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Optimization of M/M/2 Queueing Model with Working Vacations

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Abstract

The paper deals with an M/M/2 Queueing Model with working vacations and reneging of customers due to impatience. The matrix geometric method is used to find the distribution of the number of customers in the system. A cost function is constructed to obtain the optimal value of the service rate to optimize (minimize) the cost function using the Quadratic Fit Search Method (QFSM). Further, the effects on the system's performance measures using numerical analysis and graphical representation are studied.

Keywords: Queue; Working vacation; Optimization; Matrix geometric method; Quadratic fit search method.

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1. Introduction

In many real-life situations, the queueing problem can be seen in our day-to-day life, such as in banks, hospitals, airlines, telecommunication, etc. Many queueing models are designed to provide better facilities to customers in a short period with minimum cost. To receive service, customers form a queue, and when it gets long, customers start leaving from queue due to impatience. The customer either leaves the queue after staying in the queue for some time (Reneging) or decides not to join the queue (Balking), or discourages arrivals. In the Discouraged arrival queueing system, customers, due to long queues, get impatient and get discouraged from joining the queue. P. Vijaya Laxmi and Kassahun [1] studied multi-server queues with working vacations, reneging of customers, and discouraged arrivals and obtained the system's steady state and steady probabilities. V. Goswami [2] studied a discrete-time queue with reneging, balking, and working vacations and constructed a cost model to optimize service rate at a minimum cost using the quadratic fit search method. The concept of reneging and discouraged arrivals are used in this paper.

A rich literature is available on queueing theory on the concept of working vacations, and this paper also focuses on finding new results using working vacations. In working vacation, the server, instead of no service, provides the service at a lower rate than during

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regular busy periods. If a customer is in the queue, the server interrupts the vacation and starts working with the normal service rate. Then if the queue gets empty, the server takes another vacation. This vacation policy is called a multiple working vacation. Servi and Finn [3] studied and analyzed the queueing model using the concept of multiple working vacations. Baba [4] analyzed a GI/M/1 queueing model with working vacations and obtained the steady state distribution and sojourn time of the customers for this model.

In this paper, the matrix geometric method is used to obtain the distribution of the number of customers in the system. Zhang and Xu [5] derived the steady state distribution of the number of customers in the M/M/1 queueing model with multiple working vacations and N-policy. Yang *et al*. [6] considered the F- policy M/M/1/K/WV queueing system and obtained the steady state probability vector using the matrix geometric method. Further, optimization is investigated using the direct search method and the Quasi-Newton method. Laxmi *et al*. [7] analyzed a multiple working vacation queue under the N policy with reneging, balking, and vacation interruption. They obtained steady-state length distribution at arbitrary epochs, and cost analysis was carried out using the quadratic fit search method and particle swarm optimization. Chakravarthy *et al.* [8] studied *MAP/PH/1* queueing model using the matrix geometric method under certain conditions like server breakdown, vacations, repairs, and backup servers and established results for steady-state analysis with illustrative examples. Gupta and Kumar [9] investigated an M/M/1 retrial queueing model with working vacation and vacation interruption due to breakdown and repair. Mean system size and probabilities in various server states are obtained using the probability-generating function technique. Further, an optimal value of slow service rate is estimated using a quadratic fit search approach. Bouchentouf *et al.* [10] analyzed an M/M/1 queueing model under single and multiple vacation policies with balking, reneging, and multi-phase random environment and obtained steady-state probabilities using the probability generating function method.

In this paper, a cost model is constructed to find the minimum cost function for the optimal value of regular busy period service rate using the quadratic fit search method (QFSM). The rest of the paper is arranged in the following sequence: Section 2 describes the queueing model used in this paper. Section 3 deals with the steady-state of the model, steady-state probabilities, and the rate matrix R is derived. In section 4, some system performance measures are established for this model, followed by its numerical and graphical analysis in section 5. In section 6, a cost model is constructed, and for the optimization of the cost function, an algorithm of the quadratic fit search method is discussed and applied to the cost function. The paper is concluded in section 7.

