JSR Publications

Available Online

J. Sci. Res. **15** (1), 111-119 (2023)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

Solution of Linear Volterra Integral Equation of Second Kind via Rishi Transform

S. Aggarwal[1], R. Kumar[2], J. Chandel[2*]

[1]Department of Mathematics, National Post Graduate College, Barhalganj, Gorakhpur-273402, Uttar Pradesh, India

[2]Department of Mathematics, D.S. College, Aligarh, Uttar Pradesh, India

Received 18 June 2022, accepted in final revised form 27 September 2022

Abstract

The solution of various problems of engineering and science can easily determined by representing these problems in integral equations. There are numerous analytical and numerical methods which can be used for solving different kinds of integral equations. In this paper, authors used recently developed integral transform "Rishi Transform" for obtaining the analytical solution of linear Volterra integral equation of second kind (LVIESK). For this, the kernel of LVIESK has assumed a convolution type kernel. Five numerical examples are considered for demonstrating the complete procedure of determining the solution. Results of these problems suggest that Rishi transform provides the exact analytical solution of LVIESK without doing complicated calculation work.

Keywords: Analytical Solution; Rishi Transform; Inverse Rishi Transform; Convolution; Volterra Integral Equation.

doi: http://dx.doi.org/10.3329/jsr.v15i1.60337 J. Sci. Res. **15** (1), 111-119 (2023)

MATHEMATICS SUBJECT CLASSIFICATION: 35A22; 44A05; 44A35; 45A05; 45D05

1. Introduction

Volterra integral equations have numerous applications in the diverse areas of Mechanics [1-2]; linear viscoelasticity [3]; hereditary phenomena [4]; renewal theory [5]; particle size statistics [6]; theory of superfludity [7]; damped vibration of a string [8]; heat transfer problem [9]; geometric probability [10]; viscoelastic stress analysis [11]; population dynamics [12-13] and study of epidemics [14]. Various differential and partial differential equations with initial conditions can be transformed into a single or multiple Volterra integral equations [15].

There are numerous analytical and numerical methods available for treatment of Volterra integral equations such as Laplace transform [16]; Mohand transform [17]; Aboodh transform [18]; Kamal transform [19]; Laplace-Carson transform (Mahgoub

* *Corresponding author*: jyotsnaraghuvanshi5@gmail.com

transform) [20]; Anuj transform [21]; Taylor's series method [22,23]; Runge-Kutta methods [24,25]; piecewise polynomial collocation method [26]; spline method [27,28]; implicit methods [29]; backward differentiation type method [30]; modified Runge-Kutta method [31] and finite difference method [32].

In the recent years, researchers developed numerous new integral transforms (Shehu [33]; Sumudu [34]; Natural [35]; Elzaki [36]; Aboodh [37]; Mahgoub [38]; Kamal [39]; ZZ [40]; Mohand [41]; Sadik [42]; Shehu [43]; Sawi [44]; Upadhyay [45]; Jafari [46]) and used them to handle the problems of Science and Engineering. Higazy *et al*. [47] introduced a new decomposition method "Sawi decomposition method" to determine the solution of Volterra integral equation. Aggarwal with other scholars [48-55] introduced the relations of duality among the established integral transformations. Ali *et al*. [56] determined the solution of fractional Volterra-Fredholm integro-differential equations under mixed boundary conditions by using the HOBW method.

Padder *et al*. [57] analyzed the tumor-immune response model by differential transformation method. Ali *et al.* [58] used HOBW method and determined the solution of the problem of nonlinear Volterra integral equations with weakly singular kernel. Higazy and Aggarwal [59] used Sawi transform and solved the complex problem of chain reaction in chemical kinetics by representing it into a system of ordinary differential equations. El-Mesady *et al*. [60] completely solved the problem of medical science by using Jafari transform. Higazy *et al*. [61] studied infections model of HIV-1 by the help of Shehu transform. Kumar *et al*. [62] developed a new integral transform "Rishi Transform" and determined the solution of first kind Volterra integral equations. Priyanka and Aggarwal [63] recently determined the solution of the model of the bacteria growth via Rishi transform.

The motive of the present paper is to determine the analytical solution of linear Volterra integral equation of second kind by using recently developed integral transform "Rishi Transform". Rishi transform is efficient for solving Volterra integral equation of second kind compare to other methods, which are available in the literature, because it provides the analytical results in compact form without any error and without doing complicated calculation work.

