Publications

Available Online

J. Sci. Res. **14** (3), 773-784 (2022)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

Existence Solutions of ABC-Fractional Differential Equations with Periodic and Integral Boundary Conditions

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Received 13 February 2022, accepted in final revised form 27 June 2022

Abstract

The nonlinear fractional differential equation (FDE) is discussed in this study. First, the research will investigate the existence and unique solution of the nonlinear differential equation to the Atangana-Baleanu fractional derivative in the sense of Caputo with the initial periodic condition, an integral boundary condition by Krasnoselskii's and Banach fixed point theorems. Then, this work will study the Hyers-Ulam stability of our problem. Finally, an example to demonstrate the use of our main theorems will be presented.

MS (2020 Mathematics Subject Classifications: 34A08, 26A33, 34G20, 34C25, 45J05)

Keywords: ABC-derivatives; Existence and uniqueness, Periodic boundary conditions; Banach fixed point theorem; Krasnoselskiia's fixed point theorem.

doi: http://dx.doi.org/10.3329/jsr.v14i3.58210 J. Sci. Res. **14** (3), 773-784 (2022)

1. Introduction

Fractional calculus has a 300-year history, with the subject's growth-focused mainly on pure mathematics. In the nineteenth century, Liouville, Riemann, Leibniz, and others conducted the first more or less systematic investigation [1,2]. Compared to integer order, fractional calculus contains some significant changes [3]. Fractional order Differential equations have lately been useful tools for modeling a wide range of phenomena in science and engineering. Control, porous media, electromagnetic, and other domains can benefit from its use [4-6]. Depending on the physical situation at hand, this theory employs a variety of boundary conditions. Integral boundary conditions are more important and used where classical boundary conditions fail to develop mathematical models. In contrast, periodic boundary conditions are widely encountered in computational science of various areas, particularly when the physical domain involved is infinite or homogeneous along with one or more directions [6,7]. Different techniques for fractional derivatives have been proposed in research investigations, including Riemann–Liouville, Caputo, Caputo–Fabrizio, Caputo–Hadamard, Grunwald–Letnikov, and Atangana–Baleanu derivatives. Many mathematicians have worked on difficulties

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concerning fractional differential equations' existence and uniqueness [8–11]. Various well-known approaches related to fixed point theory, such as Banach and Krasnoselskii's fixed point theorems [11], are frequently applied.

In this article, we consider the following ABC-fractional boundary value problem with periodic and integral boundary conditions:

$$({}^{ABC}_{\ 0}D^{\alpha}x)(t) = f\big(t, x(t)\big), \quad 1 < \alpha \leqslant 2, \quad t \in [0,T],\ T > 0, \tag{1}$$

with

$$x(0) = x(T), \tag{2}$$

$$x'(0) = \int_0^T x(s)ds. \tag{3}$$

On the other hand, the theory of FDEs has been researched, and several basic conclusions, including stability theory, have been established. The subject of stability is crucial in both physical and biological systems. One of the necessary qualitative theories of dynamical systems is the idea of stability. As a result of its applications, the theory of stability characteristics has garnered significant attention in various study domains. Many academics have studied the Ulam-Hyers stability analysis and its applications to many types of differential equations [12,13]. Ulam stability has been introduced for the functional equation by Ulam. Ulam first addressed the consistency of functional equations in a 1940 speech at the University of Wisconsin. His query was: under what conditions does an additive mapping occur, as opposed to an approximately additive mapping? In Banach spaces, Hyers provided the first solution to Ulam's question in 1941. The Ulam–Hyers stability is the name of this form of stability. Rassias provided a notable expansion of the Ulam–Hyers stability of maps by taking variables into account in 1978. [14-16].

This article is divided into four sections. In Section 2, the current study reviews various Atangana-Baleanu fractional notations and definitions. Section 3 is divided into two parts. In the first section, we use Krasnoseleskii,s fixed point theorem to demonstrate the existence of the solution to our presented problems (1)-(3). The second section uses the Banach fixed point theorem to demonstrate a unique solution. Section 4 uses Hyers-Ulam stable (HU) to demonstrate the solution's stability. Finally, an example to further clarify our findings will present.