2. Description of the Model

An M/M/2 Queueing Model with working vacations and impatient customers is considered in this paper. Whenever the size of the queue increases, customers are discouraged and reneged from the system, which results in the loss of the customer. The arrivals of customers follow a Poisson distribution with parameter λ_n where

$$\lambda_n = \begin{cases} \lambda & , n \le 2 \\ \frac{\lambda}{\min(n,N)} & , n \ge 3 \end{cases} \tag{2.1}$$

There are 2 identical servers available and infinite waiting space for the customers. The customer is served immediately if the server is available. When the system gets empty, both servers go on vacation. If a customer arrives for service, the server interrupts the vacation and serves the customer, known as the working vacation period. In working vacation, the server serves at a lower service rate in comparison to regular busy periods. Let the service rate during regular busy periods be μ_r and during working vacation periods, be μ_v where $\mu_r > \mu_v$. Both the service rates are exponentially distributed. On completion of vacation, if the server finds an empty queue, then again server goes for another working vacation. If the server finds any customer, then it starts serving with a regular busy service rate μ_r. Let the vacation period be exponentially distributed with parameter φ and the probability of reneging of customers due to impatience be q. Let reneging time be exponentially distributed with parameter β_n, where

$$\beta_n = \begin{cases} 0 & , n \le 2 \\ (\min\{n, N\} - 2)\beta & , n \ge 3 \end{cases} \tag{2.2}$$

where N in equations (2.1) and (2.2) is a fixed large integer. Here, all distributions are independent, including inter-arrival time, service time, reneging time, and vacation time. Further, service to the customers is provided based on First Come, First Served.
Let $Q(t)$ denote the number of customers in the system at time t and state variables are

$$J(t) = \begin{cases} 0 , & \text{the system is on working vacation state} \\ 1 , & \text{the system is in a regular busy period} \end{cases}$$

Then, the Markov process denoted by $\{Q(t), J(t)\}$ with the state space

$$\Omega = \{(0,0) \cup (j,n), j = 0,1; n = 1,2,3, \ldots$$

Here state $(0, n)$ shows that server is in a working vacation state when there are n customers in the queue, and state $(1, n)$ shows that the server is in a regular busy period. Further, state (1,0) shows that the system is closed as $n = 0$, i.e., there is no customer in the system for the service.
In lexicographical sequence, the state transition matrix given by Neuts [11] is written as

$$\tilde{Q} = \begin{bmatrix} A_{00} & A_{01} & & & \\ & B_0 & A_1 & C_1 & \\ & B_2 & A_2 & C_2 & \\ & \ddots & \ddots & \ddots & \\ & B_{N-1} & A_{N-1} & C_{N-1} & \\ & B_N & A_N & C_N & \\ & & & & \end{bmatrix} \tag{2.3}$$

where

$$A_{00} = -\lambda \ , \qquad A_{01} = (\lambda \, , 0)$$

$$B_0 = [\mu_v \, , \mu_r]^T$$

$$C_n = \begin{cases} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} , 1 \le n \le 2 \\ \begin{bmatrix} \frac{\lambda}{n} & 0 \\ 0 & \frac{\lambda}{n} \end{bmatrix} , 3 \le n \le N-1 \\ \begin{bmatrix} \frac{\lambda}{N} & 0 \\ 0 & \frac{\lambda}{N} \end{bmatrix} , \quad n \ge N \end{cases} \tag{2.4}$$

$$A_n = \begin{cases} \begin{bmatrix} -(\lambda + \varphi + n\mu_v) & \varphi \\ 0 & -(\lambda + n\mu_r) \end{bmatrix} , 1 \le n \le 2 \\ \begin{bmatrix} (A_n)_{11} & (A_n)_{12} \\ 0 & (A_n)_{22} \end{bmatrix} , \ 3 \le n \le N-1 \\ \begin{bmatrix} (A_N)_{11} & (A_N)_{12} \\ 0 & (A_N)_{22} \end{bmatrix} , \ n \ge N \end{cases} \tag{2.5}$$

