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A New Integral Transform "Rishi Transform" with Application

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Abstract

In this paper, authors propose a new integral transform "Rishi Transform" with application to determine the exact (analytic) solution of first kind Volterra integral equation (V.I.E.). For this purpose, authors first derived the Rishi transform of basic mathematical functions (algebraic and transcendential) and then the fundamental properties of Rishi transform is discussed, which can be used for solving ordinary differential equations (O.D.E), partial differential equations (P.D.E.), delay differential equations (D.D.E.), fractional differential equations (F.D.E.), difference equations (D.E.), integral equations (I.E.) and integro-differential equations (I.D.E.). After this, authors determined the exact (analytic) solution of general first kind V.I.E.. They have considered three numerical problems and solved them completely step by step for explaining the utility of Rishi transform. Results depict that the proposed new integral transform "Rishi Transform" provides the exact results for first kind V.I.E. without doing complicated calculation work.

Keywords: Rishi transform; Inverse Rishi transform; Convolution; Volterra integral equation.

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1. Introduction

In the current scenario, researchers have first choice as integral transforms among other mathematical methods for determining the solutions of the problems of science, social science and engineering due to their three important characteristics; first one is simplicity, second one is providing the exact results and last one is providing the results without doing complicated calculation work. Researchers developed various new integral transforms (Mahgoub transform [1]; Kamal [2]; Elzaki [3]; Aboodh [4]; Mohand [5]; Sumudu [6]; Shehu [7]; Sadik [8]; Sawi [9]; Upadhyay [10]; ZZ [11]; Natural [12]; Jafari [13]) in recent years. Aggarwal and other researchers [14-23] used various integral transforms (Laplace [14]; Kamal [15]; Mahgoub [16]; Mohand [17]; Aboodh [18]; Elzaki [19]; Shehu [20]; Sadik [21]; Sawi [22]; Sumudu [23]) and studied the famous problems of growth and decay. Higazy *et al.* [24] developed a new method by combination of two mathematical methods (Sawi transform and Decomposition method) and determined the

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solution of Volterra integral equation. They named Sawi decomposition method to this newly developed method. Aggarwal *et al*. [25] provided the duality relations of Kamal transform with other established integral transforms. Higazy *et al*. [26] studied HIV-1 infections model with the help of Shehu transform. Higazy and Aggarwal [27] applied Sawi transform for determining the exact solutions of the problem of chemical kinetics by developing its model using ordinary differential equations. El-Mesady *et al.* [28] used Jafari transform and solved a problem of medical field.

The motive of the present paper is to develop a new integral transform “Rishi Transform” with its fundamental properties and determine the solution of the first kind V.I.E. with convolution type kernel using this transform. The proposed transform “Rishi transform” is better than the other already established transforms because it provides the exact results of the problems without doing tedious computational work and spending little time. Rishi transform has a duality relation with famous and mostly used integral transform “Laplace transform”. This duality relation is given in the section 2 with the definition of this transform. This relation makes the Rishi transform valuable because all the properties of Laplace transform are achieved using this relation. The first kind V.I.E. has various applications in creep theory; electronic lithography; aero-elasticity; population dynamics; visco-elasticity; coagulation and meteorology; radio physics; inductive electric circuit; dynamics of the pendulum; biotechnology; radiation transfer; super fluidity; electromagnetism; mining engineering and acoustic engineering [29]. Devi *et al.* [30] used Elzaki transform and solved ordinary differential equations with variable coefficients. Devi and Jakhar [31] suggested a computational algorithm for solving fractional biological population model. Devi and Jakhar [32] applied Sumudu-Adomian decomposition method on fractional order Telegraph equations and determined their solutions. Kumar *et al*. [33] developed a new integral transform “Anuj transform” and solved linear Volterra integral equations of first kind. Aggarwal *et al.* [34,35] solved Volterra integral equations of first kind by applying Kamal and Aboodh transforms on them.

2. Definition of Rishi Transform

The Rishi transform of exponential order piecewise continuous function, $\omega(t)$ defined in the interval $[0,\infty)$ is given by

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^{\infty}\omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt = T(\varepsilon,\sigma),\ \varepsilon > 0, \sigma > 0 \quad (1)$$

Remark: Rishi transform has a duality relation with famous and mostly used integral transform “Laplace transform”. If $L\{\omega(t)\} = \int_0^{\infty}\omega(t)e^{-\varepsilon t}dt = \Omega(\varepsilon)$, where L is the Laplace transform operator, then $T(\varepsilon,\sigma) = \left(\frac{\sigma}{\varepsilon}\right)\Omega\left(\frac{\varepsilon}{\sigma}\right)$.

