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Solution of Inhomogeneous Linear Fractional Differential Equations Involving Jumarie Fractional Derivative

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Abstract

This paper presents a method for solving inhomogeneous linear sequential fractional differential equations with constant coefficients (ILSFDE) involving Jumarie fractional derivatives in terms of Mittag-Leffler functions. For this purpose, the fundamental properties of the Jumarie derivative and Mittag-Leffler functions are given. After this, the successive jumarie fractional derivatives of Mittag-Leffler functions, fractional cosine, and sine functions are obtained. Further, we determined the particular integrals of these functions and then found the complete solutions of ILSFDE. in terms of Mittag-Leffler functions, fractional cosine, and sine functions. We have demonstrated this developed method with a few examples of ILSFDE. This method is similar to the method for finding the complete solutions of classical differential equations with constant coefficients.

Keywords: Jumarie derivative; Mittag Leffler function; Fractional differential Equation.

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1. Introduction

Fractional differential equations (FDE) and their solutions have applications in applied sciences, biology, and engineering. These equations have proved to be valuable tools in modeling multiple physical and technical phenomena. The scope of fractional derivatives for modeling phenomena in several fields is due to its nonlocal nature, an inherent characteristic of many complex systems [1].

The concept of fractional calculus was first pioneered by Newton and Leibniz in 1695. It was in the seventeenth century when L. Hospital asked Leibniz the meaning of $\frac{d^n}{dx^n}(x)$ if $n = \frac{1}{2}$ to which Leibniz replied that it would be an apparent paradox, "$d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx:x}$" [2]. It introduced "fractional calculus," a new discipline of mathematics. Later Lacroix further explained this concept of the fractional derivative using the Gamma function [2]. Liouville and Riemann provided the first definition of the fractional order derivative near the end of the nineteenth century. In 1967 Caputo further modified it. Some definitions of fractional derivatives are given below:

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Definition 1.1. The Riemann-Liouville (R-L) left fractional derivative is defined as [3]

$$ {}_{a}\mathfrak{D}_{x}^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha+1)} \left(\frac{d}{dx}\right)^{m+1} \int_{a}^{x}(x-u)^{m-\alpha} f(u)\, du \quad (1) $$

where $m \leq \alpha < m+1$, m is positive integer.
particularly when, $0 \leq \alpha < 1$, then

$$ {}_{a}\mathfrak{D}_{x}^{\alpha}f(x) = \frac{1}{\Gamma(-\alpha+1)} \frac{d}{dx} \int_{a}^{x}(x-u)^{-\alpha} f(u)\, du \quad (2) $$

The right R-L fractional derivative is defined as;

$$ {}_{x}\mathfrak{D}_{b}^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha+1)} \left(-\frac{d}{dx}\right)^{m+1} \int_{x}^{b}(u-x)^{m-\alpha} f(u)\, du \quad (3) $$

Where $m \leq \alpha < m+1$, m is a positive integer.
The classical derivative for a constant always gives zero, but Riemann- Liouville (R-L) definitions (left & right) give a non-zero value for the derivative of a constant.
Definition 1.2. To overcome the drawback of the R-L definition (non-zero value of derivative of a constant) of fractional derivative, in 1967 Prof. Caputo modified it. According to Caputo, a fractional derivative is as follows: [4];

$$ {}_{a}\mathfrak{D}_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x}(x-u)^{n-\alpha-1}\, f^{n}(u)\, du \quad (4) $$

where $n-1 \leq \alpha < n$
According to this definition first, differentiate $f(x)$, n times, and then integrate. The FDE of Caputo's type and classical differential equation have similar initial conditions while the R-L type differential equation has initial conditions of fractional types i.e., $\lim_{x\to a} aD_{x}^{\alpha-1}f(x) = b$. Caputo's definition was also having a shortcoming in that the function $f(x)$ must be differentiable n times then the derivative of order α will exist, where $n-1 \leq \alpha < n$. Thus, this method becomes inapplicable for non-differentiable functions.

For finding the Caputo fractional derivative of a function, the function must be differentiable. Jumarie then modified the Riemann–Liouville definition of a fractional derivative to deal with non-differentiable functions.
Definition 1.3. Jumarie modified the previous definition of derivative of fractional order for the function $f(x)$ in the interval $[a, b]$ as follows [5-7];

$$ \mathfrak{D}_{\mathcal{J}}^{\alpha}f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_{a}^{x}(x-u)^{-\alpha-1} f(u)\, du, & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{a}^{x}(x-u)^{-\alpha}[f(u)-f(a)]\, du, & 0 < \alpha < 1 \\ [f^{\alpha-m}(x)]^{m}, & m \leq \alpha < m+1 \end{cases} \quad (5) $$

If $x < a$ then $f(x) - f(a) = 0$. The first line in "Eq. (5)" represents fractional integration; the second expression is for RL derivative of order $0 < \alpha < 1$ of offset function i.e $f(x) - f(a)$. The third expression is used for $\alpha > 1$. This definition exhibits all the properties of fractional derivatives consistently. This type of fractional derivative of a constant is zero, which was a notable drawback of the RL fractional derivative definition.

The classical differential equation with a non-integer order is just a specialization of FDE. There is various integral transform (Laplace transform, Mahgoub transform, Sawi transformation, Jafari transformation, Shehu transform, etc.) to solve the classical differential equation of integer order [8-12]. Some authors [13,14] have extensively studied the solutions of the fractional differential equation via integral transform (Fractional Fourier transform, Sumudu Transform, etc.). The differential transform method [15], the exponential-function approach [16], the homotopy perturbation method [17], the variation iteration method [18], and the fractional sub-equation method [19] are also a few of the methods used to solve FDE. Thus, approaches to solving FDE and their interpretations are emerging fields of applied mathematics.

There is no standard approach to solving linear sequential fractional differential equations (LSFDE) to date because the different forms of fractional derivatives give different types of solutions. The current study presents an analytical approach for the solutions of LSFDE with constant coefficients using jumarie fractional derivatives. This approach involves obtaining complementary functions and particular integrals of LSFDE.