Where

$$(A_n)_{11} = -(\frac{\lambda}{n} + \varphi + 2\mu_v + (n-2)\beta q)$$

$$(A_n)_{12} = \varphi$$

$$(A_n)_{22} = -(\frac{\lambda}{n} + 2\mu_r + (n-2)\beta q)$$

$$(A_N)_{11} = -(\frac{\lambda}{N} + \varphi + 2\mu_v + (N-2)\beta q)$$

$$(A_N)_{12} = \varphi$$

$$(A_N)_{22} = -(\frac{\lambda}{N} + 2\mu_r + (N-2)\beta q)$$

$$B_n = \begin{cases} \begin{bmatrix} n\mu_v & 0 \\ 0 & n\mu_r \end{bmatrix} , \quad 1 \le n \le 2 \\ \begin{bmatrix} 2\mu_v + (n-2)\beta q & 0 \\ 0 & 2\mu_r + (n-2)\beta q \end{bmatrix} , 3 \le n \le N-1 \\ \begin{bmatrix} 2\mu_v + (N-2)\beta q & 0 \\ 0 & 2\mu_r + (N-2)\beta q \end{bmatrix} , \quad n \ge N \end{cases} \tag{2.6}$$

3. Steady State Analysis of the Model

Theorem: The state transition rate matrix R satisfies the quadratic equation

$$R^2 B_N + R A_N + C_N = 0 \tag{3.1}$$

has a non-negative minimal solution given by

$$R = \begin{cases} \begin{bmatrix} \delta & \frac{\delta\varphi}{n\mu_r(1-\delta)} \\ 0 & \frac{\rho}{n} \end{bmatrix} , & 1 \le n \le 2 \\ \begin{bmatrix} \gamma_1 & \frac{\gamma_1\varphi}{\gamma(1-\gamma_1)} \\ 0 & \gamma_2 \end{bmatrix} , & 3 \le n \le N-1 \\ \begin{bmatrix} \sigma_1 & \frac{\sigma_1\varphi}{\sigma(1-\sigma_1)} \\ 0 & \sigma_2 \end{bmatrix}, & n \ge N \end{cases} \tag{3.2}$$

Where

$$\delta = \frac{(\lambda + \varphi + n\mu_v) - \sqrt{(\lambda + \varphi + n\mu_v)^2 - 4n\mu_v\lambda}}{2n\mu_v}$$

$$\rho = \frac{\lambda}{\mu_r} \;, \quad \gamma = 2\mu_r + (n-2)\beta q$$

$$\gamma_1 = \frac{\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2\frac{\lambda}{n}}}{2\alpha_2} \;, \qquad \gamma_2 = \frac{\eta_1 + \sqrt{\eta_1^2 - 4\gamma\frac{\lambda}{n}}}{2\gamma}$$

$$\sigma = 2\mu_r + (N-2)\beta q$$

$$\sigma_1 = \frac{\tau_1 - \sqrt{\tau_1^2 - 4\tau_2\frac{\lambda}{N}}}{2\tau_2} \;, \sigma_2 = \frac{\epsilon_1 + \sqrt{\epsilon_1^2 - 4\sigma\frac{\lambda}{N}}}{2\sigma}$$

With $\alpha_1 = (\frac{\lambda}{n} + \varphi + 2\mu_v + (n-2)\beta q)$, $\alpha_2 = 2\mu_v + (n-2)\beta q$,

$$\eta_1 = (\frac{\lambda}{n} + 2\mu_r + (n-2)\beta q \;, \tau_1 = (\frac{\lambda}{N} + \varphi + 2\mu_v + (N-2)\beta q),$$

$$\tau_2 = 2\mu_v + (N-2)\beta q, \quad \epsilon_1 = (\frac{\lambda}{N} + 2\mu_r + (N-2)\beta q)$$

Proof: The same theorem was proved for the M/M/1 Model under repair in the paper [12]. Substituting the values of B_N, C_N, A_N from equations (2.4), (2.5), and (2.6), respectively into equation (3.1) and solving it for each interval like $1 \le n \le 2$, $3 \le n \le N-1$ and $n \ge N$, the results of the theorem are obtained.

