \times 3 \times 4}]$$

$$= (12m+6)[\frac{(12m\ +\ 5)(3m\ +\ 1)(4m\ +\ 1)}{2}].$$

The above expression shows that the term inside the square bracket is an integer if and only if $3m+1$ is even, if and only if m is odd. This proves part (6a).

To prove part (6b), let m be even. Now, consider the expression:

$$C(12m+6, 12m+1) \equiv (12m+6)[\frac{(12m+5)(3m+1)(4m+1)(12m+2)}{2\times 5}]$$

$$= (12m+6)[\frac{(12m+5)(3m+1)(4m+1)(6m+1)}{5}].$$

Clearly, the term inside the square bracket is an integer if $12m+6$ is not a multiple of 5. Thus, the Diophantine equation to be considered is

$12m+6 = 5\alpha$ for some integer $\alpha \geq 1$,

whose solution is $m = 5s+2$, $s \geq 0$. Note that, since m is even, s must also be even.

Now, let $m = 10s+2$, $s \geq 0$. The expression

$$C(12m+6, 12m) \equiv (12m+6)[\frac{(12m+5)(3m+1)(4m+1)(6m+1)(12m+1)}{5\times 6}]$$

shows that $SS(12m+6) \neq 12m$ for any $m \geq 1$. So, consider the expression:

$$C(12m+6, 12m-1) \equiv (12m+6)[\frac{(12m+5)(3m+1)(4m+1)(6m+1)(12m+1)(12m)}{5\times 6\times 7}].$$

Considering the Diophantine equation $12m+6 = 7\alpha$ (for some integer $\alpha > 1$), the solution is found to be $m = 10t+3$, $t \geq 0$. This shows that, under the given condition (that $m = 10s+2$, $s \geq 0$), $12m+6$ is not divisible by 7. Hence, the term inside the square bracket is an integer. All these complete the proof of the corollary.

In the course of proving Corollary 3.4, the following result has also been proved.

Corollary 3.5: There is no integer m such that $12m+6$ is divisible by both 5 and 7.

Part (6) of Corollary 3.4 gives the following values:

$SS(18) = 14$, $SS(42) = 38$, $SS(66) = 62$, $SS(90) = 86$, $SS(114) = 110$, $SS(138) = 134$,
$SS(54) = 49$, $SS(78) = 73$, $SS(102) = 97$, $SS(126) = 121$, $SS(174) = 169$,
$SS(30) = 23$, $SS(150) = 143$, $SS(270) = 263$, $SS(390) = 383$, $SS(510) = 503$.

It may be mentioned here that, writing $m = 2u + 1$, part (6a) of Corollary 3.4 may be recast in the form

$SS(24u + 18) = 24u + 14$ for all $u \geq 1$,

which is precisely Lemma 1.3. Again, writing $m = 2u$ in part (6b) of Corollary 3.4, one gets

$$SS(24u+6) = \begin{cases} 24u+1, & if\ u \neq 5x+1, x \geq 0 \\ 24u-1, if\ u = 5y+1, y \neq 7z+5, z \geq 0 \end{cases} \tag{3.2}$$

which has been derived earlier by a different approach (see Proposition 3.1 in Islam *et al.* [4]). The expression in (3.2) needs some explanation : If $u = 5x + 1$ for some integer $x \geq 1$, then $SS(24u + 6) = 24u - 1$ provided that $24u + 6$ is not divisible by 7. Thus, the Diophantine equation to be considered is $24u + 6 = 7\alpha$ (for some integer $\alpha > 0$), whose solution is $u = 7\beta + 5$, $\beta \geq 0$. Now, the solution of the combined equation $5y + 1 = 7\beta + 5$ is $y = 7z + 5$, $z \geq 0$.

The next theorem gives the necessary and sufficient conditions such that $SS(n) = n - 6$.

Theorem 3.4: Let n (> 0) be an integer divisible by 5. Then,

$$SS(n) = n - 6 \tag{3.3}$$

if and only if $n = 60(6m + 5)$ for any integer $m \geq 0$.

Proof: Let n (> 0) be an integer divisible by 5.

Consider the expression:

$$C(n, n-6) \equiv n\left[\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{2 \times 3 \times 4 \times 5 \times 6}\right].$$

Now, the numerator of the term inside the square bracket is divisible by $2\times3\times5$ (by Lemma 2.1, coupled with Lemma 2.3). Hence, the term inside the square bracket is an integer if and only if the following two conditions are satisfied simultaneously:

(1) 8 divides $(n - 4)$, (2) 9 divides $(n - 3)$.

By Condition (1), 8 divides $(n - 4)$, so that

$n = 8\alpha + 4$ for some integer $\alpha \geq 1$,

and by Condition (2),

$n = 9\beta + 3$ for some integer $\beta \geq 1$.

Now, the solution of the combined Diophantine equation $8\alpha + 4 = 9\beta + 3$ is $\alpha = 9x + 1$, x(≥ 0) being an integer. Therefore,

$n = 8(9x + 1) + 4 = 12(6x + 1)$.

Since 5 divides n, one needs to consider the Diophantine equation

$12(6x + 1) = 5\gamma$ for some integer γ (> 1),

whose solution is $x = 5m + 4$ for any integer $m \geq 0$. Hence, finally

$n = 12[6(5m + 4) + 1] = 60(6m + 5)$.

Note that, $n - 4 = 8(45m + 37)$ is, in fact, divisible by 8.

Theorem 3.4 gives the following values:

$SS(300) = 294$, $SS(660) = 654$, $SS(1020) = 1014$, $SS(1380) = 1374$.

Note that, Theorem 3.4 may be put in the following form

$SS(60t) = 60t - 6$ if and only if $t = 6s + 5$ for any integer $s \geq 0$. (3.4)

The above result is stronger than the previous results found in Islam *et al*. [3, Lemma 3.8] and Islam and Majumdar. [4, Proposition 3.2] by different approaches.

Note that, Lemma 1.2 may be rewritten as follows:

$SS(6m) = 6m - 4$ if and only if $m = 4s + 3$, $s \geq 0$. (3.5)

Also, Theorem 3.3 may be recast in the following equivalent form :

$SS(6m) = 6m - 5$ (3.6)

if and only if 5 does not divide m, and m is of one of the following three forms:

(1) $m = 2s$, $s \geq 1$, (2) $m = 4s + 1$, $s \geq 0$, (3) $m = 2(2s + 1)$, $s \geq 0$.

Finally, Theorem 3.4 may be rewritten in the following equivalent form

$SS(60m) = 60m - 6$ if and only if $m = 6s + 5$, $s \geq 0$ being any integer. (3.7)

In view of Theorem 3.3,

$SS(60m) \leq 60m - 6$ for any $m \geq 1$,

and in view of Theorem 3.4, we have the following results.

Corollary 3.6: $SS(60m) \leq 60m - 7$ for any $m \neq 6s + 5$, $s \geq 0$.

Corollary 3.7: There is an infinite number of primes p such that $SS(60p) = 60p - 6$.

Proof: Let p be a prime of the form $p = 6s + 5$, $s \geq 0$. By Lemma 2.5, there is an infinite number of primes of this form. With such a prime p, by (3.7), $SS(60p) = 60p - 6$.

In Majumdar [10], the following result has been established.

Lemma 3.1: $SS(60m) = 60m - 7$ if m is not divisible by 7 with $m \neq 6s + 5$, $s \geq 0$.