The residual of the present paper is planned in seven sections. Section 2 provides the nomenclature of the symbols. Section 3 gives the definition of Rishi transform. Section 4 deals the inverse Rishi transform. In section 5, solution of linear Volterra integral equation of second kind is given via Rishi transform. Section 6 contains five numerical examples for better explaining the complete procedure of determining the solution of LVIESK in detail. The paper ends in section 7 with the conclusion.

2. Nomenclature of Symbols

Υ, Rishi transform operator;
Υ^{-1}, inverse Rishi transform operator;
N, the set of natural numbers;

$\in$, belongs to;
!, the usual factorial notation;
Γ, the classical Gamma function;
L, Laplace transform operator;
R, the set of real numbers

3. Definition of Rishi Transform

The Rishi transform of a piecewise continuous exponential order function $F(t), t \geq 0$ is given by [62]

$$\Upsilon\{F(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty F(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt = T(\varepsilon,\sigma),\ \varepsilon > 0, \sigma > 0 \quad (1)$$

4. Inverse Rishi Transform [62]

The inverse rishi transform of $T(\varepsilon,\sigma)$, designated by $\Upsilon^{-1}\{T(\varepsilon,\sigma)\}$, is another function $F(t)$ having the property that $\Upsilon\{F(t)\} = T(\varepsilon,\sigma)$.
Some useful operational characteristics of Rishi transform, Rishi transforms of some fundamental functions and their inverse Rishi transforms are summarized in the Tables1-3 respectively.

Table 1. Some operational characteristics of Rishi transform [62].

S. N.	Name of Characteristic	Mathematical Form
1	Linearity	$\Upsilon\{\sum_{i=1}^{n} k_i F_i(t)\} = \sum_{i=1}^{n} k_i \Upsilon\{F_i(t)\}$, where k_i are arbitrary constants
2	Change of Scale	If $\Upsilon\{F(t)\} = T(\varepsilon,\sigma)$ then $\Upsilon\{F(kt)\} = \frac{1}{k^2}T\left(\frac{\varepsilon}{k},\sigma\right)$
3	Translation	If $\Upsilon\{F(t)\} = T(\varepsilon,\sigma)$ then $\left\{\Upsilon\{e^{kt}F(t)\} = \left(\frac{\varepsilon - k\sigma}{\varepsilon}\right) T(\varepsilon - k\sigma,\sigma)\right\}$
4	Convolution	If $\Upsilon\{F_1(t)\} = T_1(\varepsilon,\sigma)$ and $\Upsilon\{F_2(t)\} = T_2(\varepsilon,\sigma)$ then $\left\{\Upsilon\{F_1(t) * F_2(t)\} = \left[\left(\frac{\varepsilon}{\sigma}\right) T_1(\varepsilon,\sigma)\, T_2(\varepsilon,\sigma)\right]\right\}$

Table 2. Some fundamental functions and their Rishi transform [62].

S. N.	$F(t), t > 0$	$\Upsilon\{F(t)\} = T(\varepsilon,\sigma)$
1	1	$\left(\frac{\sigma}{\varepsilon}\right)^2$
2	e^{lt}	$\frac{\sigma^2}{\varepsilon(\varepsilon - l\sigma)}$
3	$t^\rho, \rho \in N$	$\rho!\left(\frac{\sigma}{\varepsilon}\right)^{\rho+2}$
4	$t^\rho, \rho > -1, \rho \in R$	$\left(\frac{\sigma}{\varepsilon}\right)^{\rho+2}\Gamma(\rho + 1)$
5	$sinlt$	$\frac{l\sigma^3}{\varepsilon(\varepsilon^2 + \sigma^2 l^2)}$

6	$coslt$	$\frac{\sigma^2}{(\varepsilon^2+\sigma^2 l^2)}$
7	$sinhlt$	$\frac{l\sigma^3}{\varepsilon(\varepsilon^2-\sigma^2 l^2)}$
8	$coshlt$	$\frac{\sigma^2}{(\varepsilon^2-\sigma^2 l^2)}$

Table 3. Inverse Rishi transformations of some fundamental functions [63].