1.1. *Mittag-Leffler function*

The Swedish mathematician Gosta Mittag-Leffler introduced the Mittag-Leffler function which is the generalization of the exponential function in 1903 [20]. The Mittag-Leffler function in one parameter is defined as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}, \qquad z \in C, \qquad Re(\alpha) > 0 \tag{6}$$

The Mittag-Leffler function in two parameters is defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \qquad z, \beta \in C, \qquad Re(\alpha) > 0 \tag{7}$$

Jumarie [21] defined the complex Mittag-Leffler function in the following form

$$E_\alpha(iax^\alpha) = \cos_\alpha ax^\alpha + i\sin_\alpha ax^\alpha \tag{8}$$

$$E_\alpha(-iax^\alpha) = \cos_\alpha(ax^\alpha) - i\sin_\alpha(ax^\alpha) \tag{9}$$

On behalf of "Eq. (8)" and "Eq. (9)", it can be written as

$$\cos_\alpha x^\alpha = \frac{E_\alpha(iax^\alpha)+E_\alpha(-iax^\alpha)}{2} \text{ and } \sin_\alpha ax^\alpha = \frac{E_\alpha(iax^\alpha)-E_\alpha(-iax^\alpha)}{2i}$$

The series of these fractional functions can be expanded as follows

$$\cos_\alpha ax^\alpha = 1 - \frac{a^2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{a^4x^{4\alpha}}{\Gamma(1+4\alpha)} - \frac{a^6x^{6\alpha}}{\Gamma(1+6\alpha)} + \cdots \ldots \ldots \ldots ..$$

$$Sin_\alpha ax^\alpha = \frac{ax^\alpha}{\Gamma(1+\alpha)} - \frac{a^3x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{a^5x^{5\alpha}}{\Gamma(1+5\alpha)} - \cdots \ldots \ldots \ldots ..$$

$$\cos_\alpha x^\alpha = \sum_{r=0}^{\infty}(-1)^r \frac{a^{2r}x^{2r\alpha}}{\Gamma(1+2r\alpha)} \tag{10}$$

$$Sin_\alpha ax^\alpha = \sum_{r=0}^{\infty}(-1)^r \frac{a^{2r+1}\, x^{(2r+1)\alpha}}{\Gamma(1+(2r+1)\alpha} \tag{11}$$

Corollary 1.1. The following equalities hold, which are [22] *(a).* $D^{\alpha}[x^{\gamma}] = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \ \gamma > 0$ (12)

(b). $D^{n+\vartheta}[x^{\gamma}] = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-n-\vartheta)} x^{\gamma-n-\vartheta}, \ 0 < \vartheta < 1$ (13)

2. Linear Fractional Differential Equation

A linear fractional differential equation is a generalized form of the classical linear differential equation. If the orders of derivatives are in a sequence in a fractional differential equation, then it is called a sequential fractional differential equation. The differential equation of the form

$$a_0 \frac{d^{n\alpha}y}{dx^{n\alpha}} + a_1 \frac{d^{(n-1)\alpha}y}{dx^{(n-1)\alpha}} + a_2 \frac{d^{(n-2)\alpha}y}{dx^{(n-2)\alpha}} + \cdots \ldots . + a_{n-1} \frac{d^{\alpha}y}{dx^{\alpha}} + a_n y = \Omega(x) \quad (14)$$

is called a sequential linear fractional differential equation having constant coefficients when $a_{0,} \ a_{1,}, \ a_{2,} \ldots \ldots \ldots .. , \ a_n$ are all constants and $\Omega(x)$ is a function of x while α is a non integer.

The part $\frac{d^{n\alpha}}{dx^{n\alpha}}$ of the fractional differential coefficient $\frac{d^{n\alpha}y}{dx^{n\alpha}}$ may be regarded as Jumarie's fractional derivative, which is a modified Riemann–Liouville derivative. So, "Eq. (14)" becomes

$$\left(a_0 \mathfrak{D}_{\mathcal{J}}^{n\alpha} + a_1 \mathfrak{D}_{\mathcal{J}}^{(n-1)\alpha} + a_2 \mathfrak{D}_{\mathcal{J}}^{(n-2)\alpha} + \cdots \ldots \ldots + a_{n-1} \mathfrak{D}_{\mathcal{J}}^{\alpha} + a_n\right) y = \Omega(x) \quad (15)$$

Where $a_{0,} \ a_{1,}, \ a_{2,} \ldots \ldots \ldots .., \ a_n$ are all constants and α is non integer.

$\mathfrak{D}_{\mathcal{J}}^{n\alpha} = \mathfrak{D}_{\mathcal{J}}^{\alpha} . \mathfrak{D}_{\mathcal{J}}^{\alpha} \ldots \ldots \ldots \ldots \mathfrak{D}_{\mathcal{J}}^{\alpha}$. upto n-times.

Rewriting "Eq. (15)"

$$f(\mathfrak{D}_{\mathcal{J}}^{\alpha}) y = \Omega(x) \quad (16)$$

where $f(\mathfrak{D}_{\mathcal{J}}^{\alpha})$ is a linear fractional differential operator. The "Eq. (16)" is known as a linear fractional differential equation (LFDE) with constant coefficients of order nα [19].

If $\Omega(x) = 0$, then "Eq. (16)" is called a linear homogeneous fractional differential equation with constant coefficients and if $\Omega(x) \neq 0$, then it is called a linear inhomogeneous fractional differential equation with constant coefficients. For α=1, "Eq. (16)" becomes nth order classical ordinary differential equation (ODE).

The complete solution y(x) of "Eq. (16)" consists of two parts one is the complementary function (C.F.) y_{cf} and second is the particular integral (P. I) y_{pi}.We have $y(x) = y_{cf} + y_{pi}$.

The complementary function is the solution of a linear homogeneous fractional differential equation with constant coefficients which is $f(\mathfrak{D}_{\mathcal{J}}^{\alpha}) y = 0$ and the particular solution y_{pi} is a function that satisfies the "Eq. (16)".