The next lemma considers $SS(420m)$. Since

$$C(420m, 420m - 7) \equiv 420m\left[\frac{(420m-1)(420m-2)(420m-3)(420m-4)(420m-5)(420m-6)}{2\times3\times4\times5\times6\times7}\right]$$

$$= 420m\left[\frac{(420m-1)(210m-1)(140m-1)(105m-1)(84m-1)(70m-1)}{7}\right],$$

it follows that $SS(420m) \neq 420m - 7$ for any integer $m \geq 1$. This is, in fact, a particular case of Theorem 3.1. It then follows, by Theorem 3.2 that

$SS(420mp) \neq 420mp - 7$ for any integer $m \geq 1$, and for any prime p.

Lemma 3.2: Let $m \geq 1$ be an integer. Then,

(1) $SS(420m) = 420m - 6$, if $m = 6s + 5$, $s \geq 0$,

(2) $SS(420m) = 420m - 8$, if $m = 8s + 1$, $s \neq 3t + 2$, $t \geq 0$,

(3) $SS(420m) = 420m - 9$, if $m = 9s + 2$, s is even (including 0),

or, if $m = 9t + 4$, $t \neq 8b + 5$, $b \geq 0$,

(4) $SS(420m) = 420m - 10$, if $m = 2(10s + 7)$, $s \neq 9u + 3$, $s \neq 9v + 4$, $u \geq 0$, $v \geq 0$,

or if $m = 20t + 9$, $t \neq 3a + 1$, $t \neq 2b$, $t \neq 9c + 1$, $t \neq 9d + 2$, $a, b, c, d \geq 0$,

(5) $SS(420m) = 420m - 11$, if 11 does not divide m and $m \neq 8a + 1$, $m \neq 9b + 2$, $m \neq 9c + 4$, $m \neq 6d + 5$, $m \neq 2(10e + 7)$, $m \neq 20f + 9$, $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, $e \geq 0$, $f \geq 0$.

Proof: To prove part (1), consider the expression:

$$C(420m, 420m-6) \equiv 420m[\frac{(420m-1)(420m-2)(420m-3)(420m-4)(420m-5)}{2\times3\times4\times5\times6}]$$

$$= 420m[\frac{(420m-1)(210m-1)(140m-1)(105m-1)(84m-1)}{6}].$$

Now, in order that the term inside the square bracket is an integer, 3 must divide $140m-1$, and 2 must divide $105m-1$, leading to the two Diophantine equations
$140m-1=3\alpha$, $105m-1=2\beta$ for some integers $\alpha>0$, $\beta>0$.
The solutions of the above equations are $m=3x+2$, $x\geq 0$ and $m=2y+1$, $y\geq 0$ respectively. Then, the combined Diophantine equation is $3x+2=2y+1$, whose solution is $x=2u+1$, $u\geq 0$. Therefore, $m=3(2u+1)+2=6u+5$, $u\geq 0$.
To prove part (2), let the integer m be such that $m\neq 6u+5$, $u\geq 0$. Consider

$$C(420m, 420m-8) \equiv 420m[\frac{(420m-1)(210m-1)(140m-1)(105m-1)(84m-1)(70m-1)(420m-7)}{7\times8}].$$

Here, the term inside the square bracket is an integer if and only if $105m-1$ is divisible by 8, that is, if and only if
$105m-1=8a$ for some integer $a>0$.
The solution of the above equation is $m=8s+1$, $s\geq 0$. Considering the combined equation $8s+1=6u+5$, the solution is found to be $s=3t+2$, $t\geq 0$.
To prove part (3), let the integer m be such that $m\neq 6u+5$, $u\geq 0$, $m\neq 8s+1$, $s\geq 0$. Consider the expression:

$$C(420m, 420m-9) \equiv 420m[\frac{(420m-1)(210m-1)(140m-1)(105m-1)(84m-1)(70m-1)(60m-1)(420m-8)}{8\times9}]$$

$$= 420m[\frac{(420m-1)(210m-1)(140m-1)(105m-1)(84m-1)(70m-1)(60m-1)(105m-2)}{2\times9}].$$

In order to find the condition such that the term inside the square bracket is an integer, first note that one of $105m-1$ and $105m-2$ is even. Thus, it is sufficient to find the condition such that the numerator of the term inside the square bracket is divisible by 9. Here, there are two possibilities, namely, either 9 divides $70m-1$, or else 9 divides $140m-1$. In the first case,
$140m-1=9\alpha$ for some integer $\alpha>0$,
whose solution is $m=9u+2$, $u\geq 0$. Now, the solution of the equation $9u+2=6x+5$ is $u=2\beta+1$, $\beta\geq 0$, while the solution of the equation $9u+2=8s+1$ is $u=8a+7$, $a\geq 0$. Note that if u is restricted to even values (including 0), then $u\neq 8a+7$ for any $a\geq 0$.
The second possibility leads to the Diophantine equation $70m-1=9\beta$ for some integer $\beta>0$,
with the solution $m=9v+4$, $v\geq 0$. Note that, the combined equation $9v+4=6x+5$ has no solution (by virtue of Lemma 2.4); also, note that, the combined equation $9v+4=8s+1$ has the solution $v=8b+5$, $b\geq 0$.
To prove part (4), let the integer m be such that $m\neq 6a+5$, $a\geq 0$, $m\neq 9b+2$, $b\geq 0$, $m\neq 9c+4$, $c\geq 0$. Consider the expression below:

$$C(420m, 420m-10) \equiv 420m[\frac{(420m-1)(210m-1)(140m-1)(105m-1)(84m-1)(70m-1)(60m-1)(105m-2)(420m-9)}{2\times9\times10}]$$

$$= 420m[\frac{(420m-1)(210m-1)(140m-1)(105m-1)(84m-1)(70m-1)(60m-1)(105m-2)(140m-3)}{2\times 3\times 10}].$$

Now, one of $140m - 1$, $70m - 1$ and $140m - 3$ is divisible by 3 (by Lemma 2.3). Therefore, the term inside the square bracket is an integer if $84m - 1$ is divisible by 5 and either $105m - 2$, or else $105m - 1$ is divisible by 4. The first possibility leads to the Diophantine equations

$84m - 1 = 5\alpha$, $105m - 2 = 4\beta$ for some integers $\alpha > 0$, $\beta > 0$,

whose solutions are $m = 5u + 4$, $u \geq 0$, and $m = 4v + 2$, $v \geq 0$ respectively. The solution of the combined Diophantine equation $5u + 4 = 4v + 2$ is $u = 4s + 2$, $s \geq 0$, so that

$m = 5(4s + 2) + 4 = 2(10s + 7)$, $s \geq 0$.

Now, the second possibility gives rise to the equation $105m - 1 = 4\gamma$ for some integer $\gamma > 0$, whose solution is $m = 4w + 1$, $w \geq 0$. The solution of the combined equation $5u + 4 = 4w + 1$ is $u = 4t + 1$, $t \geq 0$, so that $m = 5(4t + 1) + 4 = 20t + 9$.