2. Mathematical Tools

The current research presents several definitions, lemmas, and theorems relevant to our main results in this section, which will be required in the next section.
For $\omega > 0$, the left Riemann-Liouville integral is defined as [17].

$$({}_0I^{\omega}\theta)(t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\theta(s)ds.$$

For $0 \leq \omega \leq 1$, the left Riemann-Liouville fractional derivative is given as [17].

$$({}_0D^{\omega}\theta)(t) = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\omega)}\int_0^t (t-s)^{-\omega}\theta(s)ds\right).$$

For $0 \leq \omega \leq 1$, the Caputo fractional derivative is read as [16].

$$({}_{0}^{C}D^{\omega}\theta)(t)=\frac{1}{\Gamma(1-\omega)}\int_{0}^{t}(t-s)^{-\omega}\theta'(s)ds.$$

Definition 2.1 [18]. Let $\in[0,1]$, $\theta'\in H'(a,b)$, where $a\leq b$, then the Caputo AB-derivative is

$$({}_{0}^{ABC}D^{\omega}\theta)(t)=\frac{\beta(\omega)}{1-\omega}\int_{0}^{t}\theta'(s)E_{\omega}\left[-\omega\frac{(t-s)^{\omega}}{1-\omega}\right]ds.$$

Where E_{ω} is the Mittag-Leffler function, $B(\omega)$ is a positive normalizing function satisfying $B(0)=B(1)=1$.
The associated fractional integral of the Caputo AB-derivative is defined by

$$({}_{0}^{AB}I^{\omega}\theta)(t)=\frac{1-\omega}{\beta(\omega)}\theta(t)+\frac{\omega}{\beta(\omega)}({}_{0}I^{\omega}\theta)(t)$$

where ${}_{0}I^{\omega}$ is the left Riemann-Liouville integral

Lemma 2.2 [19] The ABC fractional derivative and AB fractional integral of the function $\theta(t)$ satisfies the following formula given

$${}_{a}^{AB}I_{\tau}^{\omega}({}^{ABC}{}_{a}D_{\tau}^{\omega}\theta(t))=\theta(t)-\theta(a)$$

Lemma 2.3 [20]. For $\theta(t)$ defined on $[a,b]$, $\omega\in(\vartheta,\vartheta+1]$, for some $\vartheta\in N$ we have

I. $\left({}_{a}^{ABR}D^{\omega}\,{}_{a}^{AB}I^{\omega}\theta\right)(t)=\theta(t).$

II. $\left({}_{a}^{AB}I^{\omega}\,{}_{a}^{ABR}D^{\omega}\theta\right)(t)=\theta(t)-\sum_{\xi=0}^{\vartheta-1}\frac{\theta^{\xi}(a)}{\xi!}(t-a)^{\xi}.$

III. $\left({}_{a}^{AB}I^{\omega}\,{}_{a}^{ABC}D^{\omega}\theta\right)(t)=\theta(t)-\sum_{\xi=0}^{\vartheta}\frac{\theta^{\xi}(a)}{\xi!}(t-a)^{\xi}.$

Theorem 2.4 [21] **Arzela Fixed Point Theorem**. Let Θ be a compact Hausdorff metric space. Then $\Omega\subset H(\Theta)$ is said to be relatively compact whenever Ω is equicontinuous and bounded uniformly.

3. Mean Results

In this part, the well-known fixed-point theorems are used to show the existence and uniqueness of solutions to ABC-fractional boundary value problem (1)-(3), by Krasnoselskii's and Banach's fixed point. First, the following theorem, which is critical for obtaining the existence of solution results is shown.

Theorem 3.1. Let $f:[0,T]\to R$ be a continuous function, and a function $x(t)$ is a solution of the following ABC-fractional

$$({}^{ABC}_{0}D^{\alpha}x)(t)=f\big(t,x(t)\big),\qquad 1<\alpha\leqslant 2,\ t\in[0,T],\qquad T>0$$

With

$$x(0)=x(T),x'(0)=\int_{0}^{T}x(s)ds$$

iff $x(t)$ is the solution of the following integral equation:

$$x(t)=\int_{0}^{T}G(t,s)f(s,x(s))ds$$

Where,

$$G(t,s) = \begin{cases} \frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T} + \frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}(T-s)^{\alpha-1} \\ -\frac{(2-\alpha)}{\beta(\alpha-1)T}(T-s) - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}(T-s)^{\alpha}, \quad t \le s \le T \\ \frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T} + \frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}(T-s)^{\alpha-1} \\ -\frac{(2-\alpha)}{\beta(\alpha-1)T}(T-s) - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}(T-s)^{\alpha} + \frac{2-\alpha}{\beta(\alpha-1)} + \\ \frac{\alpha-1}{\beta(\alpha-1)\Gamma(\alpha)}(t-s)^{\alpha-1}, \qquad 0 \le s \le t \end{cases}$$

Proof. Consider the following fractional differential equation:

$$({}^{ABC}_{\ \ 0}D^{\alpha}x)(t) = f\big(t,x(t)\big), \quad 1 < \alpha \leqslant 2, \quad t \in [0,T], \quad T > 0$$