S. N.	$T(\varepsilon,\sigma)$	$F(t)=\Upsilon^{-1}\{T(\varepsilon,\sigma)\}$
1	$\left(\frac{\sigma}{\varepsilon}\right)^2$	1
2	$\frac{\sigma^2}{\varepsilon(\varepsilon-l\sigma)}$	e^{lt}
3	$\left(\frac{\sigma}{\varepsilon}\right)^{\rho+2}, \rho\in N$	$\frac{t^\rho}{\rho!}$
4	$\left(\frac{\sigma}{\varepsilon}\right)^{\rho+2}, \rho>-1, \rho \in R$	$\frac{t^\rho}{\Gamma(\rho+1)}$
5	$\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2 l^2)}$	$\frac{sinlt}{l}$
6	$\frac{\sigma^2}{(\varepsilon^2+\sigma^2 l^2)}$	$coslt$
7	$\frac{\sigma^3}{\varepsilon(\varepsilon^2-\sigma^2 l^2)}$	$\frac{sinhlt}{l}$
8	$\frac{\sigma^2}{(\varepsilon^2-\sigma^2 l^2)}$	$coshlt$

5. Solution of Linear Volterra Integral Equation of Second Kind via Rishi Transform

The general form of LVIESK is given by [1]

$$\Theta(t)=F(t)+\lambda\int_0^t K(t,\zeta)\,\Theta(\zeta)d\zeta \tag{2}$$

where

$$\left.\begin{aligned}\Theta(t)&=unknown\ function\\ F(t)&=known\ function\\ K(t,\zeta)&=kernel\ of\ integral\ equation\\ \lambda&=a\ non-zero\ parameter\end{aligned}\right\}$$

In this work, we have assumed that the kernel of equation (2) is a convolution type kernel, i.e. $K(t,\zeta)=K(t-\zeta)$. So equation (2) takes the following form

$$\Theta(t)=F(t)+\lambda\int_0^t K(t-\zeta)\,\Theta(\zeta)d\zeta \tag{3}$$

Operating Rishi transform on equation (3), we get

$$\Upsilon\{\Theta(t)\}=\Upsilon\{F(t)\}+\lambda\Upsilon\left\{\int_0^t K(t-\zeta)\,\Theta(\zeta)d\zeta\right\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\}=\Upsilon\{F(t)\}+\lambda\Upsilon\{K(t)*\Theta(t)\} \tag{4}$$

Use of convolution theorem in equation (4) gives

$$\Upsilon\{\Theta(t)\} = \Upsilon\{F(t)\} + \lambda\left(\frac{\varepsilon}{\sigma}\right)\Upsilon\{K(t)\}\Upsilon\{\Theta(t)\}$$

$$\Upsilon\{\Theta(t)\} = \frac{\Upsilon\{F(t)\}}{\left[1-\lambda\left(\frac{\varepsilon}{\sigma}\right)\Upsilon\{K(t)\}\right]} \quad (5)$$

After operating inverse Rishi transform on equation (5), the solution of equation (2) is given by

$$\Theta(t) = \Upsilon^{-1}\left\{\frac{\Upsilon\{F(t)\}}{\left[1-\lambda\left(\frac{\varepsilon}{\sigma}\right)\Upsilon\{K(t)\}\right]}\right\}$$

6. Numerical Examples

This section contains five numerical examples for better explaining the complete procedure of determining the solution of LVIESK with convolution type kernel in detail.

Example: 1 Consider the following LVIESK with convolution type kernel given by [33] as

$$\Theta(t) = sint + 2\int_0^t e^{(t-\zeta)}\,\Theta(\zeta)d\zeta \quad (6)$$

Operating Rishi transform on equation (6), we get

$$\Upsilon\{\Theta(t)\} = \Upsilon\{sint\} + 2\Upsilon\left\{\int_0^t e^{(t-\zeta)}\,\Theta(\zeta)d\zeta\right\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \Upsilon\{sint\} + 2\Upsilon\{e^t * \Theta(t)\} \quad (7)$$

Using convolution theorem in equation (7), we have

$$\Upsilon\{\Theta(t)\} = \Upsilon\{sint\} + 2\left(\frac{\varepsilon}{\sigma}\right)\Upsilon\{e^t\}\Upsilon\{\Theta(t)\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)} + 2\left(\frac{\varepsilon}{\sigma}\right)\left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right]\Upsilon\{\Theta(t)\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \frac{1}{5}\left[\frac{\sigma^2}{\varepsilon(\varepsilon-3\sigma)}\right] - \frac{1}{5}\left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right] + \frac{2}{5}\left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right] \quad (8)$$