3. Rishi Transform of Frequently used Functions

Case 1: Consider the function $\omega(t)$, defined by $\omega(t) = 1, t > 0$ then by the definition of Rishi transform,

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty \omega(t) e^{-\left(\frac{\varepsilon}{\sigma}\right)t} dt = T(\varepsilon, \sigma)$$

$$\Rightarrow R\{\omega(t)\} = R\{1\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t} .1dt = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t} dt$$

$$\Rightarrow R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right) \left[\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)}\right]_0^\infty = \left(\frac{\sigma}{\varepsilon}\right)^2$$

$$\Rightarrow R\{1\} = \left(\frac{\sigma}{\varepsilon}\right)^2$$

Case 2: Consider the function $\omega(t)$, defined by $\omega(t) = e^{lt}, t > 0$ then by the definition of Rishi transform,

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty \omega(t) e^{-\left(\frac{\varepsilon}{\sigma}\right)t} dt = T(\varepsilon, \sigma)$$

$$\Rightarrow R\{\omega(t)\} = R\{e^{lt}\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t} e^{lt} dt = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}-l\right)t} dt = \left(\frac{\sigma}{\varepsilon}\right) \left[\frac{e^{-\left(\frac{\varepsilon}{\sigma}-l\right)t}}{-\left(\frac{\varepsilon}{\sigma}-l\right)}\right]_0^\infty$$

$$\Rightarrow R\{\omega(t)\} = \frac{\sigma^2}{\varepsilon(\varepsilon - l\sigma)}$$

$$\Rightarrow R\{e^{lt}\} = \frac{\sigma^2}{\varepsilon(\varepsilon - l\sigma)}$$

Case 3: Consider the function $\omega(t)$, defined by $\omega(t) = t, t > 0$ then by the definition of Rishi transform,

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty \omega(t) e^{-\left(\frac{\varepsilon}{\sigma}\right)t} dt = T(\varepsilon, \sigma)$$

$$\Rightarrow R\{\omega(t)\} = R\{t\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t} t dt = \left(\frac{\sigma}{\varepsilon}\right) \left[t.\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)}\right]_0^\infty - \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty 1.\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)} dt$$

$$\Rightarrow R\{\omega(t)\} = 0 + \left(\frac{\sigma}{\varepsilon}\right)^2 \int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t} dt = \left(\frac{\sigma}{\varepsilon}\right)^2 \int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t} dt = \left(\frac{\sigma}{\varepsilon}\right)^2 \left[\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)}\right]_0^\infty = \left(\frac{\sigma}{\varepsilon}\right)^3$$

$$\Rightarrow R\{t\} = \left(\frac{\sigma}{\varepsilon}\right)^3.$$

Case 4: Consider the function $\omega(t)$, defined by $\omega(t) = t^2, t > 0$ then by the definition of Rishi transform,

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty \omega(t) e^{-\left(\frac{\varepsilon}{\sigma}\right)t} dt = T(\varepsilon, \sigma)$$

$$\Rightarrow R\{\omega(t)\} = R\{t^2\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t}.t^2 dt = \left(\frac{\sigma}{\varepsilon}\right)\left[t^2.\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)}\right]_0^\infty - \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty 2t.\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)}dt$$

$$\Rightarrow R\{\omega(t)\} = 0 + 2\left(\frac{\sigma}{\varepsilon}\right)^2\int_0^\infty t.e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt = 2\left(\frac{\sigma}{\varepsilon}\right)^2\left[\left\{t.\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)}\right\}_0^\infty - \int_0^\infty 1.\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)}dt\right]$$

$$\Rightarrow R\{\omega(t)\} = 2\left(\frac{\sigma}{\varepsilon}\right)^2\left[0 + \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt\right] = 2\left(\frac{\sigma}{\varepsilon}\right)^3\int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt = 2\left(\frac{\sigma}{\varepsilon}\right)^3\left[\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)}\right]_0^\infty$$

$$= 2\left(\frac{\sigma}{\varepsilon}\right)^4$$

$$\Rightarrow R\{t^2\} = 2\left(\frac{\sigma}{\varepsilon}\right)^4.$$

Case 5: Consider the function $\omega(t)$, defined by $\omega(t) = t^\rho, t > 0, \rho \in N,$ then by using the principle of mathematical induction, we have