By using Lemma 2.3, obtain that

$$x(t) = x(0) + x'(0)t + \frac{2-\alpha}{\beta(\alpha-1)}\int_0^t f\big(s,x(s)\big)ds + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s,x(s)\big)ds. \tag{4}$$

Now, using the boundary conditions $x(0) = x(T)$, $x'(0) = \int_0^T x(s)ds$ and with the necessity $f(0) = 0$, we get

$$x(0) = \left(\frac{T^2}{2} - 1\right)\left(\frac{2-\alpha}{\beta(\alpha-1)T}\right)\int_0^T f\big(s,x(s)\big)ds + \left(\frac{T^2}{2} - 1\right)\left(\frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)T}\right)\int_0^T (T-s)^{\alpha-1} f\big(s,x(s)\big)ds - \frac{(2-\alpha)}{\beta(\alpha-1)T}\int_0^T (T-s)f\big(s,x(s)\big)ds - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}\int_0^T (T-s)^{\alpha} f\big(s,x(s)\big)ds.$$

and

$$x'(0) = \frac{-(2-\alpha)}{\beta(\alpha-1)T}\int_0^T f\big(s,x(s)\big)ds - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)T}\int_0^T (T-s)^{\alpha-1} f\big(s,x(s)\big)ds,$$

Putting the values of (0) , $x'(0)$ in equation (4), find that

$$x(t) = \frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T}\int_0^T f\big(s,x(s)\big)ds + \frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\int_0^T (T-s)^{\alpha-1} f\big(s,x(s)\big)ds - \frac{(2-\alpha)}{\beta(\alpha-1)T}\int_0^T (T-s)f\big(s,x(s)\big)ds - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}\int_0^T (T-s)^{\alpha} f\big(s,x(s)\big)ds + \frac{2-\alpha}{\beta(\alpha-1)}\int_0^t f\big(s,x(s)\big)ds + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s,x(s)\big)ds.$$

After simplifications and replacing the value of $G(t,s)$, obtain that

$$x(t) = \int_0^T G(t,s)f(s,x(s))ds.$$

This proves the theorem.

3.1. *Existences solution using Krasnoselskii's fixed point theorem*

In this part, one of the key fixed-point theorems was utilized to demonstrate the solution contains at least one fixed point. This theorem guarantees the fixed point's sum of compact and a contractive operator. This study, start with Krasnoselskii's fixed point theorem.

Theorem 3.2.[22] Let γ be a closed convex nonempty subset of a Banach space $(K, \|\cdot\|)$. Suppose that U_1 and U_2 map γ into K s.t

(i) $U_1 x + U_2 y \in \gamma \quad (\forall x, y \in \gamma)$.

(ii) U_1 is continuous and $U_1\gamma$ is contained in a compact set.

(iii) U_2 is a contraction with constant $\alpha < 1$.

Then there is a $y \in \gamma$ with $U_1 y + U_2 y = y$.

For easier understanding assume the following:

Let $W = C([0,T], R)$ denote the Banach space of continuous function $x: [0,T] \longrightarrow$ R with norm $\| x \| = sup_{t\in[0,T]}\, x(t)|$. Assume that $\forall\, t \in [0,T]$ the following assumptions are correct.

(a_1) $|f(t, x_1(t)) - f(t, x_2(t))| < L|x_1 - x_2|$.

(a_2) $|f(t,x)| \leq P_1(t) + c_1|x|^{h_1}$.

(a_3) $\sup\limits_{t\in[0,T]} |f(t,0)| \leqslant M$.

Where $L \geq 0$, $h_1 < 1$, $c_1 \geq 0$ and $p_1 \in W$.

Theorem 3.3. Assume that all assumptions (a_1), (a_2) and

(a4) $$\Omega_1 = L\left[\frac{(\alpha-1)(T^2+2T+2)}{2\beta(\alpha-1)\Gamma(\alpha)}\cdot\frac{T^{\alpha-1}}{\alpha} + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}\cdot\frac{T^{\alpha+1}}{\alpha} + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\frac{T^{\alpha}}{\alpha}\right] < 1$$

hold, then (1)-(3) has at least one solution on $[0,T]$.