3.1. *Steady-state probabilities*

Let steady-state probability be denoted by $P_{j,n}$ which means that the system is in state j (where $j = 0,1$), and n is the number of customers in the system.
Let $P_0 = P_{00}$ and $P_n = [\,P_{0,n}\,, P_{1,n}\,]$, $n = 1,2,3, \ldots$
In matrix form,

$$PQ = 0$$

where

$$P = [P_0,\, P_1,\, P_2, \ldots, P_N, \qquad P_{N+1}, \ldots]$$

and Q is given by equation (2.3)
From above, we have the following system of equations

$$P_0 A_{00} + P_1 B_0 = 0 \tag{3.1.1}$$

$$P_0 e_1 C_1 + P_1 A_1 + P_2 B_2 = 0 \tag{3.1.2}$$
$$P_{n-1} C_{n-1} + P_n A_n + P_{n+1} B_{n+1} = 0 \ , \ \ 2 \le n \le N \tag{3.1.3}$$
$$P_{n-1} C_N + P_n A_N + P_{n+1} B_N = 0 \ , \ n \ge N \tag{3.1.4}$$

where $e_1 = (1,0)$

As we know that the sum of probabilities is always equal to one.

Hence, we can write

$$P_{00} + \sum_{n=1}^{\infty} P_{0n} + \sum_{n=1}^{\infty} P_{1n} = 1$$

This implies,

$$P_0 = P_{00} = 1 - \sum_{n=1}^{\infty} P_{0n} - \sum_{n=1}^{\infty} P_{1n}$$

where

$$P_{0n} = P_n e_1^* = P_n (1,0)^T \text{ and } P_{1n} = P_n e_2^* = P_n (0,1)^T$$

Hence, by using equations (3.1.1) to (3.1.4) and the probability law of sum, we obtain

$$P_n = P_{n-1} R^n \ \ , \ \ 1 \le n \le N$$

and $P_n = P_N R^{n-N}$, $n \ge N+1$

with

$$R^k = \begin{bmatrix} \sigma_1^k & \frac{\sigma_1 \varphi \, (\sum_{i=1}^{k} \sigma_1^{k-i} \sigma_2^{i-1}}{\sigma (1-\sigma_1)} \\ 0 & \sigma_2^k \end{bmatrix} , \ k = 1,2,3, \dots \tag{3.1.5}$$

4. System Performance Measures

The following system performance measures for this queueing system are obtained here in this section.

(i) The probability that the system is in a regular busy period (P_r) and the probability that the system is in a working vacation period (P_v) is given by

$$P_v = \sum_{n=0}^{\infty} P_{0,n} \quad , \quad P_r = \sum_{n=1}^{\infty} P_{1,n}$$

(ii) Expected system size (L_s) is given by

$$L_s = L_{sv} + L_{sr} = \sum_{n=0}^{\infty} n P_{0,n} + \sum_{n=1}^{\infty} n P_{1,n}$$

where L_{sv} is the expected system size during the working vacation period and L_{sr} is the expected system size during the regular busy period.