$$R\{t^\rho\} = \rho!\left(\frac{\sigma}{\varepsilon}\right)^{\rho+2}.$$

Case 6: Consider the function $\omega(t)$, defined by $\omega(t) = t^\rho, t > 0, \rho > -1,$ then by the definition of Rishi transform, $R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt = T(\varepsilon, \sigma)$

$$\Rightarrow R\{\omega(t)\} = R\{t^\rho\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t}.t^\rho dt$$

$$= \left(\frac{\sigma}{\varepsilon}\right)\left[t^\rho.\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)}\right]_0^\infty - \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \rho t^{\rho-1}.\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left(-\frac{\varepsilon}{\sigma}\right)}dt$$

$$\Rightarrow R\{\omega(t)\} = 0 + \rho\left(\frac{\sigma}{\varepsilon}\right)^2\int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t}t^{\rho-1}dt$$

$$\Rightarrow R\{\omega(t)\} = \rho\left(\frac{\sigma}{\varepsilon}\right)^2\left[\frac{\Gamma(\rho)}{\left(\frac{\varepsilon}{\sigma}\right)^\rho}\right] = \left(\frac{\sigma}{\varepsilon}\right)^{\rho+2}[\rho\Gamma(\rho)] = \left(\frac{\sigma}{\varepsilon}\right)^{\rho+2}\Gamma(\rho+1)$$

$$\Rightarrow R\{t^\rho\} = \left(\frac{\sigma}{\varepsilon}\right)^{\rho+2}\Gamma(\rho+1)$$

Case 7: Consider the function $\omega(t)$, defined by $\omega(t) = sinlt, t > 0$ then by the definition of Rishi transform, $R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt = T(\varepsilon, \sigma)$

$$\Rightarrow R\{\omega(t)\} = A\{sinlt\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t}.sinltdt$$

$$\Rightarrow R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\left[\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left\{\left(-\frac{\varepsilon}{\sigma}\right)^2 + l^2\right\}}\left\{-\left(\frac{\varepsilon}{\sigma}\right)sinlt - lcoslt\right\}\right]_0^\infty$$

$$\Rightarrow R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\left[0 - \frac{1}{\left\{\left(\frac{\varepsilon}{\sigma}\right)^2 + l^2\right\}}(0 - l)\right] = \frac{l\sigma^3}{\varepsilon(\varepsilon^2 + \sigma^2 l^2)}$$

$$\Rightarrow R\{sinlt\} = \frac{l\sigma^3}{\varepsilon(\varepsilon^2 + \sigma^2 l^2)}$$

Case 8: Consider the function $\omega(t)$, defined by $\omega(t) = coslt, t > 0$ then by the definition of Rishi transform, $R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty \omega(t) e^{-\left(\frac{\varepsilon}{\sigma}\right)t} dt = T(\varepsilon, \sigma)$

$$\Rightarrow R\{\omega(t)\} = A\{coslt\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t} . coslt dt$$

$$\Rightarrow R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right) \left[\frac{e^{-\left(\frac{\varepsilon}{\sigma}\right)t}}{\left\{\left(-\frac{\varepsilon}{\sigma}\right)^2 + l^2\right\}} \left\{-\left(\frac{\varepsilon}{\sigma}\right) coslt + lsinlt\right\} \right]_0^\infty$$

$$\Rightarrow R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right) \left[0 - \frac{1}{\left\{\left(\frac{\varepsilon}{\sigma}\right)^2 + l^2\right\}} \left\{-\left(\frac{\varepsilon}{\sigma}\right) + 0\right\} \right] = \frac{\sigma^2}{(\varepsilon^2 + \sigma^2 l^2)}$$

$$\Rightarrow R\{coslt\} = \frac{\sigma^2}{(\varepsilon^2 + \sigma^2 l^2)}$$

Case 9: Consider the function $\omega(t)$, defined by $\omega(t) = sinhlt, t > 0$ then by the definition of Rishi transform,