Proof. Consider $\beta_r = \{x \in W: \| x \| \leqslant r\}$, where

$M_1(|p_1(t) \| + c_1|r|^{h_1})) \leq r$, $r = \max\{M_1(|p_1(t) \| + c_1|r|^{h_1})\}$ and

$$M_1 = \left[\frac{(2-\alpha)(T^2+2T+2)}{2\beta(\alpha-1)} + \frac{(\alpha-1)(T^2+2T+2)T^{\alpha-1}}{2\beta(\alpha-1)\Gamma(\alpha+1)} + \frac{3(2-\alpha)T}{2\beta(\alpha-1)} + \frac{(\alpha-1)T^{\alpha}}{\beta(\alpha-1)\Gamma(\alpha+2)} + \frac{(\alpha-1)T^{\alpha}}{\beta(\alpha-1)\Gamma(\alpha+1)}\right].$$

Now, define the operators $U1$ and U_2 on β_r as

$$(U_1x)(t) = \frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T}\int_0^T f(s, x(s))ds - \frac{(2-\alpha)}{\beta(\alpha-1)T}\int_0^T (T-s)f(s,x(s))ds$$

$$+\frac{2-\alpha}{\beta(\alpha-1)}\int_0^t f(s,x(s))ds,$$

and

$$(U_2X)(t) = \frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\int_0^T (T-s)^{\alpha-1} f(s,x(s))ds - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)\mathrm{T}}\int_0^T (T-$$

$$s)^{\alpha} f(s,x(s))ds + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds.$$

For $x, y \in \beta_r$, consider

$$\|U_1x+U_2y\| = \Big\|\frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T}\int_0^T f(s,x(s))ds - \frac{(2-\alpha)}{\beta(\alpha-1)T}\int_0^T (T-s)f(s,x(s))ds + \frac{2-\alpha}{\beta(\alpha-1)}\int_0^t f(s,x(s))ds + \frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\int_0^T (T-s)^{\alpha-1}f(s,y(s))ds - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}\int_0^T (T-s)^{\alpha}f(s,y(s))ds + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds\Big\|.$$

$$\leqslant \frac{(2-\alpha)(T^2+2T+2)}{2\beta(\alpha-1)}(||p_1(t)||+c_1|r|^{h_1}) + \frac{(\alpha-1)(T^2+2T+2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\frac{T^\alpha}{\alpha}(||p_1(t)||+c_1|r|^{h_1}) + \frac{2-\alpha}{\beta(\alpha-1)T}\cdot\frac{T^2}{2}(||p_1(t)||+c_1|r|^{h_1}) + \frac{(2-\alpha)T}{\beta(\alpha-1)}(||p_1(t)||+c_1|r|^{h_1}) + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}\frac{T^{\alpha+1}}{\alpha+1}\left(\left|\left|p_1(t)\right|\right|+c_1|r|^{h_1}\right) + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\ \frac{T^\alpha}{\alpha}(||p_1(t)||+c_1|r|^{h_1}).$$

$$\leqslant \left[\frac{(2-\alpha)(T^2+2T+2)}{2\beta(\alpha-1)} + \frac{(\alpha-1)(T^2+2T+2)T^{\alpha-1}}{2\beta(\alpha-1)\Gamma(\alpha+1)} + \frac{3(2-\alpha)T}{2\beta(\alpha-1)} + \frac{(\alpha-1)T^\alpha}{\beta(\alpha-1)\Gamma(\alpha+2)} + \frac{(\alpha-1)T^\alpha}{\beta(\alpha-1)\Gamma(\alpha+1)}\right].(||p_1(t)||+c_1|r|^{h_1}),$$

Therefore, obtaining that

$$\|U_1x+U_2y\| \leqslant M_1(||p_1(t)||+c_1|r|^{h_1}) \leqslant r.$$

Hence $U_1(x)+U_2(y)\in\beta_r$.

Now, to show that the operator U_2 is a contraction mapping, using (a_4),

$$\|U_2(x)-U_2(y)\| \leqslant \sup_{t\in[0,T]}|U_2x-U_2y|.$$

$$\|U_2(x)-U_2(y)\| = \Big\|\frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\int_0^T (T-s)^{\alpha-1}f(s,x(s))ds - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}\int_0^T (T-s)^{\alpha}f(s,x(s))ds + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))ds - \Big[\frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\int_0^T (T-s)^{\alpha-1}f(s,y(s))ds - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}\int_0^T (T-s)^{\alpha}f(s,y(s))ds + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds\Big]\Big\|.$$

$$\|U_2(x)-U_2(y)\| \leqslant \frac{(\alpha-1)(T^2+2T+2)L}{2\beta(\alpha-1)\Gamma(\alpha)T}\frac{T^\alpha}{\alpha}\|x-y\| + \frac{(\alpha-1)L}{\beta(\alpha-1)\Gamma(\alpha+1)T}\frac{T^{\alpha+1}}{\alpha}\|x-y\| + \frac{(\alpha-1)L}{\beta(\alpha-1)\Gamma(\alpha)}\frac{T^\alpha}{\alpha}\|x-y\|.$$

$$= L\left[\frac{(\alpha-1)(T^2+2T+2)L}{2\beta(\alpha-1)\Gamma(\alpha)T}\frac{T^\alpha}{\alpha} + \frac{(\alpha-1)L}{\beta(\alpha-1)\Gamma(\alpha+1)T}\frac{T^{\alpha+1}}{\alpha} + \frac{(\alpha-1)L}{\beta(\alpha-1)\Gamma(\alpha)}\frac{T^\alpha}{\alpha}\right]\|x-y\|,$$

Therefore, find that

$$\|U_2(x)-U_2(y)\| \leqslant \Omega_1\|x-y\|.$$

Then U_2 is a contraction mapping.