The y_{cf} for $f(\mathfrak{D}_{\mathcal{J}}^{\alpha}) y = 0$ is expressed by having Mittag-Leffler functions and fractional type of sine and cosine functions as follows (i). If $y_1, y_2, y_3, \dots, y_n$ are linearly independent solutions of $f(\mathfrak{D}_{\mathcal{J}}^{\alpha})y = 0$, then the linear combination $y = C_1y_1 + C_2y_2 + C_3y_3 + \cdots \dots + C_ny_n$ is also its solution for the arbitrary constants C_I where $i = 1,2,3, \dots \dots, n$.

(ii). If $m_1, m_2, m_3, \dots, m_n$ are n real and distinct roots of the auxiliary equation of homogeneous LFDE with constant coefficients, then its solution is

$$Y(x) = C_1E_{\alpha}(m_1x^{\alpha}) + C_2E_{\alpha}(m_2x^{\alpha}) + \cdots \dots \dots \dots \dots + C_nE_{\alpha}(m_nx^{\alpha})$$

(iii). If the roots of the auxiliary equation of homogeneous LFDE with constant coefficients have r repeated roots $(m_1 = m_2 = m_3 = \cdots \dots = m_r)$ for $1 \le r \le n$, then its solution is

$$y(x) = C_1E_{\alpha}(m_1x^{\alpha}) + x^{\alpha}C_2E_{\alpha}(m_1x^{\alpha}) + x^{2\alpha}C_3E_{\alpha}(m_1x^{\alpha}) \dots + x^{(r-1)\alpha}C_rE_{\alpha}(m_1x^{\alpha})$$

(iv). If the roots of the auxiliary equation are complex $a \pm ib$, then its solution is

$$y(x) = E_{\alpha}(ax^{\alpha})[A\cos_{\alpha}(bx^{\alpha}) + B\sin_{\alpha}(bx^{\alpha})$$

3. Particular Integral

The particular integral for "Eq. (16)" is

$$y(x) = \frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{\alpha})}\Omega(x) \tag{17}$$

The particular integral depends on the nature of the function $\Omega(x)$. The function $\Omega(x)$ can be in various forms. In this section, we will find the particular integral for different forms of $\Omega(x)$ which are the Mittag-Leffler function, fractional cosine, and sine functions.

Theorem 3.1. If $\Omega(x) = E_{\alpha}(mx^{\alpha})$, then $\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{\alpha})}\Omega(x) = \frac{1}{f(m)}E_{\alpha}(mx^{\alpha})$, provided that $f(m) \neq 0$.

Proof. By the definition of the Mittag-Leffler function

$$E_{\alpha}(mx^{\alpha}) = 1 + \frac{mx^{\alpha}}{\Gamma(1+\alpha)} + \frac{m^2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{m^3x^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \dots \dots \dots \dots \dots \tag{18}$$

Using the modified definition, of a fractional derivative of Jumarie type [22] we have

$$\mathfrak{D}_{\mathcal{J}}^{\alpha}(1) = 0\,, \quad 0 < \alpha < 1$$

Using Jumarie's equalities [10]

$$\mathfrak{D}_{\mathcal{J}}^{\alpha}(x^{n\alpha}) = \frac{\Gamma(n\alpha+1)}{\Gamma\{\alpha(n-1)+1}x^{(n-1)\alpha} \;. \tag{19}$$

$$\mathfrak{D}_{\mathcal{J}}^{\alpha}[E_{\alpha}(mx^{\alpha})] = D_J^{\alpha}\left[1 + \frac{mx^{\alpha}}{\Gamma(1+\alpha)} + \frac{m^2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{m^3x^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \dots \dots \dots \dots \dots\right]$$

$$\mathfrak{D}_{\mathcal{J}}^{\alpha}[E_{\alpha}(mx^{\alpha})] = m\left[1 + \frac{mx^{\alpha}}{\Gamma(1+\alpha)} + \frac{m^2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{m^3x^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \dots \dots \dots \dots \dots\right]$$

$$D_J^{\alpha}[E_{\alpha}(mx^{\alpha})] = mE_{\alpha}(mx^{\alpha}) \tag{20}$$

$$\mathfrak{D}_{\mathcal{J}}^{2\alpha}[E_{\alpha}(mx^{\alpha})] = D_J^{\alpha}D_J^{\alpha}E_{\alpha}(mx^{\alpha})$$

$$\mathfrak{D}_{\mathcal{J}}^{2\alpha}[E_\alpha(mx^\alpha)] = m^2 E_\alpha(mx^\alpha)$$

$$\mathfrak{D}_{\mathcal{J}}^{n\alpha}[E_\alpha(mx^\alpha)] = \mathfrak{D}_{\mathcal{J}}^{\alpha}\mathfrak{D}_{\mathcal{J}}^{\alpha}\mathfrak{D}_{\mathcal{J}}^{\alpha}\mathfrak{D}_{\mathcal{J}}^{\alpha}\mathfrak{D}_{\mathcal{J}}^{\alpha} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathfrak{D}_{\mathcal{J}}^{\alpha}[E_\alpha(mx^\alpha)]$$

$$\mathfrak{D}_{\mathcal{J}}^{n\alpha}[E_\alpha(mx^\alpha)] = m^n E_\alpha(mx^\alpha)$$

Let $f(\mathfrak{D}_{\mathcal{J}}^{\alpha})E_\alpha(mx^\alpha) = \left(a_0\mathfrak{D}_{\mathcal{J}}^{n\alpha} + a_1\mathfrak{D}_{\mathcal{J}}^{(n-1)\alpha} + a_2\mathfrak{D}_{\mathcal{J}}^{(n-2)\alpha} + \cdots + a_{n-1}\mathfrak{D}_{\mathcal{J}}^{\alpha} + a_n\right)E_\alpha(mx^\alpha)$

$= a_0\mathfrak{D}_{\mathcal{J}}^{n\alpha}E_\alpha(mx^\alpha) + a_1\mathfrak{D}_{\mathcal{J}}^{(n-1)\alpha}E_\alpha(mx^\alpha) + \cdots \ldots.. + a_{n-1}\mathfrak{D}_{\mathcal{J}}^{\alpha}E_\alpha(mx^\alpha) + a_nE_\alpha(mx^\alpha)$