(iii) Expected number of customers served (E_s) be

$$E_s = \mu_v P_v + \mu_r P_b$$

(iv) The expected waiting time in the system (E_w) be

$$E_w = \frac{L_s}{\lambda}$$

(v) The average Loss Rate (LR) is given by

LR= Average Discouragement Rate+ Average Reneging Rate

$$= \sum_{n=2}^{\infty} (\lambda - \lambda_n) P_n e_{11} + \sum_{n=2}^{\infty} \beta_n q\, P_n e_{11}$$

where $e_{11} = (1,1)^T$

5. Numerical Analysis

In this section, the effect of some parameters on various system performance measures of the queueing system will be studied. The data used is randomly chosen for the numerical analysis.

Let $\mu_v = 6,\ \mu_r = 8\ ,\ \varphi = 0.5\ , \beta = 3, q = 0.6\ , N = 45$

Table 5.1. Effect of **λ** on Various Performance Measures.

λ	P_0	L_s	E_w	E_s	LR
30	0.1471	3.4892	0.1130	6.8123	2.5832
35	0.1428	3.5324	0.1092	6.7098	3.9621
40	0.1396	3.8294	0.1051	6.6941	6.4034
45	0.1311	4.0121	0.1033	6.5824	9.1124
50	0.1258	4.1111	0.1026	6.5723	11.2348
55	0.1201	4.2129	0.1005	6.4296	13.6141
60	0.1183	4.3648	0.0985	6.3891	16.8132
65	0.1153	4.5621	0.0980	6.3342	18.9189
70	0.1129	4.6634	0.0978	6.3041	21.3481

Let $\lambda = 35,\ \mu_v = 3, \varphi = 0.5, \beta = 3, q = 0.6, N = 45$

Table 5.2. Effect of μ_r on various performance measures.

μ_r	P_0	L_s	E_w	E_s	LR
4	0.1549	4.6015	0.1261	3.0421	4.9724
5	0.1560	4.5389	0.1250	3.1480	4.9428
6	0.1579	4.5012	0.1241	3.1562	4.9132
7	0.1591	4.4987	0.1235	3.1791	4.8920
8	0.1601	4.4732	0.1226	3.1820	4.8631
9	0.1621	4.4720	0.1218	3.1947	4.7620
10	0.1638	4.4719	0.1215	3.2012	4.5265

From Table 5.1, it is evident that an increase in the arrival of customers (**λ**) increases the expected system size (L_s) and average loss rate (LR). It is also found that an increase in arrival rate has no significant impact on the values of P_0, E_w *and* E_s. The results are shown graphically in Figs. 5.1 and 5.2

From Table 5.2, it is evident that, due to an increase in the service rate in the regular busy period, there is an increase in the probability of an empty state (P_0), which is logical as a higher service rate will improve the speed of service, customers will get service in a short period, and so the probability of an empty system will increase. Further, the expected queue size (L_s) and expected waiting time in the system (E_s) decreases due to an increase in the service rate. Due to the improved service rate, there is also a decrease in the average loss rate (LR). Effects of the average service rate in the regular busy periods are shown graphically in Figs. 5.3 and 5.4.

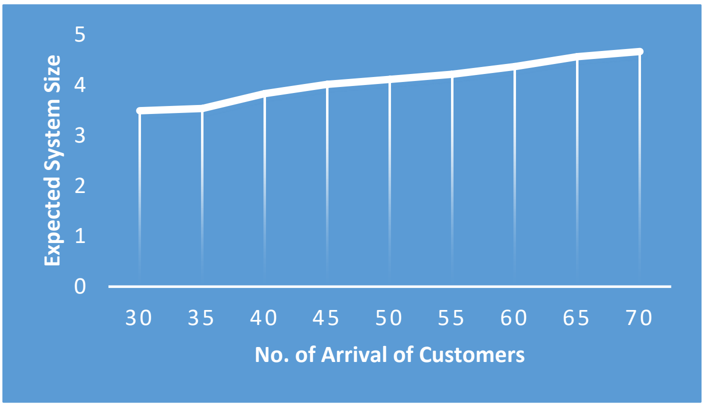

Fig. 5.1. Effect of arrivals (**λ**) on expected system Size (L_s).