The operator $U_1$is continuous, also $U_1$is uniformly bounded on β_r as

$$\|U_1(x)(t)\| \le \frac{(2-\alpha)[T^2+5T+2]}{2\beta(\alpha-1)}(\| P_1|| + c_1|r|^{h_1}).$$

Now, to prove that the compactness of the operator U_1 is continuous, the view of (a_1) was used.

$$\|(U_1x)(t_1) - (U_1x)(t_2)\| = \left\|\frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T}\int_0^T f(s,x(s))ds - \frac{(2-\alpha)}{\beta(\alpha-1)T}\int_0^T (T - s)f(s,x(s))ds + \frac{2-\alpha}{\beta(\alpha-1)}\int_0^{t_1} f(s,x(s))ds - \frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T}\int_0^T f(s,x(s))\mathrm{ds} + \frac{2-\alpha}{\beta(\alpha-1)T}\int_0^T (\mathrm{T}-s)f(s,x(s))\mathrm{ds} - \frac{2-\alpha}{\beta(\alpha-1)}\int_0^{t_2} f(s,x(s))ds\right\|.$$

$$= \left\|\frac{(2-\alpha)}{\beta(\alpha-1)}(\int_0^{t_1} f(s,x(s))ds - \int_0^{t_2} f(s,x(s))ds)\right\|,$$

So that, we get

$$\|(U_1x)(t_1) - (U_1x)(t_2)\| \le \frac{2-\alpha}{\beta(\alpha-1)}(\| P_1|| + c_1|r|^{h_1})\|t_1 - t_2\|.$$

Which is independent of x and tends to zero as $t_1 \to t_2$ thus, $U_1$is relatively compact on β_r. Hence, by Theorem 2.4, $U_1$is compact on β_r.
Hence the boundary value problem (1)-(3) has at least one solution on $[0,T]$.

3.2. *Unique solution using Banach fixed point theorem*

The purpose of this section is to find a unique solution to the problem (1)-(3). The unique solution is the most significant thing in any mathematical model. Having more than one solution can be useless and may not provide the required information. Therefore, this research use a well-known theorem named the Banach contraction principle to obtain a unique solution, starting with the Banach fixed point theorem.

Theorem 3.4. [23] (Banach's fixed point theorem). Let η be a non-empty closed subset of a Banach space X. Then any contraction mapping S of η into itself has a unique fixed point.

Theorem 3.5. Assume (a_1), (a_3) and

(a5) $$\Omega_2 = L\left[\frac{(2-\alpha)(T^2+2T+2)}{2\beta(\alpha-1)} + \frac{(\alpha-1)(T^2+2T+2)T^{\alpha-1}}{2\beta(\alpha-1)\Gamma(\alpha+1)} + \frac{3(2-\alpha)\mathrm{T}}{2\beta(\alpha-1))} + \frac{(\alpha-1)T^\alpha}{\beta(\alpha-1)\Gamma(\alpha+2)} + \frac{(\alpha-1)T^\alpha}{\beta(\alpha-1)\Gamma(\alpha+1)}\right] < 1$$

hold. Then (1)-(3) has a unique solution.

Proof. Consider $B_{r_0} = \{x \in W: \| x \| \leqslant r_0\}$, where $r_0 \geqslant \frac{\Omega_3}{1-\Omega_2}$, with $\Omega_3 = M\Omega_2$, then

$$\|(Ux)(t)\| = \left\|\frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T}\int_0^T f(s,x(s))ds + \frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\int_0^T (T-s)^{\alpha-1}f(s,x(s))ds -\right.$$

$\frac{2-\alpha}{\beta(\alpha-1)T}\int_0^T(T-s)f\big(s,x(s)\big)ds+\frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)\mathrm{T}}\int_0^T(T\ -s)^{\alpha}f\big(s,x(s)\big)ds+$

$\frac{2-\alpha}{\beta(\alpha-1)}\int_0^t f\big(s,x(s)\big)ds+\frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\big(s,x(s)\big)ds\Big\|.$