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty \omega(t) e^{-\left(\frac{\varepsilon}{\sigma}\right)t} dt = T(\varepsilon, \sigma)$$

$$\Rightarrow R\{\omega(t)\} = A\{sinhlt\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t} . \left[\frac{e^{lt} - e^{-lt}}{2}\right] dt$$

$$\Rightarrow R\{\omega(t)\} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right) \left[\int_0^\infty e^{-(\frac{\varepsilon}{\sigma} - l)t} dt - \int_0^\infty e^{-(\frac{\varepsilon}{\sigma} + l)t} dt \right]$$

$$\Rightarrow R\{\omega(t)\} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right) \left[\left[\frac{e^{-(\frac{\varepsilon}{\sigma} - l)t}}{-(\frac{\varepsilon}{\sigma} - l)}\right]_0^\infty - \left[\frac{e^{-(\frac{\varepsilon}{\sigma} + l)t}}{-(\frac{\varepsilon}{\sigma} + l)}\right]_0^\infty \right]$$

$$\Rightarrow R\{\omega(t)\} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right) \left[\left\{0 + \frac{1}{\left(\frac{\varepsilon}{\sigma} - l\right)}\right\} - \left\{0 + \frac{1}{\left(\frac{\varepsilon}{\sigma} + l\right)}\right\} \right]$$

$$\Rightarrow R\{\omega(t)\} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right) \left[\frac{1}{\left(\frac{\varepsilon}{\sigma} - l\right)} - \frac{1}{\left(\frac{\varepsilon}{\sigma} + l\right)} \right] = \frac{l\sigma^3}{\varepsilon(\varepsilon^2 - \sigma^2 l^2)}$$

$$\Rightarrow R\{sinhlt\} = \frac{l\sigma^3}{\varepsilon(\varepsilon^2 - \sigma^2 l^2)}$$

Case 10: Consider the function $\omega(t)$, defined by $\omega(t) = coshlt, t > 0$ then by the definition of Rishi transform, $R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty \omega(t) e^{-\left(\frac{\varepsilon}{\sigma}\right)t} dt = T(\varepsilon, \sigma)$

$$\Rightarrow R\{\omega(t)\} = R\{coshlt\} = \left(\frac{\sigma}{\varepsilon}\right) \int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t} . \left[\frac{e^{lt} + e^{-lt}}{2}\right] dt$$

$$\Rightarrow R\{\omega(t)\} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right) \left[\int_0^\infty e^{-(\frac{\varepsilon}{\sigma} - l)t} dt + \int_0^\infty e^{-(\frac{\varepsilon}{\sigma} + l)t} dt \right]$$

$$\Rightarrow R\{\omega(t)\} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)\left[\left[\frac{e^{-(\frac{\varepsilon}{\sigma}-l)t}}{-(\frac{\varepsilon}{\sigma}-l)}\right]_0^\infty + \left[\frac{e^{-(\frac{\varepsilon}{\sigma}+l)t}}{-(\frac{\varepsilon}{\sigma}+l)}\right]_0^\infty\right]$$

$$\Rightarrow R\{\omega(t)\} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)\left[\left\{0 + \frac{1}{\left(\frac{\varepsilon}{\sigma}-l\right)}\right\} + \left\{0 + \frac{1}{\left(\frac{\varepsilon}{\sigma}+l\right)}\right\}\right]$$

$$\Rightarrow R\{\omega(t)\} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)\left[\frac{1}{\left(\frac{\varepsilon}{\sigma}-l\right)} + \frac{1}{\left(\frac{\varepsilon}{\sigma}+l\right)}\right] = \frac{\sigma^2}{(\varepsilon^2-\sigma^2 l^2)}$$

$$\Rightarrow R\{coshlt\} = \frac{\sigma^2}{(\varepsilon^2-\sigma^2 l^2)}$$

4. Some Salient Properties of Rishi Transform

This section contains some salient properties of Rishi transform.

4.1. *Linearity property of Rishi transform*

If $R\{\omega_i(t)\} = T_i(\varepsilon,\sigma)$ then $R\{\sum_{i=1}^n \alpha_i\,\omega_i(t)\} = \alpha_i \sum_{i=1}^n R\{\omega_i(t)\} = \alpha_i \sum_{i=1}^n T_i(\varepsilon,\sigma)$, where α_i are arbitrary constants.