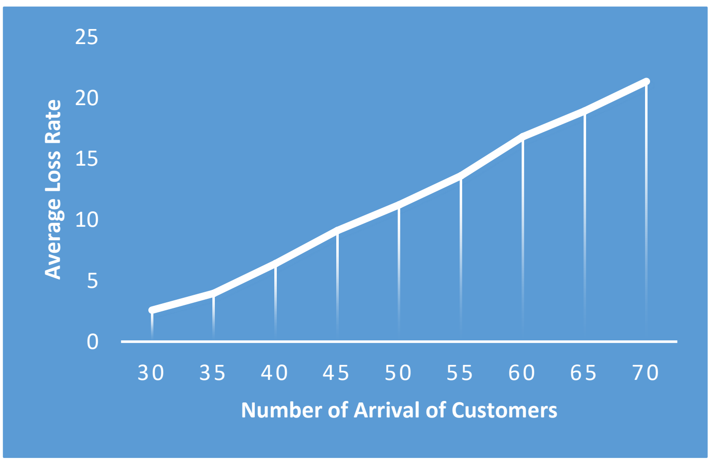

Fig. 5.2. Effect of arrivals (**λ**) on average loss rate (LR).

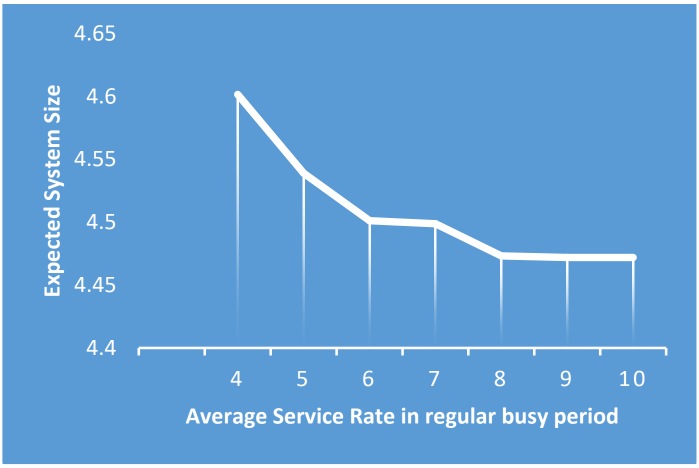

Fig. 5.3. Effect of average service rate in a regular busy period (μ_r) on expected system size (L_s).

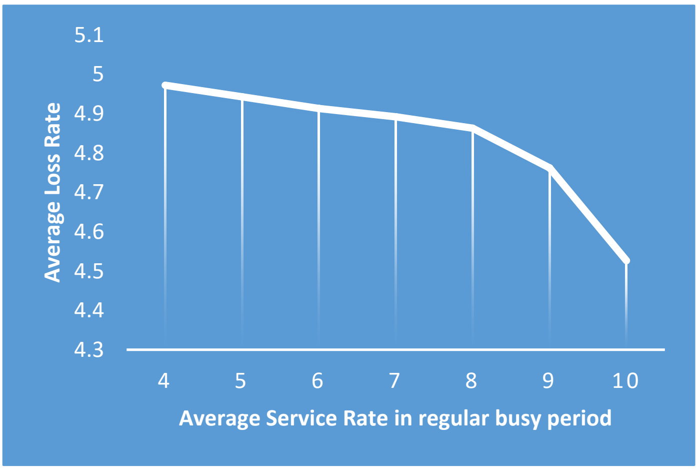

Fig. 5.4. Effect of average service rate in the regular busy period on average loss rate.