After operating inverse Rishi transform on equation (8), the solution of equation (6) is given by

$$\Theta(t) = \Upsilon^{-1}\left\{\frac{1}{5}\left[\frac{\sigma^2}{\varepsilon(\varepsilon-3\sigma)}\right] - \frac{1}{5}\left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right] + \frac{2}{5}\left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right]\right\}$$

$$\Rightarrow \Theta(t) = \frac{1}{5}\Upsilon^{-1}\left\{\left[\frac{\sigma^2}{\varepsilon(\varepsilon-3\sigma)}\right]\right\} - \frac{1}{5}\Upsilon^{-1}\left\{\left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right]\right\} + \frac{2}{5}\Upsilon^{-1}\left\{\left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right]\right\}$$

$$\Rightarrow \Theta(t) = \left(\frac{1}{5}e^{3t} - \frac{1}{5}cost + \frac{2}{5}sint\right).$$

Remark: The same analytical solution obtained using Rishi transform as given [33] but without any tedious calculation work.

Example: 2 Consider the following LVIESK with convolution type kernel given [33] as

$$\Theta(t) = cost + sint - \int_0^t \Theta(\zeta)\,d\zeta \quad (9)$$

Operating Rishi transform on equation (9), we get

$$\Upsilon\{\Theta(t)\} = \Upsilon\{cost\} + \Upsilon\{sint\} - \Upsilon\left\{\int_0^t \Theta(\zeta)\,d\zeta\right\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \Upsilon\{cost\} + \Upsilon\{sint\} - \Upsilon\{1 * \Theta(t)\} \quad (10)$$

Using convolution theorem in equation (10), we have

$$\Upsilon\{\Theta(t)\} = \Upsilon\{cost\} + \Upsilon\{sint\} - \left(\frac{\varepsilon}{\sigma}\right)\Upsilon\{1\}\Upsilon\{\Theta(t)\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right] + \left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right] - \left(\frac{\varepsilon}{\sigma}\right)\left(\frac{\sigma}{\varepsilon}\right)^2 \Upsilon\{\Theta(t)\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right] \quad (11)$$

After operating inverse Rishi transform on equation (11), the solution of equation (9) is given by

$$\Theta(t) = \Upsilon^{-1}\left\{\left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right]\right\}$$

$$\Rightarrow \Theta(t) = (cost).$$

Remark: The same analytical solution obtained using Rishi transform as given [33] but without doing any complicated calculation work.

Example: 3 Consider the following LVIESK with convolution type kernel given [23] as

$$\Theta(t) = t - \int_0^t (t-\zeta)\Theta(\zeta)\,d\zeta \quad (12)$$

Operating Rishi transform on equation (12), we get

$$\Upsilon\{\Theta(t)\} = \Upsilon\{t\} - \Upsilon\left\{\int_0^t (t-\zeta)\Theta(\zeta)\,d\zeta\right\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \Upsilon\{t\} - \Upsilon\{t * \Theta(t)\} \quad (13)$$

Using convolution theorem in equation (13), we have

$$\Upsilon\{\Theta(t)\} = \Upsilon\{t\} - \left(\frac{\varepsilon}{\sigma}\right)\Upsilon\{t\}\Upsilon\{\Theta(t)\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \left(\frac{\sigma}{\varepsilon}\right)^3 - \left(\frac{\varepsilon}{\sigma}\right)\left(\frac{\sigma}{\varepsilon}\right)^3 \Upsilon\{\Theta(t)\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right] \quad (14)$$

After operating inverse Rishi transform on equation (14), the solution of equation (12) is given by

$$\Theta(t) = \Upsilon^{-1}\left\{\left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right]\right\}$$

$$\Rightarrow \Theta(t) = (sint).$$

Remark: Rishi transform provides the exact analytical solution of the above problem without doing large computational work while Taylor's series method [23] provides the approximate result of this problem.