Proof: Using (1), we obtain

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$

$$\Rightarrow R\left\{\sum_{i=1}^n \alpha_i\,\omega_i(t)\right\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \left[\sum_{i=1}^n \alpha_i\,\omega_i(t)\right]e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$

$$\Rightarrow R\left\{\sum_{i=1}^n \alpha_i\,\omega_i(t)\right\} = \alpha_i\sum_{i=1}^n \left[\left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega_i(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt\right]$$

$$\Rightarrow R\left\{\sum_{i=1}^n \alpha_i\,\omega_i(t)\right\} = \alpha_i\sum_{i=1}^n R\{\omega_i(t)\}$$

$\Rightarrow R\{\sum_{i=1}^n \alpha_i\,\omega_i(t)\} = \alpha_i \sum_{i=1}^n T_i(\varepsilon,\sigma)$, where α_i are arbitrary constants.

4.2. *Scaling property of Rishi transform*

If $R\{\omega(t)\} = T(\varepsilon,\sigma)$ then $R\{\omega(kt)\} = \frac{1}{k^2}T\left(\frac{\varepsilon}{k},\sigma\right)$.

Proof: Using (1), we obtain

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$

$$\Rightarrow R\{\omega(kt)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega(kt)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$

Put $kt = p \Rightarrow kdt = dp$ in above equation, we have

$$R\{\omega(kt)\} = \frac{1}{k}\left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega(p)e^{-\left(\frac{\varepsilon}{\sigma}\right)\left(\frac{p}{k}\right)}dp$$

$$\Rightarrow R\{\omega(kt)\} = \frac{1}{k^2}\left(\frac{\sigma}{\varepsilon/k}\right)\int_0^\infty e^{-\left(\frac{\varepsilon}{k}\right)\left(\frac{p}{\sigma}\right)}\omega(p)dp$$
$$\Rightarrow R\{\omega(kt)\} = \frac{1}{k^2}T\left(\frac{\varepsilon}{k},\sigma\right)$$

4.3. ***Translation property of Rishi transform***

If $R\{\omega(t)\} = T(\varepsilon,\sigma)$ then $R\{e^{kt}\omega(t)\} = \left(\frac{\varepsilon-k\sigma}{\varepsilon}\right) T(\varepsilon-k\sigma,\sigma)$.
Proof: Using (1), we obtain

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$
$$\Rightarrow R\{e^{kt}\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty e^{kt}\omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$
$$\Rightarrow R\{e^{kt}\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega(t)e^{-\left(\frac{\varepsilon}{\sigma}-k\right)t}dt$$
$$\Rightarrow R\{e^{kt}\omega(t)\} = \left(\frac{\varepsilon-k\sigma}{\varepsilon}\right)\left(\frac{\sigma}{\varepsilon-k\sigma}\right)\int_0^\infty \omega(t)e^{-\left(\frac{\varepsilon-k\sigma}{\sigma}\right)t}dt = \left(\frac{\varepsilon-k\sigma}{\varepsilon}\right)T(\varepsilon-k\sigma,\sigma)$$

4.4. ***Faltung (Convolution) property of Rishi transform***

If $R\{\omega_1(t)\} = T_1(\varepsilon,\sigma)$ and $R\{\omega_2(t)\} = T_2(\varepsilon,\sigma)$ then $R\{\omega_1(t)*\omega_2(t)\} = \left(\frac{\varepsilon}{\sigma}\right)R\{\omega_1(t)\}R\{\omega_2(t)\} = \left(\frac{\varepsilon}{\sigma}\right) T_1(\varepsilon,\sigma)\, T_2(\varepsilon,\sigma)$, where faltung of $\omega_1(t)$ and $\omega_2(t)$ is denoted by $\omega_1(t)*\omega_2(t)$ and it is defined by $\omega_1(t)*\omega_2(t) = \int_0^t \omega_1(t-u)\,\omega_2(u)du = \int_0^t \omega_1(u)\,\omega_2(t-u)du.$
Proof: Using (1), we obtain

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$
$$\Rightarrow R\{\omega_1(t)*\omega_2(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t}\,[\omega_1(t)*\omega_2(t)]dt$$
$$\Rightarrow R\{\omega_1(t)*\omega_2(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t}\left[\int_0^t \omega_1(t-u)\,\omega_2(u)du\right]dt$$

After reversing the order of integration, we get

$$R\{\omega_1(t)*\omega_2(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega_2(u)\left[\int_u^\infty \omega_1(t-u)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}\,dt\right]du$$