To complete the proof of part (3), it remains to find the conditions such that the conditions of part (1) and part (2) are not satisfied. Clearly, the combined equation $2(10s + 7) = 6x + 5$ has no solution; also, the equation $2(10s + 7) = 8y + 1$ has no solution. Now, considering the Diophantine equation $2(10s + 7) = 9u + 2$, the solution is found to be $s = 9a + 3$, $a \geq 0$, while the solution of the equation $2(10s + 7) = 9v + 4$ is $s = 9b + 4$, $b \geq 0$. Next, we have to consider the following combined equations:

$20t + 9 = 6a + 5$, $20t + 9 = 8b + 1$, $20t + 9 = 9c + 2$, $20t + 9 = 9d + 4$;

the solutions of the above equations are $t = 3a + 1$, $a \geq 0$, $t = 2b$, $b \geq 0$, $t = 9c + 1$, $t = 9d + 2$ respectively.

Finally, to prove part (5), let the integer m be such that all the conditions in parts (1) – (4) are violated. Consider the expression:

$$C(420m, 420m - 11) \equiv 420m[\frac{(420m-1)(210m-1)(140m-1)(105m-1)(84m-1)(70m-1)(60m-1)(105m-2)(140m-3)(420m-10)}{2\times 3\times 10\times 11}]$$

$$= 420m[\frac{(420m-1)(210m-1)(140m-1)(105m-1)(84m-1)(70m-1)(60m-1)(105m-2)(140m-3)(42m-1)}{2\times 3\times 11}].$$

Now, one of $105m - 1$ and $105m - 2$ is even; also, one of $140m - 1$, $70m - 2$, and $140m - 3$ is divisible by 3. Therefore, if m is not divisible by 11, then the term inside the square bracket is an integer.

Lemma 3.2 gives the following values:

$SS(2100) = 2094$, $SS(4620) = 4614$, $SS(7140) = 7134$, $SS(9660) = 9654$, $SS(12180) = 12174$,

$SS(420) = 412$, $SS(3780) = 3772$, $SS(10500) = 10492$, $SS(13860) = 13852$,

$SS(840) = 831$, $SS(8400) = 8391$, $SS(15960) = 15951$, $SS(23520) = 23511$,

$SS(1680) = 1671$, $SS(5460) = 5451$, $SS(9240) = 9231$, $SS(13020) = 13011$,

$SS(5880) = 5870$, $SS(14280) = 14270$, $SS(22680) = 22670$, $SS(28980) = 28970$,

$SS(1260) = 1249$, $SS(2520) = 2509$, $SS(2940) = 2929$, $SS(3360) = 3349$, $SS(4200) = 4189$.

Some consequences of Lemma 3.2 are given below.

Lemma 3.3: There is an infinite number of primes p such that $SS(420p) = 420p - 6$.

Proof: Let p be the prime of the form $p = 6s + 5$, $s \geq 0$. By Lemma 2.5, there is an infinite number of primes of the prescribed form. With this p, by part (1) of Lemma 3.2, $SS(420p) = 420p - 6$.

Lemma 3.4: There is an infinite number of primes p such that $SS(420p) = 420p - 8$.

Proof: Let p be the prime of the form $p = 8s + 1$, $s \neq 3t + 2$, $t \geq 0$. With this p, by part (2) of Lemma 3.2, $SS(420p) = 420p - 6$. Clearly, there is an infinite number of such primes.

Lemma 3.5: There is an infinite number of primes p such that $SS(420p) = 420p - 9$.

Proof: Let p be the prime of the form $p = 9s + 4$, $s \neq 8t + 5$, $t \geq 0$. Then, by part (3) of Lemma 3.2, $SS(420p) = 420p - 9$. Note that there is an infinite number of such p.

The following results, involving the function $SS(840m)$, are evident from Lemma 3.2.

Corollary 3.8: $SS(840m) \neq 840m - 6$ for any integer m (≥ 1).

Corollary 3.9: $SS(840m) \neq 840m - 8$ for any integer m (≥ 1).

Lemma 3.2 may be exploited to find the expressions of $SS(840m)$, as is done below.

Corollary 3.10: Let $m \geq 1$ be an integer. Then,

(1) $SS(840m) = 840m - 9$, if $m = 9s + 1$, $s \geq 0$, or, if $m = 9t + 2$, $t \geq 0$,

(2) $SS(840m) = 840m - 10$, if $m = 10s + 7$, $s \neq 9u + 3$, $s \neq 9v + 4$, $u \geq 0$, $v \geq 0$,

(3) $SS(840m) = 840m - 11$, if 11 does not divide m and $m \neq 9a + 1$, $m \neq 9b + 2$, $m \neq 10c + 7$, $a \geq 0$, $b \geq 0$, $c \geq 0$.

Proof: We may find $SS(840m)$ by replacing m by $2m$ in $SS(420m)$.

By part (3) of Lemma 3.2,

$SS(840m) = 840m - 9$ if $2m = 9s + 2$, $s \geq 0$, or if $2m = 9t + 4$, $t \geq 0$.

Now, the solution of the Diophantine equation $2m = 9s + 2$ is $m = 9x + 1$, $x \geq 0$, while the equation $2m = 9t + 4$ has the solution $m = 9y + 2$, $y \geq 0$.

Replacing m by $2m$ in part (4) of Lemma 3.2 and noting that only the first of the two conditions can hold true, one gets

$SS(840m) = 840m - 10$ if $2m = 2(10s + 7)$, $s \geq 0$.

Here, s is such that $m = 10s + 7$ does not assume the values given in part (1) of Corollary 3.9. Thus, the equations to be considered are $10s + 7 = 9x + 1$ and $10s + 7 = 9y + 2$, whose solutions are $s = 9u + 3$, $u \geq 0$ and $s = 9v + 4$, $v \geq 0$ respectively.

To prove part (3), consider the following expression for $C(840m, 840m - 11)$:

$$840m\left[\frac{(840m-1)(840m-2)(840m-3)(840m-4)(840m-5)(840m-6)(840m-7)(840m-8)(840m-9)(840m-10)}{2\times3\times4\times5\times6\times7\times8\times9\times10\times11}\right]$$

$$= 840m\left[\frac{(840m-1)(420m-1)(280m-1)(210m-1)(168m-1)(140m-1)(120m-1)(105m-1)(280m-3)(84m-1)}{3\times11}\right].$$

Now, since m is not divisible by 11, the numerator of the term inside the square bracket is divisible by 11; also, one of $280m - 1$, $140m - 1$, and $280m - 3$ is divisible by 3. Thus, the term inside the square bracket is an integer.

The results in Corollary 3.10 match with those found by Islam *et al*. [4, Lemma 3.4 – Lemma 3.6] by following a different approach.

Part (5) of Lemma 3.2 suggests that the next function to be considered is $SS(4620m)$. In this connection, the following results can be established using Lemma 3.2, noting that $4620m$ is 11 times $420m$.

Lemma 3.6: Let $m \geq 1$ be an integer. Then,

(1) $SS(4620m) = 4620m - 6$, if $m = 6s + 1$, $s \geq 0$,
(2) $SS(4620m) = 4620m - 8$, if $m = 8s + 3$, $s \neq 3t + 2$, $t \geq 0$,
(3) $SS(4620m) = 4620m - 9$, if $m = 9s + 1$, s is odd,
or, if $m = 9t + 2$, $t \neq 8b + 1$, $b \geq 0$,
(4) $SS(4620m) = 4620m - 10$, if $m = 2(10s + 7)$, $s \neq 9u + 3$, $s \neq 9v + 7$, $u \geq 0$, $v \geq 0$,
or if $m = 20t + 19$, $t \neq 2a$, $t \neq 3b$, $t \neq 9c + 5$, $a, b, c \geq 0$.