$f(\mathfrak{D}_{\mathcal{J}}^{\alpha})E_\alpha(mx^\alpha) = [a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots \ldots\ldots. + a_{n-1}m + a_n]E_\alpha(mx^\alpha)$

$$f(\mathfrak{D}_{\mathcal{J}}^{\alpha})E_\alpha(mx^\alpha) = f(m)E_\alpha(mx^\alpha)$$

Operating $\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{\alpha})}$, we get $\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{\alpha})}\, f(\mathfrak{D}_{\mathcal{J}}^{\alpha})E_\alpha(mx^\alpha) = \frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{\alpha})}\, f(m)E_\alpha(mx^\alpha)$

$$E_\alpha(mx^\alpha) = f(m)\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{\alpha})}\, E_\alpha(mx^\alpha)$$

$$\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{\alpha})}\, E_\alpha(mx^\alpha) = \frac{1}{f(m)}\, E_\alpha(mx^\alpha), \text{ provided } f(m) \neq 0 \tag{21}$$

Theorem 3.2. If $\Omega(x) = \cos_\alpha ax^\alpha$, then $\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})}\,\Omega(x) = \frac{1}{f(-a^2)}\cos_\alpha ax^\alpha$, provided that $f(-a^2) \neq 0$.

Proof. We have

$$\cos_\alpha ax^\alpha = 1 - \frac{a^2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{a^4x^{4\alpha}}{\Gamma(1+4\alpha)} - \frac{a^6x^{6\alpha}}{\Gamma(1+6\alpha)} + \cdots \ldots\ldots\ldots..$$

$$\mathfrak{D}_{\mathcal{J}}^{\alpha}[\cos_\alpha x^\alpha] = \mathfrak{D}_{\mathcal{J}}^{\alpha}\left[1 - \frac{a^2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{a^4x^{4\alpha}}{\Gamma(1+4\alpha)} - \frac{a^6x^{6\alpha}}{\Gamma(1+6\alpha)} + \cdots \ldots\ldots\ldots..\right]$$

Using Corollary 1.1.

$$= -\frac{a^2\Gamma(1+2\alpha)}{\Gamma(2\alpha-\alpha+1)}\frac{x^{2\alpha-\alpha}}{\Gamma(1+2\alpha)} + \frac{a^4\,\Gamma(1+4\alpha)}{\Gamma(1+3\alpha)}\frac{x^{4\alpha-\alpha}}{\Gamma(1+4\alpha)} - \frac{a^6\,\Gamma(1+6\alpha)}{\Gamma(1+5\alpha)}\frac{x^{6\alpha-\alpha}}{\Gamma(1+6\alpha)} + \cdots \ldots\ldots\ldots..$$

$$= -a\left[\frac{ax^\alpha}{\Gamma(1+\alpha)} - \frac{a^3x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{a^5x^{5\alpha}}{\Gamma(1+5\alpha)} + \cdots \ldots\ldots\ldots..\right]$$

$$\mathfrak{D}_{\mathcal{J}}^{\alpha}[\cos_\alpha ax^\alpha] = -a\sin{}_\alpha ax^\alpha$$

$$\mathfrak{D}_{\mathcal{J}}^{2\alpha}[\cos_\alpha ax^\alpha] = -a\mathfrak{D}_{\mathcal{J}}^{\alpha}[\sin_\alpha ax^\alpha]$$

$$\mathfrak{D}_{\mathcal{J}}^{2\alpha}[\cos_\alpha ax^\alpha] = (-a^2)\cos_\alpha ax^\alpha$$

$$\mathfrak{D}_{\mathcal{J}}^{3\alpha}[\cos_\alpha ax^\alpha] = a^3 sin_\alpha ax^\alpha$$

$$\mathfrak{D}_{\mathcal{J}}^{4\alpha}[\cos_\alpha ax^\alpha] = a^4\cos_\alpha ax^\alpha$$

$$\mathfrak{D}_{\mathcal{J}}^{(2\alpha)2}[\cos_\alpha ax^\alpha] = (-a^2)^2\cos_\alpha ax^\alpha$$

$$\mathfrak{D}_{\mathcal{J}}^{(2\alpha)n}[\cos_\alpha ax^\alpha] = (-a^2)^n \cos_\alpha ax^\alpha$$

Hence $f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})\cos_\alpha ax^\alpha = f(-a^2)\cos_\alpha ax^\alpha$

Operating $\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})}$

$$\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})} f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})\cos_\alpha ax^\alpha = \frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})} f(-a^2)\cos_\alpha ax^\alpha$$

$$\cos_\alpha ax^\alpha = f(-a^2)\ \frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})}\cos_\alpha ax^\alpha$$

$$\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})}\cos_\alpha ax^\alpha = \frac{1}{f(-a^2)}\ \cos_\alpha ax^\alpha, \text{ provided } f(-a^2) \neq 0 \qquad (22)$$

Theorem 3.3. If $\Omega(x) = \sin_\alpha ax^\alpha$, then $\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})}\ \Omega(x) = \frac{1}{f(-a^2)}\ \sin_\alpha ax^\alpha$, provided that $f(-a^2) \neq 0$.

Proof. we have $\mathfrak{D}_{\mathcal{J}}^{\alpha}[\ \sin_\alpha ax^\alpha] = \mathfrak{D}_{\mathcal{J}}^{\alpha}[\frac{ax^\alpha}{\Gamma(1+\alpha)} - \frac{a^3x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{a^5x^{5\alpha}}{\Gamma(1+5\alpha)} + \cdots \ldots\ldots\ldots..]$

$$=\frac{\Gamma(1+\alpha)}{\Gamma(1)}\frac{ax^0}{\Gamma(1+\alpha)} - \frac{\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)}\frac{a^3x^{3\alpha-\alpha}}{\Gamma(1+3\alpha)} + \frac{\Gamma(1+5\alpha)}{\Gamma(1+4\alpha)}\frac{a^5x^{5\alpha-\alpha}}{\Gamma(1+5\alpha)} + \cdots \ldots\ldots\ldots..$$

$$= a\left[1 - \frac{a^2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{a^4x^{4\alpha}}{\Gamma(1+4\alpha)} + \cdots \ldots\ldots\ldots..\right]$$

$$\mathfrak{D}_{\mathcal{J}}^{\alpha}[\ \sin_\alpha ax^\alpha] = a\cos_\alpha ax^\alpha$$

$$\mathfrak{D}_{\mathcal{J}}^{2\alpha}[\ \sin_\alpha ax^\alpha] = (-\ a^2)\sin_\alpha ax^\alpha$$

$$\mathfrak{D}_{\mathcal{J}}^{3\alpha}[\ \sin_\alpha ax^\alpha] = a^3\cos_\alpha ax^\alpha$$

$$\mathfrak{D}_{\mathcal{J}}^{(2\alpha)2}[\ \sin_\alpha ax^\alpha] = (-a^2)^n \sin_\alpha ax^\alpha$$