Substituting $t-u=v$ so that $dt=dv$ in the above equation, we obtain

$$R\{\omega_1(t)*\omega_2(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega_2(u)\left[\int_0^\infty \omega_1(v)e^{-\left(\frac{\varepsilon}{\sigma}\right)(v+u)}\,dv\right]du$$
$$\Rightarrow R\{\omega_1(t)*\omega_2(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega_2(u)e^{-\left(\frac{\varepsilon}{\sigma}\right)u}\left[\int_0^\infty \omega_1(v)e^{-\left(\frac{\varepsilon}{\sigma}\right)v}\,dv\right]du$$
$$\Rightarrow R\{\omega_1(t)*\omega_2(t)\} = \left(\frac{\varepsilon}{\sigma}\right)\left[\left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega_2(u)e^{-\left(\frac{\varepsilon}{\sigma}\right)u}\right]\left[\left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega_1(v)e^{-\left(\frac{\varepsilon}{\sigma}\right)v}\,dv\right]du$$
$$\Rightarrow R\{\omega_1(t)*\omega_2(t)\} = \left(\frac{\varepsilon}{\sigma}\right) T_1(\varepsilon,\sigma)\, T_2(\varepsilon,\sigma).$$

5. Rishi Transform of Derivatives

If $R\{\omega(t)\} = T(\varepsilon,\sigma)$ then

a) $R\{\omega'(t)\} = \left(\frac{\varepsilon}{\sigma}\right)T(\varepsilon,\sigma) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega(0)$

b) $R\{\omega''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)^{2}T(\varepsilon,\sigma) - \omega(0) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega'(0)$

c) $R\{\omega'''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)^{3}T(\varepsilon,\sigma) - \left(\frac{\varepsilon}{\sigma}\right)\omega(0) - \omega'(0) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega''(0)$

Proof (a): Using (1), we get

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^{\infty}\omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$

$$\Rightarrow R\{\omega'(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^{\infty}\omega'(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$

$$\Rightarrow R\{\omega'(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\left[\omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}\right]_0^{\infty} - \left(\frac{\sigma}{\varepsilon}\right)\int_0^{\infty}\left\{-\left(\frac{\varepsilon}{\sigma}\right)\right\}\omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$

$$\Rightarrow R\{\omega'(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\lim_{t\to\infty}\left[\omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}\right] - \left(\frac{\sigma}{\varepsilon}\right)\omega(0) + \int_0^{\infty}\omega(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt$$

$$= 0 - \left(\frac{\sigma}{\varepsilon}\right)\omega(0) + \left(\frac{\varepsilon}{\sigma}\right)A\{\omega(t)\}$$

$$\Rightarrow R\{\omega'(t)\} = \left(\frac{\varepsilon}{\sigma}\right)T(\varepsilon,\sigma) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega(0)$$

Proof (b): We have, $R\{\omega'(t)\} = \left(\frac{\varepsilon}{\sigma}\right)T(\varepsilon,\sigma) - \left(\frac{\sigma}{\varepsilon}\right)\omega(0) = \left(\frac{\varepsilon}{\sigma}\right)R\{\omega(t)\} - \left(\frac{\sigma}{\varepsilon}\right)\omega(0)$

$$\Rightarrow R\{\omega''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)R\{\omega'(t)\} - \left(\frac{\sigma}{\varepsilon}\right)\omega'(0)$$

$$= \left(\frac{\varepsilon}{\sigma}\right)\left[\left(\frac{\varepsilon}{\sigma}\right)R\{\omega(t)\} - \left(\frac{\sigma}{\varepsilon}\right)\omega(0)\right] - \left(\frac{\sigma}{\varepsilon}\right)\omega'(0)$$

$$\Rightarrow R\{\omega''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)^{2}R\{\omega(t)\} - \omega(0) - \left(\frac{\sigma}{\varepsilon}\right)\omega'(0)$$

$$\Rightarrow R\{\omega''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)^{2}T(\varepsilon,\sigma) - \omega(0) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega'(0)$$