$\leqslant\frac{(2-\alpha)(T^2+2T+2)}{2\beta(\alpha-1)T}\left\|\int_0^T(f(s,x(s)-f(s,0)+f(s,0))\,ds\right\|+\frac{(\alpha-1)(T^2+2T+2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\left\|\int_0^T(T-\right.$

$s)^{\alpha-1}(f(s,x(s)-f(s,0)+f(s,0))ds\Big\|+\frac{2-\alpha}{\beta(\alpha-1)T}\Big\|\int_0^T(T-s)(f(s,x(s)-$

$f(s,0)+f(s,0))ds\Big\|+\frac{\alpha-1}{\beta(\alpha-1)\Gamma(\alpha+1)T}\Big\|\int_0^T(T-s)^{\alpha}(f(s,x(s)-f(s,0)+$

$f(s,0))\,ds\Big\|+\frac{2-\alpha}{\beta(\alpha-1)}\Big\|\int_0^t(f(s,x(s)-f(s,0)+f(s,0))ds\Big\|+\frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\Big\|\int_0^t(t-$

$s)^{\alpha-1}\,(f(s,x(s)-f(s,0)+f(s,0))\mathrm{ds}\Big\|$

Hence, obtaining that

$\|(Ux)(t)\|\leq\frac{(2-\alpha)(T^2+2T+2)}{2\beta(\alpha-1)}(L\|x(t)\|+M))+\frac{(\alpha-1)(T^2+2T+2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\,\frac{T^{\alpha}}{\alpha}(L\|x(t)\|+M)+$

$\frac{2-\alpha}{\beta(\alpha-1)T}\cdot\frac{T^2}{2}(L\|x(t)\|+M)+\frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}\cdot\frac{T^{\alpha+1}}{\alpha+1}(L\|x(t)\|+M)+$

$\frac{(2-\alpha)T}{\beta(\alpha-1)}(L\|x(t)\|+M)+\frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\cdot\frac{T^{\alpha}}{\alpha}(L\|x(t)\|+M).$

$\leq$

$\frac{(2-\alpha)(T^2+2T+2)}{2\beta(\alpha-1)}+\frac{(\alpha-1)(T^2+2T+2)}{2\beta(\alpha-1)\Gamma(\alpha)}\cdot$

$\frac{T^{\alpha-1}}{\alpha}+\frac{3(2-\alpha)\mathrm{T}}{2\beta(\alpha-1)}+\frac{(\alpha-1)T^{\alpha}}{\beta(\alpha-1)\Gamma(\alpha+2)}+\frac{(\alpha-1)T^{\alpha}}{\beta(\alpha-1)\Gamma(\alpha+1)}\Big](Lr_0+M)$, so that

$\|(Ux)(t)\|\leqslant\Omega_2r_0+\Omega_3.$

This show that U is a self-mapping on β_{r_0}.

Now, operator U is shown as the contraction principle. For all $x,y\in B_{r_0}$and for all $t\in[0,T]$.

$\|(Ux)(t)-(Uy)(t)\|=\Big\|\frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T}\int_0^T f\big(s,x(s)\big)ds+\frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\int_0^T\ (T-$

$s)^{\alpha-1}f\big(s,x(s)\big)ds-\frac{(2-\alpha)}{\beta(\alpha-1)T}\int_0^T\ (T-s)f\big(s,x(s)\big)ds-\frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}\int_0^T\ (T-$

$s)^{\alpha}f\big(s,x(s)\big)ds\ +\frac{2-\alpha}{\beta(\alpha-1)}\int_0^t\ f\big(s,x(s)\big)ds+\frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\int_0^t\ (t-$

$s)^{\alpha-1}f\big(s,x(s)\big)ds\ -\Big[\frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T}\int_0^T\ f\big(s,y(s)\big)ds+\frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}\int_0^T\ (T-$

$s)^{\alpha-1}f\big(s,y(s)\big)ds - \frac{(2-\alpha)}{\beta(\alpha-1)T}\int_0^T (T-s)f\big(s,y(s)\big)ds - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}\int_0^T (T-$
$s)^{\alpha}f\big(s,y(s)\big)ds + \frac{2-\alpha}{\beta(\alpha-1)}\int_0^t f\big(s,y(s)\big)ds + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\int_0^t (\mathrm{t}-s)^{\alpha-1}f\big(s,y(s)\big)ds\Big]\Big\|.$

$\leq \frac{(2-\alpha)(T^2+2T+2)L}{2\beta(\alpha-1)}\|x-y\| + \frac{(\alpha-1)(T^2+2T+2)}{2\beta(\alpha-1)\Gamma(\alpha)\mathrm{T}}(L\|x-y\|)(\frac{T^\alpha}{\alpha}) + \frac{2-\alpha}{\beta(\alpha-1)T}(L\|x-$
$y\|)(\frac{T^2}{2}) + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)\mathrm{T}}(L\|x-y\|)(\frac{T^{\alpha+1}}{\alpha+1}) + \frac{(2-\alpha)T}{\beta(\alpha-1)}(L\|x-y\|) + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}(L\|x-$
$y\|)(\frac{T^\alpha}{\alpha}).$