Proof: To prove part (1) of the lemma, note that, replacing m by $11m$ in part (1) of Lemma 3.2, the condition therein becomes $11m = 6a + 5$, whose solution is $m = 6s + 1$, $s \geq 0$. Writing $11m$ in place of m in part (2) of Lemma 3.2, the condition therein takes the form $11m = 8b + 1$, whose solution is $m = 8s + 3$, $s \geq 0$. In order to exclude the possibility of the values in part (1) of Lemma 3.6, the equation to be considered is $8s + 3 = 6\alpha + 1$, whose solution is $s = 3t + 2$, $t \geq 0$.

To prove part (3), let m in part (3) of Lemma 3.2 be replaced by $11m$. Here, there are two possible cases. The first possibility is that $11m = 9\beta + 2$, whose solution is $m = 9s + 1$, $s \geq 0$. To find the restrictive conditions on s, one needs to consider the following two combined Diophantine equations:

$9s + 1 = 6b + 1$, $9s + 1 = 8c + 3$.

The solutions of the above equations are $s = 2d$, $d \geq 0$ and $s = 8e + 2$, $e \geq 0$ respectively. Note that, if s is restricted to odd values, then $s \neq 8e + 2$ for any $e \geq 0$.

In the second case, the condition becomes $11m = 9\gamma + 4$, whose solution is $m = 9t + 2$, $t \geq 0$. In this case, noting that the Diophantine equation $9t + 2 = 6b + 1$ has no solution, the restrictive condition on t is determined by the equation $9t + 2 = 8c + 3$ only. This gives the solution $t = 8z + 1$, $z \geq 0$.

It now remains to prove part (4) of the lemma. Writing $11m$ for m in part (4) of Lemma 3.2, the two Diophantine equations therein become respectively

$11m = 2(10\alpha + 7)$, $11m = 20\beta + 9$,

with respective solutions $m = 2(10s + 7)$, $s \geq 0$ and $m = 20t + 19$, $t \geq 0$. With the first solution, the conditions on s are to be found that guarantee that s cannot take the values given in parts (1), (2) and (3) of Lemma 3.6. Since none of the combined Diophantine equations $2(10s + 7) = 6a + 1$ and $2(10s + 7) = 8b + 3$ has a solution, it is sufficient to consider the following two combined equations:

$2(10s + 7) = 9c + 1$, $2(10s + 7) = 9d + 2$.

The solutions of the above equations are $s = 9u + 7$, $u \geq 0$ and $s = 9v + 3$, $v \geq 0$ respectively. With the second solution, four combined Diophantine equations are to be considered. They are

$20t + 19 = 6\alpha + 1$, $20t + 19 = 8\beta + 3$, $20t + 19 = 9\gamma + 1$, $20t + 19 = 9\theta + 2$, with the respective solutions

$t = 3a$, $a \geq 0$, $t = 2b$, $b \geq 0$, $t = 9c$, $c \geq 0$, $t = 9d + 5$, $d \geq 0$.

Note that if t is not divisible by 3, then it is also not divisible by 9.

Using Lemma 3.6, the following values may be obtained.

$SS(4620) = 4614$, $SS(32340) = 32334$, $SS(60060) = 60054$, $SS(87780) = 87774$,
$SS(13860) = 13852$, $SS(50820) = 50812$, $SS(124740) = 124732$, $SS(161700) = 161692$,

$SS(9240) = 9231$, $SS(92400) = 92391$, $SS(133980) = 133971$, $SS(175560) = 175551$,
$SS(46200) = 46191$, $SS(129360) = 129351$, $SS(212520) = 212511$,
$SS(64680) = 64670$, $SS(157080) = 157070$, $SS(180180) = 180170$, $SS(249480) = 249470$.

Consider the following expression for $C(4620m, 4620m-11)$:

$$4620m[\frac{(4620m-1)(4620m-2)(4620m-3)(4620m-4)(4620m-5)(4620m-6)(4620m-7)(4620m-8)(4620m-9)(4620m-9)}{2\times3\times4\times5\times6\times7\times8\times9\times10\times11}]$$

$$= 4620m[\frac{(4620m-1)(2310m-1)(1540m-1)(1155m-1)(924m-1)(770m-1)(660m-1)(1155m-2)(1540m-3)(462m-1)}{2\times3\times11}].$$

The above expression shows that $SS(4620m) \neq 4620m - 11$ for any integer $m \geq 1$. It may be mentioned here that this result also follows from Theorem 3.1.

Lemma 3.6 is supplemented by the following two results.

Lemma 3.7: $SS(4620m) = 4620m - 12$ if $m = 6(12s + 5)$, $s \neq 5t + 2$, $t \geq 0$.

Proof: Consider the following expression for $C(4620m, 4620m-12)$:

$$4620m[\frac{(4620m-1)(2310m-1)(1540m-1)(1155m-1)(924m-1)(770m-1)(660m-1)(1155m-2)(1540m-3)(462m-1)(4620m-11)}{2\times3\times11\times12}]$$

$$= 4620m[\frac{(4620m-1)(2310m-1)(1540m-1)(1155m-1)(924m-1)(770m-1)(660m-1)(1155m-2)(1540m-3)(462m-1)(420m-1)}{2\times3\times12}]$$

Now, the term inside the square bracket is an integer if $1155m - 2$ is divisible by 8 and $1540m - 3$ is divisible by 9. Thus, the following two Diophantine equations result :
$1155m - 2 = 8\alpha$, $1540m - 3 = 9\beta$ for some integers $\alpha > 0$, $\beta > 0$.

The solutions of the above equations are $m = 8u + 6$, $u \geq 0$ and $m = 9v + 3$, $v \geq 0$ respectively. Now, considering the combined Diophantine equation $8u + 6 = 9v + 3$, the solution is found to be $u = 9s + 3$, $s \geq 0$, so that $m = 8(9s + 3) + 6 = 6(12s + 5)$. It now remains to find the condition(s) on s in order to exclude the common values shared by the values given in Lemma 3.6. Since none of the equations

$6(12s + 5) = 6a + 1$, $6(12s + 5) = 8b + 3$, $6(12s + 5) = 9c + 1$, $6(12s + 5) = 9d + 2$,
$6(12s + 5) = 20\alpha + 19$

has a solution, it is sufficient to consider the equation $6(12s + 5) = 2(10\beta + 7)$ only. Now, the solution of the equation is $s = 5t + 2$, $t \geq 0$.

Lemma 3.7 shows that, though $SS(4620m) = 4620m - 12$ for an infinite number of m, these values are distributed rather sparsely. The first few values, obtained from Lemma 3.7, are listed below.

$SS(138600) = 138588$, $SS(471240) = 471228$, $SS(1136520) = 1136508$.

Lemma 3.8: Let m be an integer not divisible by 13 such that $m \neq 6a + 1$, $m \neq 9b + 1$, $m \neq 9c + 2$, $m \neq 8d + 3$, $m \neq 2(10e + 7)$, $m \neq 20f + 19$, $m \neq 6(12g + 5)$; $a, b, c, d, e, f, g \geq 0$. Then, $SS(4620m) = 4620m - 13$.