Proof (c): We have, $R\{\omega''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)^{2}T(\varepsilon,\sigma) - \omega(0) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega'(0)$

$$\Rightarrow R\{\omega'''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)^{2}R\{\omega(t)\} - \omega(0) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega'(0)$$

$$\Rightarrow R\{\omega'''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)^{2}R\{\omega'(t)\} - \omega'(0) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega''(0)$$

$$\Rightarrow R\{\omega'''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)^{2}\left[\left(\frac{\varepsilon}{\sigma}\right)R\{\omega(t)\} - \left(\frac{\sigma}{\varepsilon}\right)\omega(0)\right] - \omega'(0) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega''(0)$$

$$\Rightarrow R\{\omega'''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)^{3}R\{\omega(t)\} - \left(\frac{\varepsilon}{\sigma}\right)\omega(0) - \omega'(0) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega''(0)$$

$$\Rightarrow R\{\omega'''(t)\} = \left(\frac{\varepsilon}{\sigma}\right)^{3}T(\varepsilon,\sigma) - \left(\frac{\varepsilon}{\sigma}\right)\omega(0) - \omega'(0) - \left(\frac{\varepsilon}{\sigma}\right)^{-1}\omega''(0)$$

6. The Inverse Rishi Transform

If $R\{\omega(t)\} = T(\varepsilon,\sigma)$ then $\omega(t)$ is called the inverse Rishi transform of $T(\varepsilon,\sigma)$ and is written as
$\omega(t) = R^{-1}\{T(\varepsilon,\sigma)\}$, where R^{-1} is called the inverse Rishi transform operator.

7. Linearity Property of Inverse Rishi Transform

If $R^{-1}\{T_i(\varepsilon,\sigma)\} = \omega_i(t)$ then $R^{-1}\{\sum_{i=1}^{n}\alpha_i T_i(\varepsilon,\sigma)\} = \sum_{i=1}^{n}\alpha_i\, R^{-1}\{T_i(\varepsilon,\sigma)\}$, where α_i are arbitrary constants.

8. Solution of First Kind V.I.E. by Rishi Transform

The first kind V.I.E. with convolution type kernel is given by [29]

$$\theta(t) = \int_0^t K(t-u)\,\omega(u)du \tag{2}$$

where

$$\left.\begin{aligned}\omega(t) &= unknown\ function\\ \theta(t) &= known\ function\\ K(t-u) &= convolution\ type\ kernel\end{aligned}\right\}$$

Operating Rishi transform on equation (2), we get

$$R\{\theta(t)\} = R\left\{\int_0^t K(t-u)\,\omega(u)du\right\}$$

$$\Rightarrow R\{\theta(t)\} = R\{K(t) * \omega(t)\} \tag{3}$$

The use of Faltung (Convolution) theorem of Rishi transform in equation (3) gives

$$R\{\theta(t)\} = \left(\frac{\varepsilon}{\sigma}\right) R\{K(t)\}R\{\omega(t)\}$$

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\left(\frac{R\{\theta(t)\}}{R\{K(t)\}}\right) \tag{4}$$

After operating inverse Rishi transform on equation (4), the required solution of equation (2) obtain and it is given by

$$\omega(t) = R^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)\left(\frac{R\{\theta(t)\}}{R\{K(t)\}}\right)\right\}$$

9. Numerical Problems

This section contains three numerical problems for explaining the utility of Rishi transform for determining the exact (analytic) solution of first kind V.I.E. with convolution type kernel.

Problem: 1 Consider the following first kind V.I.E. with convolution type kernel given by [34] as

$$t = \int_0^t e^{(t-u)}\,\omega(u)du \tag{5}$$

Operating Rishi transform on equation (5), we get

$$R\{t\} = R\left\{\int_0^t e^{(t-u)}\,\omega(u)du\right\}$$

$$\Rightarrow \left(\frac{\sigma}{\varepsilon}\right)^3 = R\{e^t * \omega(t)\} \tag{6}$$

Using convolution theorem of Rishi transform in equation (6), we have

$$\left(\frac{\sigma}{\varepsilon}\right)^3 = \left(\frac{\varepsilon}{\sigma}\right) R\{e^t\} R\{\omega(t)\}$$

$$\Rightarrow \left(\frac{\sigma}{\varepsilon}\right)^3 = \left(\frac{\varepsilon}{\sigma}\right) \left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right] R\{\omega(t)\}$$

$$\Rightarrow R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)^4 \left[\frac{\varepsilon(\varepsilon-\sigma)}{\sigma^2}\right] = \left(\frac{\sigma}{\varepsilon}\right)^2 - \left(\frac{\sigma}{\varepsilon}\right)^3 \tag{7}$$

After operating inverse Rishi transform on equation (7), the required solution of equation (5) obtain and it is given by

$$\omega(t) = R^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^2 - \left(\frac{\sigma}{\varepsilon}\right)^3\right\} = R^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^2\right\} - R^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^3\right\} = 1 - t$$

Remark: The same exact solution obtained using Rishi transform as given in [34] but without any tedious computational work.