Example: 4 Consider the following LVIESK with convolution type kernel given [33] as

$$\Theta(t) = 1 - t - \int_0^t (t-\zeta)\Theta(\zeta)\,d\zeta \quad (15)$$

Operating Rishi transform on equation (15), we get

$$\Upsilon\{\Theta(t)\} = \Upsilon\{1\} - \Upsilon\{t\} - \Upsilon\left\{\int_0^t (t-\zeta)\Theta(\zeta)\,d\zeta\right\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \Upsilon\{1\} - \Upsilon\{t\} - \Upsilon\{t * \Theta(t)\} \quad (16)$$

Using convolution theorem in equation (16), we have

$$\Upsilon\{\Theta(t)\} = \Upsilon\{1\} - \Upsilon\{t\} - \left(\frac{\varepsilon}{\sigma}\right)\Upsilon\{t\}\Upsilon\{\Theta(t)\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \left(\frac{\sigma}{\varepsilon}\right)^2 - \left(\frac{\sigma}{\varepsilon}\right)^3 - \left(\frac{\varepsilon}{\sigma}\right)\left(\frac{\sigma}{\varepsilon}\right)^3 \Upsilon\{\Theta(t)\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right] - \left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right] \quad (17)$$

After operating inverse Rishi transform on equation (17), the solution of equation (15) is given by

$$\Theta(t) = \Upsilon^{-1}\left\{\left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right]\right\} - \Upsilon^{-1}\left\{\left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right]\right\}$$

$$\Rightarrow \Theta(t) = (cost - sint).$$

Remark: Rishi transform method provides the exact analytical solution of the above problem without doing large computational work is given elsewhere [33].

Example: 5 Consider the following LVIESK with convolution type kernel given by [47] as

$$\Theta(t) = t + \int_0^t \Theta(\zeta)\, d\zeta \quad (18)$$

Operating Rishi transform on equation (18), we get

$$\Upsilon\{\Theta(t)\} = \Upsilon\{t\} + \Upsilon\left\{\int_0^t \Theta(\zeta)\, d\zeta\right\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \Upsilon\{t\} + \Upsilon\{1 * \Theta(t)\} \quad (19)$$

Using convolution theorem in equation (19), we have

$$\Upsilon\{\Theta(t)\} = \Upsilon\{t\} + \left(\frac{\varepsilon}{\sigma}\right)\Upsilon\{1\}\Upsilon\{\Theta(t)\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \left(\frac{\sigma}{\varepsilon}\right)^3 + \left(\frac{\varepsilon}{\sigma}\right)\left(\frac{\sigma}{\varepsilon}\right)^2 \Upsilon\{\Theta(t)\}$$

$$\Rightarrow \Upsilon\{\Theta(t)\} = \left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right] - \left[\left(\frac{\sigma}{\varepsilon}\right)^2\right] \quad (20)$$

After operating inverse Rishi transform on equation (20), the solution of equation (18) is given by

$$\Theta(t) = \Upsilon^{-1}\left\{\left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right]\right\} - \Upsilon^{-1}\left\{\left[\left(\frac{\sigma}{\varepsilon}\right)^2\right]\right\}$$

$$\Rightarrow \Theta(t) = (e^t - 1).$$

Remark: Rishi transform provides the exact analytical solution of the above problem without doing large computational work while Sawi decomposition method [47] provides the approximate result of this problem.

7. Conclusion

In the presented paper, authors successfully obtained the solution of LVIESK using Rishi transform. The findings of the presented paper indicate that the Rishi transform is a very efficient integral transform for obtaining the exact solution of LVIESK without doing tedious and large computational work. In future, Rishi transform can be use for solving the complex problems of science and engineering which can be transformed into a single or multiple Volterra integral equations.

Data Availability

Authors of this paper confirm that the datasets that are used in this paper are available from the author upon request.