Proof: Consider the following expression for $C(4620m, 4620m-13)$:

$$4620m[\frac{(4620m-1)(2310m-1)(1540m-1)(1155m-1)(924m-1)(770m-1)(660m-1)(1155m-2)(1540m-3)(462m-1)(420m-1)(4620m-12)}{2\times3\times12\times13}]$$

$$= 4620m\left[\frac{(4620m-1)(2310m-1)(1540m-1)(1155m-1)(924m-1)(770m-1)(660m-1)(1155m-2)(1540m-3)(462m-1)(420m-1)(385m-1)}{2\times 3\times 13}\right].$$

Now, consider the numerator of the term inside the square bracket. Since 13 does not divide *m*, the numerator is divisible by 13; also, one of $1155m-1$ and $1155m-2$ is even, and one of $1540m-1$, $770m-1$, and $1540m-3$ is divisible by 3. Thus, the term inside the square bracket is an integer.

Applying Lemma 3.8, the following values are obtained.

$SS(18480)=18467$, $SS(23100)=23087$, $SS(27720)=27707$, $SS(36960)=36947$.

The following results, involving $SS(9240m)$, are evident from Lemma 3.6.

Corollary 3.11: $SS(9240m)\neq 9240m-6$ for any integer m (≥ 1).

Corollary 3.12: $SS(9240m)\neq 9240m-8$ for any integer m (≥ 1).

Lemma 3.6 – Lemma 3.8 may be employed to find $SS(9240m)$, as is done below.

Corollary 3.13: Let $m\geq 1$ be an integer. Then,

(1) $SS(9240m)=9240m-9$, if $m=9s+1$, $s\geq 0$, or, if $m=9t+5$, $t\geq 0$,

(2) $SS(9240m)=9240m-10$, if $m=10s+7$, $s\neq 9u+3$, $s\neq 9v+7$, $u\geq 0$, $v\geq 0$,

(3) $SS(9240m)=9240m-12$, if $m=3(12s+5)$, $s\neq 5x+2$, $x\geq 0$,

(4) $SS(9240m)=9240m-13$, if 13 does not divide m and $m\neq 9a+1$, $m\neq 9b+5$, $m\neq 10c+7$, $m\neq 3(12d+5)$, $a\geq 0$, $b\geq 0$, $c\geq 0$, $d\geq 0$.

Proof: The function $SS(9240m)$ may be obtained from $SS(4620m)$ by replacing m by $2m$.

Replacing m by $2m$ in part (3) of Lemma 3.6, one gets

$SS(9240m)=9240m-9$ if $2m=9x+2$, $x\geq 0$, or if $2m=9y+1$, $y\geq 0$.

Now, the solutions of the Diophantine equations are $m = 9s + 1$, $s\geq 0$ and $m = 9t + 5$, $t\geq 0$ respectively.

Replacing m by $2m$ in part (4) of Lemma 3.6, and noting that only the first of the two conditions can hold true, one gets

$SS(9240m)=9240m-10$ if $2m=2(10s+7)$, $s\geq 0$.

Thus, $m = 10s + 7$, where s is such that m does not assume the values given in part (1) of Corollary 3.13. To do so, the Diophantine equations to be considered are $10s + 7 = 9a + 1$ and $10s+7=9b+5$, whose solutions are $s=9u+3$, $u\geq 0$ and $s=9v+7$, $v\geq 0$ respectively.

Now, replacing m by $2m$ in Lemma 3.7, one gets

$SS(9240m)=9240m-12$ if $2m=6(12s+5)$, $s\geq 0$.

Here, noting that none of the equations $3(12s + 5) = 9a + 1$ and $3(12s + 5) = 9b + 5$ possesses a solution, in order to find the restrictive condition on s, it is sufficient to consider the equation $3(12s+5)=10c+7$, whose solution is $s=5t+2$, $t\geq 0$.

To prove part (4), consider the following expression for $C(9240m, 9240m-13)$:

$$9240m\left[\frac{(9240m-1)(9240m-2)(9240m-3)(9240m-4)(9240m-5)(9240m-6)(9240m-7)(9240m-8)(9240m-9)(9240m-10)(9240m-11)(9240m-12)}{2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10\times 11\times 12\times 13}\right]$$

$$= 9240m\left[\frac{(9240m-1)(4620m-1)(3080m-1)(2310m-1)(1848m-1)(1540m-1)(1320m-1)(1155m-1)(3080m-3)(924m-1)(840m-1)(770m-1)}{3\times 13}\right].$$

Now, one of $3080m-1$, $1540m-1$, and $3080m-3$ is divisible by 3; moreover, one of the factors in the numerator of the term inside the square bracket is divisible by 13. Thus, the term inside the square bracket is an integer.

Using Corollary 3.13, the following values may be obtained.

$SS(9240) = 9231$, $SS(46200) = 46191$, $SS(92400) = 92391$, $SS(129360) = 129351$,

$SS(64680) = 64670$, $SS(157080) = 157070$, $SS(2494800) = 249470$, $SS(434820) = 434270$,

$SS(138600) = 138588$, $SS(471240) = 471228$, $SS(1136520) = 1136508$,

$SS(18480) = 18467$, $SS(27720) = 27707$, $SS(36960) = 36947$, $SS(55440) = 55427$.

Corollary 3.13 shows that

$9240m - 9 \le SS(9240m) \le 9240m - 13$ for any integer $m \ge 1$,

with $SS(9240m) \ne 9240m - 11$ for any integer $m \ge 1$.

The final result of this section is the following.

Lemma 3.9: Let $m \ge 1$ be an integer. Then,

(1) $SS(60060m) = 60060m - 6$, if $m = 6s + 1$, $s \ge 0$,

(2) $SS(60060m) = 60060m - 8$, if $m = 8s + 7$, $s \ne 3a$, $a \ge 0$,

(3) $SS(60060m) = 60060m - 9$, if $m = 9s + 5$, $s \ne 8a + 2$, $a \ge 0$,
or, if $m = 9t + 7$, $t \ge 1$ is odd,

(4) $SS(60060m) = 60060m - 10$, if $m = 20s + 3$, s is even, $s \ne 3a + 2$, $s \ne 9b + 2$, $s \ne 9c + 8$, $a \ge 0$, $b \ge 0$, $c \ge 0$,
or if $m = 2(10t + 9)$, $t \ne 9u + 7$, $t \ne 9v + 8$, $u, v \ge 0$

(5) $SS(60060m) = 60060m - 12$, if $m = 6(12s + 5)$, $s \ne 5a + 4$, $a \ge 0$.

Proof: To prove part (1), note that, replacing m by $13m$ in part (1) of Lemma 3.6, the condition therein becomes $13m = 6a + 1$, whose solution is $m = 6s + 1$, $s \ge 0$.

Writing $13m$ in place of m in part (2) of Lemma 3.6, the condition therein takes the form $13m = 8b + 3$, whose solution is $m = 8s + 7$, $s \ge 0$. Note that, the solution of the equation $8s + 7 = 6a + 1$ is $s = 3x$, $x \ge 0$.

Next, let m in part (3) of Lemma 3.6 be replaced by $13m$. Here, there are two possible cases. The first possibility is that $13m = 9\beta + 1$, whose solution is $m = 9t + 7$, $t \ge 0$. Here, one needs to consider the following two combined Diophantine equations:

$9t + 7 = 6a + 1$, $9t + 7 = 8c + 7$.