Problem: 2 Consider the following first kind V.I.E. with convolution type kernel given by [33] as

$$t + t^2 = \int_0^t cos(t-u)\, \omega(u) du \tag{8}$$

Operating Rishi transform on equation (8), we get

$$R\{t+t^2\} = R\left\{\int_0^t cos(t-u)\, \omega(u) du\right\}$$

$$\Rightarrow R\{t\} + R\{t^2\} = R\{cost * \omega(t)\} \tag{9}$$

Using convolution theorem of Rishi transform in equation (9), we have

$$\left(\frac{\sigma}{\varepsilon}\right)^3 + 2\left(\frac{\sigma}{\varepsilon}\right)^4 = \left(\frac{\varepsilon}{\sigma}\right) R\{cost\} R\{\omega(t)\}$$

$$\Rightarrow \left(\frac{\sigma}{\varepsilon}\right)^3 + 2\left(\frac{\sigma}{\varepsilon}\right)^4 = \left(\frac{\varepsilon}{\sigma}\right) \left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right] R\{\omega(t)\}$$

$$\Rightarrow R\{\omega(t)\} = \left[\left(\frac{\sigma}{\varepsilon}\right)^4 + 2\left(\frac{\sigma}{\varepsilon}\right)^5\right] \left[\frac{(\varepsilon^2+\sigma^2)}{\sigma^2}\right] = \left(\frac{\sigma}{\varepsilon}\right)^2 + 2\left(\frac{\sigma}{\varepsilon}\right)^3 + \left(\frac{\sigma}{\varepsilon}\right)^4 + 2\left(\frac{\sigma}{\varepsilon}\right)^5 \tag{10}$$

After operating inverse Rishi transform on equation (10), the required solution of equation (8) obtain and it is given by

$$\omega(t) = R^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^2 + 2\left(\frac{\sigma}{\varepsilon}\right)^3 + \left(\frac{\sigma}{\varepsilon}\right)^4 + 2\left(\frac{\sigma}{\varepsilon}\right)^5\right\}$$

$$\Rightarrow \omega(t) = R^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^2\right\} + 2R^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^3\right\} + R^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^4\right\} + 2R^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^5\right\}$$

$$\Rightarrow \omega(t) = 1 + 2t + \frac{t^2}{2} + \frac{t^3}{3}$$

Remark: The same exact solution obtained using Rishi transform as given in [33] but spending a little time.

Problem: 3 Consider the following first kind V.I.E. with convolution type kernel given by [35] as

$$sint = \int_0^t e^{(t-u)}\, \omega(u) du \tag{11}$$

Operating Rishi transform on equation (11), we get

$$R\{sint\} = R\left\{\int_0^t e^{(t-u)}\, \omega(u) du\right\}$$

$$\Rightarrow R\{sint\} = R\{e^t * \omega(t)\} \tag{12}$$

Using convolution theorem of Rishi transform in equation (12), we have

$$\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)} = \left(\frac{\varepsilon}{\sigma}\right) R\{e^t\} R\{\omega(t)\}$$

$$\Rightarrow \frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)} = \left(\frac{\varepsilon}{\sigma}\right)\left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right] R\{\omega(t)\}$$

$$\Rightarrow R\{\omega(t)\} = \left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)} - \frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right] \quad (13)$$

After operating inverse Rishi transform on equation (13), the required solution of equation (11) obtain and it is given by

$$\omega(t) = R^{-1}\left\{\frac{\sigma^2}{(\varepsilon^2+\sigma^2)} - \frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right\}$$

$$\Rightarrow \omega(t) = R^{-1}\left\{\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right\} - R^{-1}\left\{\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right\}$$

$$\Rightarrow \omega(t) = cost - sint$$

Remark: The same exact solution obtained using Rishi transform as given in [35] but spending a little time and without large computational work.

10. Conclusion

In the present paper, authors successfully introduced a new integral transform "Rishi Transform" and obtained the exact solution of first kind V.I.E. with convolution type kernel. Authors also presented the fundamental properties (linearity; scaling; translation; convolution) of the proposed transform with its inverse transform. The findings of this paper suggest that the proposed transform (Rishi Transform) provides the exact results without doing complicated calculation work. In future, Rishi transform can be considered to solve various complex problems of science, medicine and engineering by developing their mathematical models.

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