References

1. A. M. Wazwaz, Linear and Nonlinear Integral Equations: Methods and Applications (Higher Education Press, Beijing, 2011). https://doi.org/10.1007/978-3-642-21449-3
2. A. Jerri, Introduction to Integral Equations with Applications (Wiley, New York, 1999).
3. C. M. Dafermos, J. Differential Equat. **7**, 554 (1970). https://doi.org/10.1016/0022-0396(70)90101-4
4. N. Distefano, J. Math. Anal. Appl. **23**, 365 (1968). https://doi.org/10.1016/0022-247X(68)90074-7
5. W. Feller, Ann. Math. Stat. **12**, 243 (1941). https://doi.org/10.1214/aoms/1177731708
6. P. L. Goldsmith, Brit. J. Appl. Phys. **18**, 813 (1967). https://doi.org/10.1088/0508-3443/18/6/317
7. N. Levinson, J. Math. Anal. Appl. **1**, 1 (1960). https://doi.org/10.1016/0022-247X(60)90028-7
8. S. P. Lin, J. Fluid Mech. **72**, 787 (1975). https://doi.org/10.1017/S002211207500328X
9. W. R. Mann and F. Wolf, Quart. Appl. Math. **9**, 163 (1951). https://doi.org/10.1090/qam/42596
10. J. R. Phillips, Biometrika **53**, 365 (1966). https://doi.org/10.2307/2333643
11. T. G. Rogers and E. H. Lee, Quart. Appl. Math. **22**, 117 (1964). https://doi.org/10.1090/qam/167050
12. C. C. Shilepsky, J. Math. Anal. Appl. **48**, 764 (1974). https://doi.org/10.1016/0022-247X(74)90150-4
13. K. E. Swick, SIAM J. Appl. Math. **40**, 266 (1981). https://doi.org/10.1137/0140023
14. F. J. S. Wang, SIAM J. Math. Anal. **9**, 529 (1978). https://doi.org/10.1137/0509034
15. R. P. Kanwal, Linear Integral Equations (Birkhauser, Boston, 1997). https://doi.org/10.1007/978-1-4612-0765-8
16. R. Chauhan and S. Aggarwal, J. Adv. Res. Appl. Math. Stat. **4**, 1 (2019).
17. S. Aggarwal, N. Sharma, and R. Chauhan, Int. J. Res. Adv. Technol. **6**, 3098 (2018).
18. S. Aggarwal, N. Sharma, and R. Chauhan, Asian Resonance 7, 156 (2018).
19. S. Aggarwal, R. Chauhan, and N. Sharma, Int. J. Latest Technol. Eng. Manag. Appl. Sci. **7**, 138 (2018).
20. S. Aggarwal, R. Chauhan, and N. Sharma, Asian Resonance 7, 46 (2018).
21. A. Kumar, S. Bansal, and S. Aggarwal, Design Eng. **2021**, 12741 (2021).
22. A. Goldfine, Math. Comp. **31**, 691 (1977). https://doi.org/10.1090/S0025-5718-1977-0440970-4
23. S. Aggarwal, K. Bhatnagar, and A. Dua, Int. J. Res. Innov. Appl. Sci. **5**, 32 (2020).
24. C. T. H. Baker, Runge-Kutta Methods for Volterra Integral Equations of the Second Kind, Lecture Notes in Mathematics 630 (Springer-Verlag, Berlin, 1978).
25. H. Brunner, E. Hairer, and S. P. Norsett, Math. Comp. **19**, 147 (1982). https://doi.org/10.1090/S0025-5718-1982-0658219-8
26. H. Brunner and M. D. Evans, J. Inst. Math. Appl. **20**, 415 (1977). https://doi.org/10.1093/imamat/20.4.415
27. M. E. A. EL Tom, J. Inst. Math. Appl. **8**, 354 (1971). https://doi.org/10.1093/imamat/8.3.354
28. H. S. Hung, The Numerical Solution of Differential and Integral Equations by Spline Functions, Tech. Summary Rept. 1053 (Math. Res. Center, Univ. Wisconsin, Madison, 1970).
29. L. Garey, Implicit Methods for Volterra Integral Equations of the Second Kind - *Proc. Second Manitoba Conference on Numerical Mathematics* (1972). https://doi.org/10.1007/BF01932304
30. P. J. V. D. Houwen and H. J. J. T. Riele, Numer. Math. **37**, 205 (1981). https://doi.org/10.1007/BF01398253