6. Cost Model

Practically, it is always seen that minimization of operating cost is demanded in each and every application of queueing models. Now, the following cost factors are applicable to construct a total expected cost function per unit of time in this queueing model.
The total expected cost function per unit of time is given by

$$F(x) = C_1 L_s + C_2 E_s + C_3 LR + 2C_4$$

(Here, in the cost function, 2 is used as there are 2 servers)
Where

$C_1 =$ Cost to hold each customer per unit of time in the system
$C_2 =$ Cost to serve each customer per unit of time

$C_3 =$ Cost of losing customers per unit of time
$C_4 =$ Cost per server

The main objective of this cost model is to obtain the minimum total expected cost per unit of time for the optimum value of x which is μ_r. To achieve this objective, an optimization technique is used, which is known as Quadratic Fit Search Method (QFSM).

6.1. *Algorithm for QFSM*

In this algorithm, all other parameters are kept constant. The following steps are involved in this algorithm:

Step 1: Start by selecting a 3-point pattern (x_0, x_1, x_2) with stopping tolerance k and set the iteration $i = 1$.
Step 2: Approximate optimal solution is x_1 if $|x_2 - x_0| \leq k$
Step 3: Calculate the optimum value x which is the quadratic fit given by

$$x = \frac{1}{2}\left[\frac{f(x_0)(x_1^2 - x_2^2) + f(x_1)(x_2^2 - x_0^2) + f(x_2)(x_0^2 - x_1^2)}{f(x_0)(x_1 - x_2) + f(x_1)(x_2 - x_0) + f(x_2)(x_0 - x_1)}\right]$$

If $x < x_1$, then move to step 5, and if $x > x_1$, then move to step 6.
Step 4: Now x must overlap with current x_1. If x_1 is away from x_0 than from x_2, adjust left $x \leftarrow x_1 - \frac{k}{2}$ and move to step 5. Otherwise, perturb right $x \leftarrow x_1 + \frac{k}{2}$ and move to step 6.
Step 5: Left: If $f(x_1)$ is superior than $f(x)$ (i.e.; less in case of minimization and greater in case of maximization), then update $x_0 \leftarrow x$, otherwise replace $x_2 \leftarrow x_1, x_1 \leftarrow x$. Another way, move from iteration i to $i + 1$ and return to step 2.
Step 6: Right: If $f(x_1)$ is superior than $f(x)$ (i.e., less in case of minimization and greater in case of maximization), then update $x_2 \leftarrow x$, otherwise replace $x_0 \leftarrow x_1, x_1 \leftarrow x$. Another way, move from iteration i to $i + 1$ and return to step 2.

Table 6.1. Search for optimum service rate (μ_r). ($\lambda = 35, \mu_v = 0.8, \varphi = 0.5, \beta = 3, q = 0.6, N = 45$).

i	x_0	x_1	x_2	x	$f(x)$
1	5.5	5.75	6.0	5.78	315.036
2	5.75	5.78	6.0	5.77324	315.036
3	5.75	5.77324	5.78	5.77321	315.036
4	5.75	5.77321	5.77324	5.77321	315.036
5	5.75	5.77321	5.77321	5.77322	315.038
6	5.75	5.77321	5.77322	5.77321	315.039
7	5.77321	5.77321	5.77322	5.77320	315.039
8	5.77321	5.77321	5.77322	5.77320	315.039

It is clear from Table 6.1 that using QFSM, the minimum total expected cost $F(\mu_r) =$ 315.039 is obtained after 8 iterations at the optimum regular busy service rate $\mu_r =$ 5.77320.

7. Conclusion

An M/M/2 queueing model with working vacations and reneging of customers is studied in this paper. A matrix geometric method is used to find the model's steady-state and steady-state probabilities. Further, the various system performance measures are also analyzed numerically and graphically. The result shows that an increase in arrival rate increases the expected system size and average loss rate, whereas an improved service rate reduces the expected queue size, expected waiting time, and average loss rate, although the probability of an empty state increases. A cost function is also constructed to obtain the optimal (minimum) cost function corresponding to the optimal service rate through Quadratic Fit Search Method (QFSM). The investigation shows that after some iterations, the minimum total expected cost is obtained at the optimum regular busy service rate.

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