Hence $f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})\sin_\alpha ax^\alpha = f(-a^2)\sin_\alpha ax^\alpha$

Operating $\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})}$

$$\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})} f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})\sin_\alpha ax^\alpha = \frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})} f(-a^2)\sin_\alpha ax^\alpha$$

$$\sin_\alpha ax^\alpha = f(-a^2)\ \frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})}\sin_\alpha ax^\alpha$$

$$\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{2\alpha})}\sin_\alpha ax^\alpha = \frac{1}{f(-a^2)}\ \sin_\alpha ax^\alpha, \text{ provided } f(-a^2) \neq 0 \qquad (23)$$

Theorem 3.4. If $\Omega(x) = E_\alpha(mx^\alpha)\,\psi(x)$, then

$$\frac{1}{f(D_J^\alpha)}\ \Omega(x) = E_\alpha(mx^\alpha)\ \frac{1}{f(D_J^\alpha+m)}\ \psi(x)$$

Proof Let $V(x)$ is a function of variable x, then by differentiation

$$\mathfrak{D}_{\mathcal{J}}^{\alpha}[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, V(x)] = \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x) + V(x)\mathfrak{D}_{\mathcal{J}}^{\alpha}\, \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})$$

$$= \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x) + V(x)\, \mathrm{m}\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})$$

$$= \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\left[\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x) + V(x)\mathrm{m}\,\right]$$

$$= \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\left(\mathfrak{D}_{\mathcal{J}}^{\alpha}\, + \mathrm{m}\right)V(x)$$

$$\mathfrak{D}_{\mathcal{J}}^{2\alpha}[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, V(x)] = \mathfrak{D}_{\mathcal{J}}^{\alpha}\mathfrak{D}_{\mathcal{J}}^{\alpha}[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, V(x)]$$

$$= \mathfrak{D}_{\mathcal{J}}^{\alpha}\, [\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x) + V(x)\mathrm{m}\, \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})$$

$$= \mathfrak{D}_{\mathcal{J}}^{\alpha}\left[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x)\right] + \mathrm{m}\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, [V(x)\, \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})]$$

$$= \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\mathfrak{D}_{\mathcal{J}}^{\alpha}\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x) + \{\mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x)\}\mathfrak{D}_{\mathcal{J}}^{\alpha}\, \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})] + \mathrm{m}[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x) + V(x)\mathrm{m}\, \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})]$$

$$= \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\mathfrak{D}_{\mathcal{J}}^{2\alpha}\, V(x) + \{\mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x)\}\, \mathrm{m}\, \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})] + m\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x) + V(x)m^2\, \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})]$$

$$= \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\left[\mathfrak{D}_{\mathcal{J}}^{2\alpha}\, V(x) + \mathrm{m}\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x)\, + m\, \mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x) + m^2\, V(x)\right]$$

$$= \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\left[\mathfrak{D}_{\mathcal{J}}^{2\alpha}\, V(x) + 2m\mathfrak{D}_{\mathcal{J}}^{\alpha}\, V(x) + m^2\, V(x)\,\right]$$

$$= \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\left[\mathfrak{D}_{\mathcal{J}}^{2\alpha}\, + 2m\mathfrak{D}_{\mathcal{J}}^{\alpha}\, + m^2\right]V(x)$$

$$\mathfrak{D}_{\mathcal{J}}^{2\alpha}[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, V(x)] = \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \left(\mathfrak{D}_{\mathcal{J}}^{\alpha} + m\right)^2 V(x)$$

Similarly, $\mathfrak{D}_{\mathcal{J}}^{2\alpha}[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, V(x)] = \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \left(\mathfrak{D}_{\mathcal{J}}^{\alpha} + m\right)^3 V(x)$

In general, $\mathfrak{D}_{\mathcal{J}}^{\mathrm{n}\alpha}[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, V(x)] = \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \left(\mathfrak{D}_{\mathcal{J}}^{\alpha} + m\right)^n V(x)$

$$\mathrm{f}(\mathfrak{D}_{\mathcal{J}}^{\alpha})[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, V(x)] = \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \, f\left(\mathfrak{D}_{\mathcal{J}}^{\alpha} + m\right)V(x)$$

Operating $\frac{1}{\mathrm{f}(\mathfrak{D}_{\mathcal{J}}^{\alpha})}$, we have $\frac{1}{\mathrm{f}(\mathfrak{D}_{\mathcal{J}}^{\alpha})}\mathrm{f}(\mathfrak{D}_{\mathcal{J}}^{\alpha})[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, V(x)] = \frac{1}{\mathrm{f}(\mathfrak{D}_{\mathcal{J}}^{\alpha})}\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \, f\left(\mathfrak{D}_{\mathcal{J}}^{\alpha} + m\right)V(x)$

$$\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, V(x) = \frac{1}{\mathrm{f}(\mathfrak{D}_{\mathcal{J}}^{\alpha})}\left[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \, f\left(\mathfrak{D}_{\mathcal{J}}^{\alpha} + m\right)V(x)\right] \tag{24}$$

Now let $f\left(\mathfrak{D}_{\mathcal{J}}^{\alpha} + m\right)V(x) = \psi(x)$, i.e., $V(x) = \frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{\alpha}+m)}\psi(x)$

From “Eq. (24)”

$$\frac{1}{\mathrm{f}(\mathfrak{D}_{\mathcal{J}}^{\alpha})}[\mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\, \psi(x)] = \mathrm{E}_{\alpha}(\mathrm{m}x^{\alpha})\frac{1}{f(\mathfrak{D}_{\mathcal{J}}^{\alpha}+m)}\psi(x) \tag{25}$$

4. Results and Discussion

The analytical and graphical solutions of some ILSFDE are provided below and compared with graphs of solutions of the differential equations of integer order.