The solutions of the above equations are $s = 2d$, $d \ge 0$ and $s = 8e$, $e \ge 0$ respectively.

The second possibility is that $13m = 9\gamma + 2$, whose solution is $m = 9s + 5$, $s \ge 0$. Here, the Diophantine equation $9s + 5 = 6a + 1$ has no solution (by Lemma 2.4), while the solution of the equation $9s + 5 = 8c + 7$ is $s = 8y + 2$, $y \ge 0$.

To prove part (4) of the lemma, replacing m by $13m$ for m in part (4) of Lemma 3.6, the two Diophantine equations therein become respectively

$13m = 2(10\alpha + 7)$, $13m = 20\beta + 19$, with respective solutions $m = 2(10t + 9)$, $t \geq 0$ and $m = 20s + 3$, $s \geq 0$. Now, none of the combined Diophantine equations $2(10t + 9) = 6a + 1$ and $2(10t + 9) = 8c + 7$ has a solution, while the solutions of the following two combined equations

$$2(10t + 9) = 9d + 5,\ 2(10t + 9) = 9e + 7,$$

are $t = 9u + 7$, $u \geq 0$ and $t = 9v + 8$, $v \geq 0$ respectively. With the second solution $m = 20s + 3$, the following four equations are to be considered:

$$20s + 3 = 6a + 1,\ 20s + 3 = 8c + 7,\ 20s + 3 = 9d + 5,\ 20s + 3 = 9e + 7,$$

whose solutions are respectively

$$s = 3a + 2,\ a \geq 0,\ s = 2b + 1,\ b \geq 0,\ s = 9c + 8,\ c \geq 0,\ s = 9d + 2,\ d \geq 0.$$

To prove part (5) of the lemma, replacing m by $13m$ in Lemma 3.7, the condition therein becomes $13m = 6(12a + 5)$, whose solution is $m = 6(12s + 5)$. Here, the Diophantine equations to be considered are

$$6(20s + 5) = 6a + 1,\ 6(20s + 5) = 8c + 7,\ 6(20s + 5) = 9d + 5,\ 6(20s + 5) = 9e + 7,$$

$$6(20s + 5) = 20f + 3,\ 6(20s + 5) = 2(10g + 9).$$

The solution of the last equation is $s = 5t + 4$, $t \geq 0$, while none of the remaining equations possesses a solution.

Using Lemma 3.6, the following values may be obtained.

$SS(60060) = 60054$, $SS(420420) = 420414$, $SS(780780) = 780774$, $SS(1141140) = 1141134$,

$SS(900900) = 9008922$, $SS(1381380) = 1381372$, $SS(2342340) = 2342332$,

$SS(30030) = 300291$, $SS(840840) = 840831$, $SS(960960) = 960951$,

$SS(180180) = 180170$, $SS(1081080) = 1081070$, $SS(2282280) = 2282270$,

$SS(1801800) = 1801788$, $SS(6126120) = 6126108$, $SS(10450440) = 10450428$.

Note that $SS(60060m) \neq 60060m - 13$ for any integer m (≥ 1).

4. Some Remarks

This section derives some interesting results involving the function $SS(n)$.

Lemma 4.1: Let the equation

$$SS(n + 1) = SS(n) + m \tag{4.1}$$

have a solution for some (positive) integers n and m. Then, with this m, the equation

$$SS(n + 1) = SS(n) - m + 2 \tag{4.2}$$

has also a solution.

Proof: Let, for some integer m (> 0) fixed, n_0 be a solution of the equation (4.1). Then, n_0 must be even, for otherwise, n_0 is odd, so that

$$SS(n_0 + 1) \leq n_0 - 2,\ SS(n_0) = n_0 - 2,$$

violating the equation (4.1). Hence, n_0 must be even with

$$SS(n_0 + 1) = n_0 - 1,\ SS(n_0) = n_0 - m - 1.$$

Now, $SS(n_0 - 1) = n_0 - 3$, so that

$$SS(n_0) - SS(n_0 - 1) = 2 - m,$$

which shows that $n = n_0 - 1$ is a solution of the equation (4.2).

Lemma 4.2: The equation

$$SS(n+1) = SS(n) - 1 \tag{4.3}$$

has an infinite number of solutions.

Proof: Let $N = 24m + 18$, $m \geq 0$ being any integer. Then, by Lemma 1.3, $SS(N) = N - 4$, and by Lemma 1.1, $SS(N + 1) = N - 1$, so that $N = 24m + 18$ is a solution of the equation $SS(n + 1) = SS(n) + 3$. Therefore, by Lemma 4.1, $n = 24m + 17$, $m \geq 0$, is a solution of the equation (4.3). Since there is an infinite number of such n, the lemma is proved.

Lemma 4.3: The equation

$$SS(n+1) = SS(n) - 2 \tag{4.4}$$

has an infinite number of solutions.

Proof: Letting $N = 12m$ ($m \geq 0$ being an integer not divisible by 5), by Theorem 3.3, $SS(N) = N - 5$. Since $SS(N + 1) = N - 1$, such an N is a solution of the equation $SS(n + 1) = SS(n) + 4$. Then, by Lemma 4.1, $n = 12m - 1$ ($m \geq 0$ being an integer not divisible by 5) is a solution of the equation (4.4). Clearly, there is an infinite number of such n.

A second solution of the equation (4.4) is given as follows: Let $n = 24m + 12$ with $m \neq 5s + 2$, $s \geq 0$ being an integer. By part (3) of Lemma 3.3, $SS(n) = n - 5$, so that $SS(n + 1) = SS(n) + 5$. Therefore, by Lemma 4.1, $n = 24m + 11$, $m \neq 5s + 2$, $s \geq 0$, is a solution of the equation (4.4). Note that there is an infinite number of solutions of (4.5).

Lemma 4.4: The equation

$$SS(n+1) = SS(n) - 3 \tag{4.5}$$

has an infinite number of solutions.

Proof: Let $N = 60m$, $m = 6s + 5$, $s \geq 0$. By (3.6), $SS(N) = N - 6$. Thus, $SS(N + 1) = SS(N) + 5$. Therefore, by Lemma 4.1, $n = 60m - 1$, $m = 6s + 5$, $s \geq 0$, is a solution of the equation (4.5). Clearly, there is an infinite number of solutions of (4.5).

Lemma 4.5: The equation

$$SS(n+1) = SS(n) - 4 \tag{4.6}$$

has an infinite number of solutions.

Proof: Let $N = 12m + 6$, $m = 2(5s + 1)$ (for any integer $s \geq 0$). By part (6b) of Corollary 3.4, $SS(N) = N - 7$, so that $SS(N + 1) = SS(N) + 6$. Then, by Lemma 4.1, $n = 12m + 5$, $m = 2(5s + 1)$, ($s \geq 0$) is a solution of the equation (4.6).