$\leq$
$L\Big[\frac{(2-\alpha)(T^2+2T+2)}{2\beta(\alpha-1)} + \frac{(\alpha-1)(T^2+2T+2)T^{\alpha-1}}{2\beta(\alpha-1)\Gamma\,(\alpha+1)} + \frac{3(2-\alpha)T}{2\beta(\alpha-1)} + \frac{(\alpha-1)T^\alpha}{\beta(\alpha-1)\Gamma\,(\alpha+2)} +$
$\frac{(\alpha-1)T^\alpha}{\beta(\alpha-1)\Gamma\,(\alpha+1))}\Big]\|x-y\|.$

$\|(Ux)(t)-(Uy)(t)\| \leqslant \Omega_2\|x-y\|.$

Since $\Omega_2 < 1$, by Banach fixed point theorem, the problem (1)-(3) is a unique solution.

4. Hyers-Ulam Stability

In this part, the stability will discuss. After that, it was clear that the problem (1)-(3) had a unique solution in section (3.2). We use Hyers-Ulam Stability for our problems (1)-(3). The notes and definitions that follow will be beneficial for our primary result.

Definition 4.1.[24]**.** The problem (1)-(3) is said to be Hyers-Ulam stable if there exists a real number $H_g > 0$ such that for each $\epsilon > 0$ and for each solution $z \in W$ of the inequality

$$\left|\left({}_0^{\mathrm{ABC}}D^\alpha x\right)(t) - f(t,z(t))\right| \leq \epsilon,\ \text{ for all } t \in [0,T]$$

there exists a solution $x \in W$ of the problem (1)-(3) such that

$$|z(t)-x(t)| \leq H_g\epsilon,\ t \in [0,T].$$

Remark 4.2. [24] A function $z \in W$ is a solution of the inequality

$$\left|\left({}_0^{\mathrm{ABC}}D^\alpha x\right)(t) - f(t,z(t))\right| \leq \epsilon,\ \text{ for all } t \in [0,T]$$

if and only if there exists a function $C \in W$ (which depends on z) such that:

(i) $|C(t)| \leq \epsilon, \quad \forall t \in [0,\mathrm{T}].$

(ii) $\left({}_0^{\mathrm{ABC}}D^\alpha z\right)(t) = f\big(t,z(t)\big) + C(t), \qquad t \in [0,\mathrm{T}].$

Remark 4.3. From Theorem 3.1.

$$x(t) = \int_0^T G(t,s)f(s,x(s))ds.$$

where

$$G(t,s) = \begin{cases} \frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T} + \frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}(T-s)^{\alpha-1} \\ -\frac{(2-\alpha)}{\beta(\alpha-1)T}(T-s) - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}(T-s)^{\alpha}, \quad t \le s \le T \\ \frac{(2-\alpha)(T^2-2t-2)}{2\beta(\alpha-1)T} + \frac{(\alpha-1)(T^2-2t-2)}{2\beta(\alpha-1)\Gamma(\alpha)T}(T-s)^{\alpha-1} \\ -\frac{(2-\alpha)}{\beta(\alpha-1)T}(T-s) - \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T}(T-s)^{\alpha} + \frac{2-\alpha}{\beta(\alpha-1)} + \\ \frac{\alpha-1}{\beta(\alpha-1)\Gamma(\alpha)}(t-s)^{\alpha-1}, \qquad 0 \le s \le t \end{cases}$$

Note that $|G(t,s)| \leqslant K$.

Theorem 4.4. Suppose $f:[0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and satisfying (a_1), then the problem (1)-(3) is Hyers-Ulam stability.

Proof. Let $x(t) \in W$ be any solution to the inequality

$$\left|\left({}_{0}^{\mathrm{ABC}}D^{\alpha}x\right)(t) - f(t,z(t))\right| \le \epsilon, \text{ for all } t \in [0,T]$$

Using Remark 4.2.

$$\left({}_{0}^{\mathrm{ABC}}D^{\alpha}x\right)(t) = f\big(t,x(t)\big) + C(t),\ t \in [0,T],$$

and, using Remark 4.3., written that

$$x(t) = \int_0^T G(t,s)f(s,x(s))ds.$$

Which implies that

$$\left|x(t) - \int_0^T G(t,s)f(s,x(s))ds\right| \le KT\epsilon.$$

Where $|G(t,s)| \leqslant K$, was defined in remark 4.3.