Example 4.1. Consider

$$\left(\mathfrak{D}_{\mathcal{J}}^{2\alpha} + 4\,\mathfrak{D}_{\mathcal{J}}^{\alpha} + 3\right)\mathrm{y(x)} = \mathrm{E}_{\alpha}(\mathrm{x}^{\alpha}) \tag{26}$$

be an inhomogeneous linear fractional differential equation and $\mathfrak{D}_{\mathcal{J}}$is jumarie fractional derivative. Let $\alpha = 0.5$,then "Eq. (26)" is

$$\left(\mathfrak{D}_{\mathcal{J}}^{2\alpha} + 4\,\mathfrak{D}_{\mathcal{J}}^{\alpha} + 3\right)\mathrm{y(x)} = \mathrm{E}_{0.5}(\mathrm{x}^{0.5})$$

The auxiliary equation of this fractional differential equation is

$\mathrm{m}^2 + 4\mathrm{m} + 3 = 0$ and roots are $\mathrm{m} = -1\,, -3$

The complementary function is $y_{com}(\mathrm{x}) = \mathrm{C}_1\mathrm{E}_{0.5}(-\mathrm{x}^{0.5}) + \mathrm{C}_2\mathrm{E}_{0.5}(-3\mathrm{x}^{0.5})$

The particular integral is $y_{pi} = \frac{1}{\left(\mathfrak{D}_{\mathcal{J}}^{\alpha}\right)^2 + 4\,\mathfrak{D}_{\mathcal{J}}^{\alpha} + 3}\,\mathrm{E}_{0.5}(\mathrm{x}^{0.5})$

Using "*Theorem 3.1*"

$$y_{pi} = \frac{1}{(1)^2 + 4(1) + 3}\,\mathrm{E}_{0.5}(\mathrm{x}^{0.5}) = \frac{1}{8}\,\mathrm{E}_{0.5}(\mathrm{x}^{0.5})$$

The complete solution of "Eq. (26)" is

$$y(x) = \mathrm{C}_1\mathrm{E}_{0.5}(-\mathrm{x}^{0.5}) + \mathrm{C}_2\mathrm{E}_{0.5}(-3\mathrm{x}^{0.5}) + \frac{1}{8}\,\mathrm{E}_{0.5}(\mathrm{x}^{0.5})$$

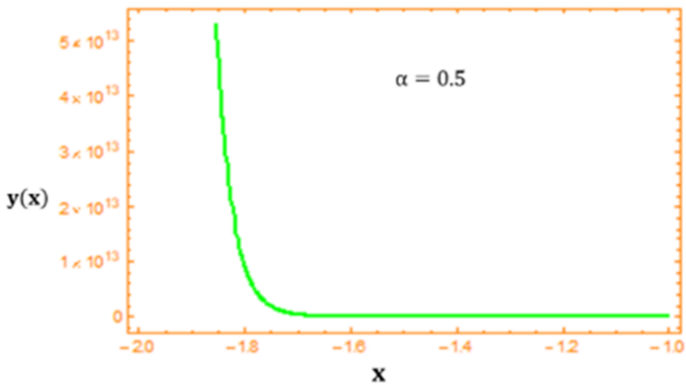

Fig. 1. The solution for "Eq. (26)" for $\alpha = 0.5$.

If $\alpha = 0.7$,then "Eq. (26)" is $\left(\mathfrak{D}_{\mathcal{J}}^{2\alpha} + 4\,\mathfrak{D}_{\mathcal{J}}^{\alpha} + 3\right)\mathrm{y(x)} = \mathrm{E}_{0.7}(\mathrm{x}^{0.7})$. The complementary function is $y_{com}(\mathrm{x}) = \mathrm{C}_1\mathrm{E}_{0.7}(-\mathrm{x}^{0.7}) + \mathrm{C}_2\mathrm{E}_{0.7}(-3\mathrm{x}^{0.7})$.

The particular integral is $y_{pi} = \frac{1}{\left(\mathfrak{D}_{\mathcal{J}}^{\alpha}\right)^2 + 4\,\mathfrak{D}_{\mathcal{J}}^{\alpha} + 3}\,\mathrm{E}_{0.7}(\mathrm{x}^{0.7})$

Using "*Theorem 3.1*"

$$y_{pi} = \frac{1}{(1)^2 + 4(1) + 3}\,\mathrm{E}_{0.7}(\mathrm{x}^{0.7}) = \frac{1}{8}\,\mathrm{E}_{0.7}(\mathrm{x}^{0.7})$$

The complete solution of "Eq. (26)" for $\alpha = 0.7$ is

$$y(x) = \mathrm{C_1 E_{0.7}(-x^{0.7}) + C_2 E_{0.7}(-3x^{0.7})} + \frac{1}{8}\,\mathrm{E_{0.7}(x^{0.7})}$$

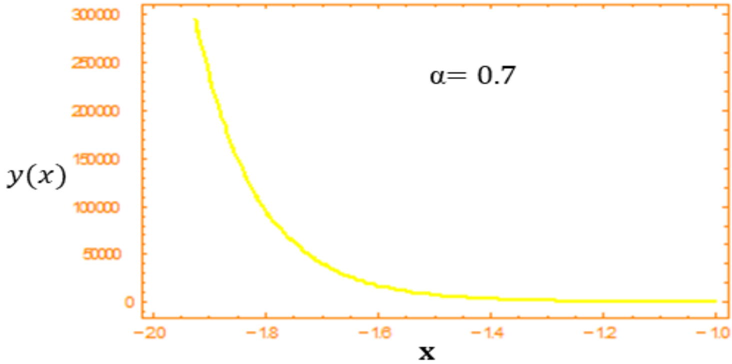

Fig. 2. The solution for “Eq. (26)” for $\alpha = 0.7$.