5. Conclusion

This paper studies a newly introduced Smarandache-type arithmetic funcation called Sandor-Smarandache function, $SS(n)$. Recently, a set of necessary and sufficient conditions has been derived such that $SS(n) = n - 4$. This study finds the necessary and sufficient conditions for $SS(n) = n - 5$ and $SS(n) = n - 6$. Theorem 3.3 involves the function $SS(12m)$. Then, the paper derives the expressions of $SS(60m)$, $SS(420m)$, $SS(4620m)$, $SS(60060m)$ and their values are appended here successively. The analysis so far reveals the following facts about the function $SS(n)$:

(1) $SS(n) = n - 5$, if $n = 12m$ ($m \geq 1$) and 5 does not divide n,
(2) $n - 6 \leq SS(n) \leq n - 7$, if $n = 60m$ ($m \geq 1$) and 7 does not divide n,
(3) $n - 6 \leq SS(n) \leq n - 11$, if $n = 420m$ ($m \geq 1$) and 11 does not divide n,

(4) $n-6 \le SS(n) \le n-13$, if $n = 4620m$ $(m \ge 1)$ and 13 does not divide n.

It thus appears that $SS(n)$ depends, to some extent, on the prime factors of n in their sequential order 2, 3, 5, …. However, if p is the largest prime factor of n, then $SS(n) \ne n-p$. It may open a new research horizon regarding Sandor-Smarandache function.

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Appendices

Table A-1. Values of $SS(60m)$, $1 \le m \le 200$.

n	*SS*(*n*)	*n*	*SS*(*n*)	*n*	*SS*(*n*)	*n*	*SS*(*n*)	*n*	*SS*(*n*)
60	*53*	2460	*2454*	4860	*4853*	7260	*7253*	9660	*9654*
120	*113*	2520	*2509*	4920	*4913*	7320	*7313*	9720	*9713*
180	*173*	2580	*2573*	4980	*4974*	7380	*7373*	9780	*9773*
240	*233*	2640	*2633*	5040	*5029*	7440	*7433*	9840	*9833*
300	*294*	2700	*2693*	5100	*5093*	7500	*7494*	9900	*9893*
360	*353*	2760	*2753*	5160	*5153*	7560	*7549*	9960	*9953*
420	*412*	2820	*2814*	5220	*5213*	7620	*7613*	10020	*10014*
480	*473*	2880	*2873*	5280	*5273*	7680	*7673*	10080	*10069*
540	*533*	2940	*2929*	5340	*5334*	7740	*7733*	10140	*10133*
600	*593*	3000	*2993*	5400	*5393*	7800	*7793*	10200	*10193*
660	*654*	3060	*3053*	5460	*5451*	7860	*7854*	10260	*10253*
720	*713*	3120	*3113*	5520	*5513*	7920	*7913*	10320	*10313*
780	*773*	3180	*3174*	5580	*5573*	7980	*7969*	10380	*10374*
840	*831*	3240	*3233*	5640	*5633*	8040	*8033*	10440	*10433*
900	*893*	3300	*3293*	5700	*5694*	8100	*8093*	10500	*10492*
960	*953*	3360	*3349*	5760	*5753*	8160	*8153*	10560	*10553*
1020	*1014*	3420	*3413*	5820	*5813*	8220	*8214*	10620	*10613*
1080	*1073*	3480	*3473*	5880	*5870*	8280	*8273*	10680	*10673*
1140	*1133*	3540	*3534*	5940	*5933*	8340	*8333*	10740	*10734*
1200	*1193*	3600	*3593*	6000	*5993*	8400	*8391*	10800	*10793*
1260	*1249*	3660	*3653*	6060	*6054*	8460	*8453*	10860	*10853*
1320	*1313*	3720	*3713*	6120	*6113*	8520	*8513*	10920	*10909*
1380	*1374*	3780	*3772*	6180	*6173*	8580	*8574*	10980	*10973*
1440	*1433*	3840	*3833*	6240	*6233*	8640	*8633*	11040	*11033*
1500	*1493*	3900	*3894*	6300	*6289*	8700	*8693*	11100	*11094*
1560	*1553*	3960	*3953*	6360	*6353*	8760	*8753*	11160	*11153*
1620	*1613*	4020	*4013*	6420	*6414*	8820	*8809*	11220	*11213*
1680	*1671*	4080	*4073*	6480	*6473*	8880	*8873*	11280	*11273*
1740	*1734*	4140	*4133*	6540	*6533*	8940	*8934*	11340	*11329*
1800	*1793*	4200	*4189*	6600	*6593*	9000	*8993*	11400	*11393*
1860	*1853*	4260	*4254*	6660	*6653*	9060	*9053*	11460	*11454*
1920	*1913*	4320	*4313*	6720	*6709*	9120	*9113*	11520	*11513*
1980	*1973*	4380	*4373*	6780	*6774*	9180	*9173*	11580	*11573*
2040	*2033*	4440	*4433*	6840	*6833*	9240	*9231*	11640	*11633*
2100	*2094*	4500	*4493*	6900	*6893*	9300	*9294*	11700	*11693*
2160	*2153*	4560	*4553*	6960	*6953*	9360	*9353*	11760	*11749*
2220	*2213*	4620	*4614*	7020	*7013*	9420	*9413*	11820	*11814*
2280	*2273*	4680	*4673*	7080	*7073*	9480	*9473*	11880	*11873*
2340	*2333*	4740	*4733*	7140	*7134*	9540	*9533*	11940	*11933*
2400	*2393*	4800	*4793*	7200	*7193*	9600	*9593*	12000	*11993*

Table A-2. Values of $SS(420m)$, $1 \le m \le 200$.

n	$SS(n)$	n	$SS(n)$	n	$SS(n)$	n	$SS(n)$	n	$SS(n)$
420	412	17220	17214	34020	34012	50820	50812	67620	67614
840	831	17640	17629	34440	34429	51240	51229	68040	68029
1260	1249	18060	18049	34860	34854	51660	51649	68460	68449
1680	1671	18480	18467	35280	35269	52080	52069	68880	68871
2100	2094	18900	18889	35700	35691	52500	52494	69300	69287
2520	2509	19320	19309	36120	36109	52920	52909	69720	69711
2940	2929	19740	19734	36540	36529	53340	53329	70140	70134
3360	3349	20160	20149	36960	36947	53760	53751	70560	70549
3780	3772	20580	20572	37380	37374	54180	54172	70980	70972
4200	4189	21000	20989	37800	37789	54600	54591	71400	71389
4620	4614	21420	21409	38220	38209	55020	55014	71820	71809
5040	5029	21840	21829	38640	38631	55440	55427	72240	72229
5460	5451	22260	22254	39060	39049	55860	55849	72660	72654
5880	5870	22680	22670	39480	39471	56280	56270	73080	73070
6300	6289	23100	23087	39900	39894	56700	56689	73500	73491
6720	6709	23520	23511	40320	40309	57120	57109	73920	73907
7140	7134	23940	23932	40740	40732	57540	57534	74340	74332
7560	7549	24360	24351	41160	41149	57960	57949	74760	74749
7980	7969	24780	24774	41580	41567	58380	58371	75180	75174
8400	8391	25200	25189	42000	41989	58800	58789	75600	75589
8820	8809	25620	25609	42420	42414	59220	59209	76020	76009
9240	9231	26040	26029	42840	42829	59640	59629	76440	76431
9660	9654	26460	26449	43260	43251	60060	60054	76860	76849
10080	10069	26880	26869	43680	43669	60480	60469	77280	77271
10500	10492	27300	27294	44100	44092	60900	60892	77700	77694
10920	10909	27720	27707	44520	44509	61320	61311	78120	78109
11340	11329	28140	28131	44940	44934	61740	61729	78540	78527
11760	11749	28560	28549	45360	45349	62160	62151	78960	78949
12180	12174	28980	28970	45780	45770	62580	62574	79380	79370
12600	12589	29400	29389	46200	46191	63000	62989	79800	79789
13020	13011	29820	29814	46620	46609	63420	63409	80220	80214
13440	13429	30240	30229	47040	47031	63840	63829	80640	80629
13860	13852	30660	30652	47460	47454	64260	64252	81060	81052
14280	14270	31080	31071	47880	47870	64680	64670	81480	81470
14700	14694	31500	31489	48300	48289	65100	65094	81900	81889
15120	15109	31920	31911	48720	48709	65520	65509	82320	82309
15540	15529	32340	32334	49140	49129	65940	65931	82740	82734
15960	15951	32760	32749	49560	49549	66360	66349	83160	83147
16380	16369	33180	33169	49980	49974	66780	66769	83580	83569
16800	16791	33600	33589	50400	50389	67200	67189	84000	83991