31. P. J. V. D. Houwen, P. H. M. Wolkenfelt, and C. T. H. Baker, IMA J. Numer. Anal. **1**, 303 (1981). https://doi.org/10.1093/imanum/1.3.303
32. P. Linz, The Numerical Solution of Volterra Integral Equations by Finite Difference Methods, Tech. Summary Rept. 825 (Math. Res. Center, Univ. Wisconsin, Madison, 1967).
33. S. Aggarwal, A. Vyas, and S. D. Sharma, Int. J. Latest Technol Eng. Manag. Appl. Sci. **9**, 26 (2020).
34. G. K. Watugula, Int. J. Math. Educat. Sci. Technol. **24**, 35 (1993).
35. Z. H. Khan and W. A. Khan, NUST J. Eng. Sci. **1**, 127 (2008).
36. T. M. Elzaki, Global J. Pure Appl. Math. **1**, 57 (2011).
37. K. S. Aboodh, Global J. Pure Appl. Math. **9**, 35 (2013).
38. M. A. M. Mahgoub, Adv. Theor. Appl. Math. **11**, 391 (2016).
39. K. Abdelilah and S. Hassan, Adv. Theor. Appl. Math. **11**, 451 (2016).
40. Zafar, Z. Ul Abadin, Int. J. Adv. Eng. Global Technol. **4**, 1605 (2016).
41. M. Mohand and A. Mahgoub, Adv. Theor. Appl. Math. **12**, 113 (2017).
42. L. S. Sadikali, Am. Int. J. Res. Sci. Technol. Eng. Math. **22**, 100 (2018).
43. S. Maitama and W. Zhao, Int. J. Anal. Applicat. **17**, 167 (2019).
44. M. M. A. Mahgoub, Adv. Theor. Appl. Math. **14**, 81 (2019).
45. L. M. Upadhyaya, Bul. Pure Appl. Sci. 38E, 471 (2019). https://doi.org/10.5958/2320-3226.2019.00051.1
46. H. Jafari, J. Adv. Res. **32**, 133 (2021). https://doi.org/10.1016/j.jare.2020.08.016
47. M. Higazy, S. Aggarwal, and A. N. Taher, J. Math. **2020**, 1 (2020). https://doi.org/10.1155/2020/6687134
48. S. Aggarwal and A. R. Gupta, Int. J. Recent Technol. Eng. **8**, 843 (2019).
49. S. Aggarwal and A. R. Gupta, Int. J. Recent Technol. Eng. **8**, 5978 (2019).
50. S. Aggarwal, K. Bhatnagar, and A. Dua, Int. J. Innov. Technol. Expl. Eng. **8**, 4312 (2019). https://doi.org/10.35940/ijitee.L2729.1081219
51. R. Chauhan, N. Kumar, and S. Aggarwal, Int. J. Innovative Technol. Explor. Eng. **8**, 1654 (2019).
52. S. Aggarwal and K. Bhatnagar, Int. J. Eng. Adv. Technol. **9**, 936 (2019).
53. R. Chaudhary, S. D. Sharma, N. Kumar, and S. Aggarwal, Int. J. Innov. Technol. Expl. Eng. **9**, 1465 (2019).
54. R. Mishra, S. Aggarwal, L. Chaudhary, and A. Kumar, Int. J. Innov. Technol. Expl. Eng. **9**, 153 (2020).
55. S. Aggarwal, N. Sharma, and R. Chauhan, SN Appl. Sci. **2**, 135 (2020). https://doi.org/10.1007/s42452-019-1896-z
56. M. R. Ali, A. R. Hadhoud, and H. M. Srivastava, Adv. Diff. Equat. **2019**, 1 (2019). https://doi.org/10.1186/s13662-019-2044-1
57. M.A. Padder, Afroz, and A. Khan, J. Sci. Res. **14**, 243 (2022). https://doi.org/10.3329/jsr.v14i1.55065
58. M. R. Ali, M. M. Mousa, and W. –X. Ma, Adv. Math. Phys. **2019**, ID 1705651 (2019).
59. M. Higazy and S. Aggarwal, Ain Shams Eng. J. **12**, 3173 (2021). https://doi.org/10.1016/j.asej.2021.01.027
60. A. I. El-Mesady, Y. S. Hamed, and A. M. Alsharif, Fractal Fract. **5**, 130 (2021). https://doi.org/10.3390/fractalfract5030130
61. M. Higazy, S. Aggarwal, and Y. S. Hamed, J. Math. **2020**, 1 (2020). https://doi.org/10.1155/2020/6624794
62. R. Kumar, J. Chandel, and S. Aggarwal, J. Sci. Res. **14**, 521 (2022). https://doi.org/10.3329/jsr.v14i2.56545
63. Priyanka and S. Aggarwal, J. Adv. Res. Appl. Math. Stat. **7**, 5 (2022).