Now, let $z(t) \in W$ be a unique solution to the fractional boundary value problem (1)-(3). Consider that

$$|x(t) - z(t)| = \left|x(t) - \int_0^T G(t,s)f(s,z(s))ds\right|.$$

$$|x(t) - z(t)| = \left|x(t) - \int_0^T G(t,s)f\big(s,z(s)\big)ds + \int_0^T G(t,s)f\big(s,x(s)\big)ds - \int_0^T G(t,s)f\big(s,x(s)\big)ds\right|.$$

$$|x(t) - z(t)| \le KT\epsilon + TKL|x(t) - z(t)|.$$

Taking up over t $\in [0,T]$, obtain that

$$\|x - z\| \le KT\epsilon + TKL\|x - z\|.$$

$$\|x - z\|\mathrm{TKL} \le \mathrm{KT}\epsilon.$$

$$\Rightarrow \|x - z\| \leqslant \frac{\epsilon}{L}.$$

$\|x - z\| \le H_g\epsilon$, where $\frac{1}{L} = H_g$.

Therefore, the problem (1)-(3) is Hyers-Ulam stable.

5. Example

This section contains an example to illustrate the previous theorems. Consider the following fractional differential equation

$$\left({}^{ABC}_{0}D_t^{\alpha} x\right)(\mathrm{t}) = \frac{1}{e^t+14}x(t) + \frac{1}{2(\mathrm{t}+2)^3}\sin\big(x(t)\big), \qquad 1 < \alpha \leqslant 2, \qquad t \in [0,2] \qquad (5.5)$$

with

$$x(0) = x(2), \quad x'(0) = \int_0^2 x(s)ds. \qquad (5.6)$$

such that $t \in J = [0,2]$, with the initial periodic condition $\mathrm{u}(0) = x(2)$, where ${}^{ABC}_{0}D_t^{\alpha}$ denotes the Atangana-Baleanu fractional derivative in the sense of Caputo. $(\alpha \in (1,2])$.
Here $T = 2$, $f\big(t, x(t)\big) = \frac{1}{e^t+14}x(t) + \frac{1}{2(\mathrm{t}+2)^3}\sin\big(x(t)\big)$, obtain that $L = 0.1222$

Table 1. Calculated values of Ω_1 and Ω_2 when $\alpha \in (1,2]$ and T = 2.

$\alpha \in (1,2]$	Ω_1	Ω_2	$\alpha \in (1,2]$	Ω_1	Ω_2
1.01	0.0097	0.9727	1.6	0.3945	0.6965
1.1	0.0930	0.9304	1.7	0.4196	0.6433
1.2	0.1758	0.8852	1.8	0.4332	0.5872
1.3	0.2479	0.8404	1.9	0.4356	0.5288
1.4	0.3086	0.7946	2	0.4277	0.4684
1.5	0.3575	0.7469			

From Table 1, All values of Ω_1 and Ω_2 were obtained that are less than one, where

$$\Omega_1 = L\left[\frac{(\alpha-1)(T^2+2T+2)}{2\beta(\alpha-1)\Gamma(\alpha)} \cdot \frac{T^{\alpha-1}}{\alpha} + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha+1)T} \cdot \frac{T^{\alpha+1}}{\alpha} + \frac{(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha)}\frac{T^{\alpha}}{\alpha}\right].$$

$$\Omega_2 = L\left[\frac{(2-\alpha)(T^2+2T+2)}{2\beta(\alpha-1)} + \frac{(\alpha-1)(T^2+2T+2)T^{\alpha-1}}{2\beta(\alpha-1)\Gamma(\alpha+1)} + \frac{3(2-\alpha)\mathrm{T}}{2\beta(\alpha-1))} + \frac{(\alpha-1)T^{\alpha}}{\beta(\alpha-1)\Gamma(\alpha+2)} + \frac{(\alpha-1)T^{\alpha}}{\beta(\alpha-1)\Gamma(\alpha+1)}\right].$$

Therefore, from theorem 3.3 and theorem 3.5, a unique solution existed to the nonlinear fractional differential equation (5.1) and (5.2). Also, from Theorem 4.4, the problem (5.1)-(5.2) is Hyers-Ulam stable.

6. Conclusion

The existence and stability of solutions were studied for the nonlinear differential equation's nonlinear fractional differential equation to the Atangana-Baleanu fractional derivative in the sense of Caputo with the initial periodic condition and an integral boundary condition Krasnoselskii's and Banach fixed point theorems. Also, the Hyers-Ulam stability of solutions was investigated for the nonlinear fractional differential. Finally, an example to demonstrate our main theorems was presented.

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