Table A-3. Values of $SS(4620m)$, $1 \leq m \leq 160$.

n	$SS(n)$	n	$SS(n)$	n	$SS(n)$	n	$SS(n)$
4620	*4614*	189420	*189407*	374220	*374207*	559020	*559014*
9240	*9231*	194040	*194027*	378840	*378831*	563640	*563627*
13860	*13852*	198660	*198654*	383460	*383452*	568260	*568252*
18480	*18467*	203280	*203267*	388080	*388067*	572880	*572867*
23100	*23087*	207900	*207887*	392700	*392694*	577500	*577487*
27720	*27707*	212520	*212511*	397320	*397307*	582120	*582107*
32340	*32334*	217140	*217131*	401940	*401927*	586740	*586734*
36960	*36947*	221760	*221747*	406560	*406547*	591360	*591351*
41580	*41567*	226380	*226374*	411180	*411167*	595980	*595967*
46200	*46191*	231000	*230987*	415800	*415787*	600600	*600586*
50820	*50812*	235620	*235612*	420420	*420414*	605220	*605212*
55440	*55427*	240240	*240223*	425040	*425031*	609840	*609827*
60060	*60054*	244860	*244847*	429660	*429647*	614460	*614454*
64680	*64670*	249480	*249470*	434280	*434270*	619080	*619070*
69300	*69287*	254100	*254094*	438900	*438887*	623700	*623687*
73920	*73907*	258720	*258711*	443520	*443507*	628320	*628311*
78540	*78527*	263340	*263327*	448140	*448134*	632940	*632931*
83160	*83147*	267960	*267947*	452760	*452747*	637560	*637547*
87780	*87774*	272580	*272572*	457380	*457372*	642180	*642174*
92400	*92391*	277200	*277187*	462000	*461991*	646800	*646787*
97020	*97007*	281820	*281814*	466620	*466611*	651420	*651407*
101640	*101627*	286440	*286427*	471240	*471228*	656040	*656027*
106260	*106247*	291060	*291047*	475860	*475854*	660660	*660643*
110880	*110867*	295680	*295671*	480480	*480463*	665280	*665267*
115500	*115494*	300300	*300291*	485100	*485087*	669900	*669894*
120120	*120103*	304920	*304907*	489720	*489707*	674520	*674511*
124740	*124732*	309540	*309534*	494340	*494332*	679140	*679132*
129360	*129351*	314160	*314147*	498960	*498947*	683760	*683747*
133980	*133971*	318780	*318767*	503580	*503574*	688380	*688367*
138600	*138588*	323400	*323387*	508200	*508191*	693000	*692987*
143220	*143214*	328020	*328007*	512820	*512807*	697620	*697614*
147840	*147827*	332640	*332627*	517440	*517427*	702240	*702227*
152460	*152447*	337260	*337254*	522060	*522047*	706860	*706847*
157080	*157070*	341880	*341871*	526680	*526670*	711480	*711471*
161700	*161692*	346500	*346492*	531300	*531294*	716100	*716092*
166320	*166307*	351120	*351107*	535920	*535907*	720720	*720703*
170940	*170934*	355740	*355727*	540540	*540523*	725340	*725334*
175560	*175551*	360360	*360343*	545160	*545151*	729960	*729947*
180180	*180170*	364980	*364974*	549780	*549771*	734580	*734570*
184800	*184787*	369600	*369587*	554400	*554387*	739200	*739187*

Table A-4. Values of $SS(60060m)$, $1 \le m \le 120$.

n	*SS*(*n*)	*n*	*SS*(*n*)	*n*	*SS*(*n*)
60060	*60054*	2462460	*2462451*	4864860	*4864843*
120120	*120103*	2522520	*2522503*	4924920	*4924903*
180180	*180170*	2582580	*2582574*	4984980	*4984970*
240240	*240223*	2642640	*2642623*	5045040	*5045023*
300300	*300291*	2702700	*2702686*	5105100	*5105094*
360360	*360343*	2762760	*2762745*	5165160	*5165151*
420420	*420414*	2822820	*2822812*	5225220	*5225212*
480480	*480463*	2882880	*2882865*	5285280	*5285271*
540540	*540523*	2942940	*2942934*	5345340	*5345323*
600600	*600586*	3003000	*3002991*	5405400	*5405383*
660660	*660643*	3063060	*3063041*	5465460	*5465454*
720720	*720703*	3123120	*3123111*	5525520	*5525503*
780780	*780774*	3183180	*3183164*	5585580	*5585565*
840840	*840831*	3243240	*3243223*	5645640	*5645626*
900900	*900892*	3303300	*3303294*	5705700	*5705692*
960960	*960951*	3363360	*3363343*	5765760	*5765743*
1021020	*1021006*	3423420	*3423403*	5825820	*5825814*
1081080	*1081070*	3483480	*3483470*	5885880	*5885870*
1141140	*1141134*	3543540	*3543531*	5945940	*5945923*
1201200	*1201183*	3603600	*3603583*	6006000	*6005983*
1261260	*1261245*	3663660	*3663654*	6066060	*6066046*
1321320	*1321303*	3723720	*3723703*	6126120	*6126108*
1381380	*1381372*	3783780	*3783772*	6186180	*6186174*
1441440	*1441423*	3843840	*3843823*	6246240	*6246231*
1501500	*1501494*	3903900	*3903883*	6306300	*6306283*
1561560	*1561543*	3963960	*3963946*	6366360	*6366351*
1621620	*1621603*	4024020	*4024014*	6426420	*6426403*
1681680	*1681665*	4084080	*4084071*	6486480	*6486463*
1741740	*1741723*	4144140	*4144124*	6546540	*6546534*
1801800	*1801788*	4204200	*4204191*	6606600	*6606584*
1861860	*1861854*	4264260	*4264252*	6666660	*6666652*
1921920	*1921911*	4324320	*4324303*	6726720	*6726703*
1981980	*1981963*	4384380	*4384374*	6786780	*6786771*
2042040	*2042031*	4444440	*4444423*	6846840	*6846823*
2102100	*2102083*	4504500	*4504483*	6906900	*6906894*
2162160	*2162143*	4564560	*4564545*	6966960	*6966943*
2222220	*2222214*	4624620	*4624611*	7027020	*7027004*
2282280	*2282270*	4684680	*4684670*	7087080	*7087070*
2342340	*2342332*	4744740	*4744734*	7147140	*7147132*
2402400	*2402383*	4804800	*4804783*	7207200	*7207183*