If $\alpha = 1$,then “Eq. (26)” is $(D^2 + 4\,D + 3)\mathrm{y(x)} = e^x$. The complementary function is $y_{com}(\mathrm{x}) = \mathrm{C_1}e^{-x} + \mathrm{C_2}e^{-3x}$.

The particular integral is $y_{pi} = \frac{1}{D^2+4\,D+3}\quad e^x$

Using “*Theorem 3.1*”

$$y_{pi} = \frac{1}{(1)^2 + 4(1) + 3}\;e^x = \frac{1}{8}\;e^x$$

The complete solution of “Eq. (26)” for $\alpha = 1$ is

$$y(x) = \mathrm{C_1}e^{-x} + \mathrm{C_2}e^{-3x} + \frac{1}{8}\;e^x$$

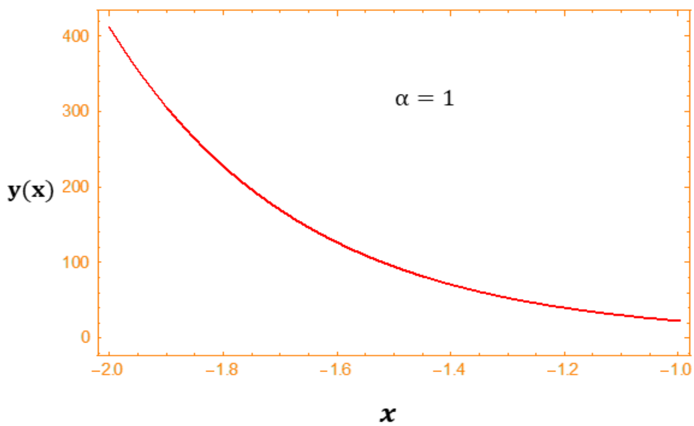

Fig. 3. The solution for “Eq. (26)” for $\alpha = 1$.

Figs. 1, 2 and 3 show that as the value of α approaches 1 then the graphs of fractional differential equations $\left(\mathfrak{D}_{\mathcal{J}}^{2(0.5)} + 4\,\mathfrak{D}_{\mathcal{J}}^{0.5} + 3\right)\mathrm{y(x)} = \mathrm{E}_{0.5}(\mathrm{x}^{0.5})$ and $\left(\mathfrak{D}_{\mathcal{J}}^{2(0.7)} + 4\,\mathfrak{D}_{\mathcal{J}}^{0.7} + 3\right)\mathrm{y(x)} = \mathrm{E}_{0.7}(\mathrm{x}^{0.7})$ coincide with the graph of the solution of a classical differential equation $(D^2 + 4\,D + 3)\mathrm{y(x)} = e^x$.

Example 4.2. Consider

$$\left(\mathfrak{D}_{\mathcal{J}}^{2\alpha} + \mathfrak{D}_{\mathcal{J}}^{\alpha} - 2\right)\mathrm{y(x)} = \mathrm{E}_{\alpha}(\mathrm{x}^{\alpha}) \tag{27}$$

be a fractional differential equation. Let $\alpha = 0.5$, then "Eq. (27)" can be written as

$$\left(\mathfrak{D}_{\mathcal{J}}^{2\alpha} + \mathfrak{D}_{\mathcal{J}}^{\alpha} - 2\right)\mathrm{y(x)} = \mathrm{E}_{0.5}(\mathrm{x}^{0.5})$$

The auxiliary equation is $\mathrm{m}^2 + \mathrm{m} - 2 = 0$ and roots are $\mathrm{m} = 1\,, -2$

The C.F. is $y_{com}(\mathrm{x}) = \mathrm{C}_1\mathrm{E}_{0.5}(\mathrm{x}^{0.5}) + \mathrm{C}_2\mathrm{E}_{0.5}(-2\mathrm{x}^{0.5})$

The P.I. is $y_{pi} = \frac{1}{\left(\mathfrak{D}_{\mathcal{J}}^{\alpha}\right)^2 + \mathfrak{D}_{\mathcal{J}}^{\alpha} - 2}\,\mathrm{E}_{0.5}(\mathrm{x}^{0.5}) = \frac{1}{(\mathfrak{D}_{\mathcal{J}}^{\alpha} - 1)(\mathfrak{D}_{\mathcal{J}}^{\alpha} + 2)}\,\mathrm{E}_{0.5}(\mathrm{x}^{0.5})$

Using "*Theorem 3.1*"

$$= \frac{1}{\left(\mathfrak{D}_{\mathcal{J}}^{\alpha} - 1\right)(3)}\,\mathrm{E}_{0.5}(\mathrm{x}^{0.5}) = \mathrm{E}_{0.5}(\mathrm{x}^{0.5})\frac{1}{\left(\mathfrak{D}_{\mathcal{J}}^{\alpha}\right)(3)}$$

$$= \frac{1}{3}\mathrm{E}_{0.5}(\mathrm{x}^{0.5})\frac{1}{\left(\mathfrak{D}_{\mathcal{J}}^{0.5}\right)} = \frac{1}{3}\mathrm{E}_{0.5}(\mathrm{x}^{0.5})\frac{\mathfrak{D}_{\mathcal{J}}^{0.5}1}{\mathfrak{D}_{\mathcal{J}}} = \frac{1}{3}\mathrm{E}_{0.5}(\mathrm{x}^{0.5})\mathfrak{D}_{\mathcal{J}}^{0.5}(x)$$

$$y_{pi} = \frac{2}{3}\sqrt{\frac{x}{\pi}}\,\mathrm{E}_{0.5}(\mathrm{x}^{0.5})$$

The complete solution of "Eq. (27)" is

$$y(x) = \mathrm{C}_1\mathrm{E}_{0.5}\left(\mathrm{x}^{\frac{1}{2}}\right) + \mathrm{C}_2\mathrm{E}_{0.5}(-2\mathrm{x}^{0.5}) + \frac{2}{3}\sqrt{\frac{x}{\pi}}\,\mathrm{E}_{0.5}(\mathrm{x}^{0.5})$$

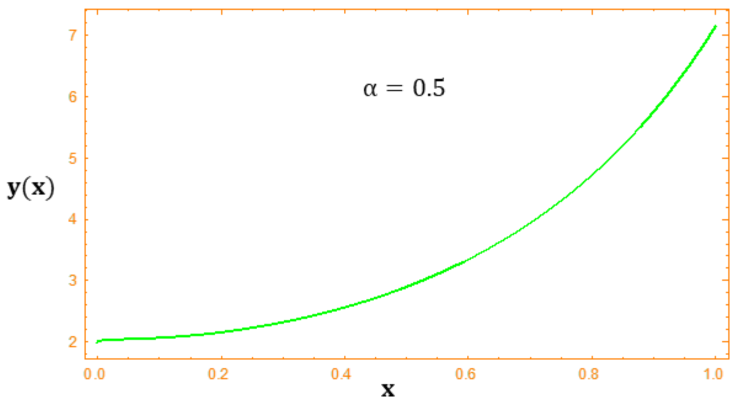

Fig. 4. The solution for "Eq. (27)" for $\alpha = 0.5$.

If $\alpha = 0.7$, then "Eq. (27)" can be written as

$$\left(\mathfrak{D}_{\mathcal{J}}^{2\alpha} + \mathfrak{D}_{\mathcal{J}}^{\alpha} - 2\right)\mathrm{y(x)} = \mathrm{E}_{0.7}(\mathrm{x}^{0.7})$$

The C.F. is $y_{com}(\mathrm{x}) = \mathrm{C}_1\mathrm{E}_{0.7}(\mathrm{x}^{0.7}) + \mathrm{C}_2\mathrm{E}_{\frac{1}{2}}(-2\mathrm{x}^{0.7})$

The P.I. is $y_{pi} = \frac{1}{(\mathfrak{D}_{\mathcal{J}}^{\alpha})^2 + \mathfrak{D}_{\mathcal{J}}^{\alpha} - 2}\ \mathrm{E}_{0.7}(\mathrm{x}^{0.7}) = \frac{1}{(\mathfrak{D}_{\mathcal{J}}^{\alpha}-1)(\mathfrak{D}_{\mathcal{J}}^{\alpha}+2)}\ \mathrm{E}_{0.7}(\mathrm{x}^{0.7})$

Using "*Theorem 3.1*"

$$y_{pi} = \frac{1}{(\mathfrak{D}_{\mathcal{J}}^{\alpha}-1)(3)}\ \mathrm{E}_{0.7}(\mathrm{x}^{0.7}) = \mathrm{E}_{0.7}(\mathrm{x}^{0.7})\frac{1}{(\mathfrak{D}_{\mathcal{J}}^{0.7})(3)} = \frac{1}{3}\mathrm{E}_{0.7}(\mathrm{x}^{0.7})\frac{1}{(\mathfrak{D}_{\mathcal{J}}^{0.7})}$$

$$y_{pi} = \frac{1}{3}\mathrm{E}_{0.7}(\mathrm{x}^{0.7})\frac{\mathfrak{D}_{\mathcal{J}}^{0.3}}{\mathfrak{D}_{\mathcal{J}}^{0.7}\mathfrak{D}_{\mathcal{J}}^{0.3}} = \frac{1}{3}\mathrm{E}_{0.7}(\mathrm{x}^{0.7})\frac{\mathfrak{D}_{\mathcal{J}}^{0.3}}{\mathfrak{D}}\ 1 = \frac{1}{3}\mathrm{E}_{0.7}(\mathrm{x}^{0.7})\ \mathfrak{D}_{\mathcal{J}}^{0.3}\ (x)$$

Using *Corollary 1.1.(a)*

$$y_{pi} = \frac{1}{3}\frac{\Gamma(1.3)}{\Gamma(1.7)}\mathrm{x}^{0.7}\ \mathrm{E}_{0.7}(\mathrm{x}^{0.7})$$

The complete solution of "Eq. (27)" for $\alpha = 0.7$ is

$$y(x) = \mathrm{C}_1\mathrm{E}_{0.7}(\mathrm{x}^{0.7}) + \mathrm{C}_2\mathrm{E}_{\frac{1}{2}}(-2\mathrm{x}^{0.7}) + \frac{1}{3}\frac{\Gamma(1.3)}{\Gamma(1.7)}\mathrm{x}^{0.7}\ \mathrm{E}_{0.7}(\mathrm{x}^{0.7})$$

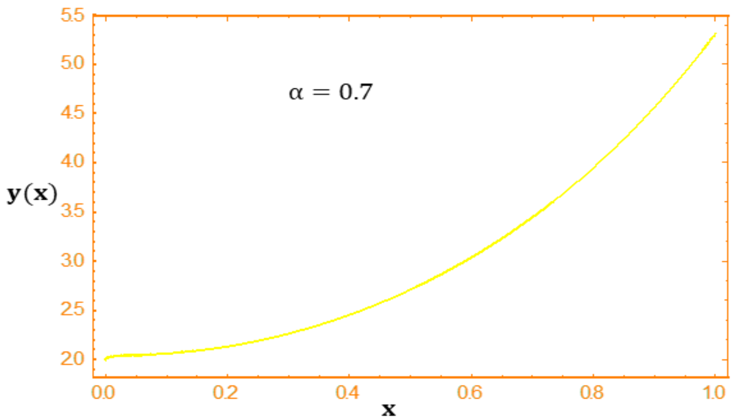

Fig. 5. The solution for "Eq. (27)" for $\alpha = 0.7$.

If $\alpha = 1$, then "Eq. (27)" can be written as

$$(\mathfrak{D}^2 + \mathfrak{D} - 2)\mathrm{y}(\mathrm{x}) = e^x$$

The C.F. is $y_{com}(\mathrm{x}) = \mathrm{C}_1 e^x + \mathrm{C}_2 e^{-2x}$

The P.I. is $y_{pi} = \frac{1}{\mathfrak{D}^2+\mathfrak{D}-2}\ e^x = x\frac{1}{2\mathfrak{D}+1}\ e^x = \frac{x}{3}\ e^x$

The complete solution of "Eq. (27)" for $\alpha = 1$ is

$$y(x) = \mathrm{C}_1 e^x + \mathrm{C}_2 e^{-2x} + \frac{x}{3}\ e^x$$

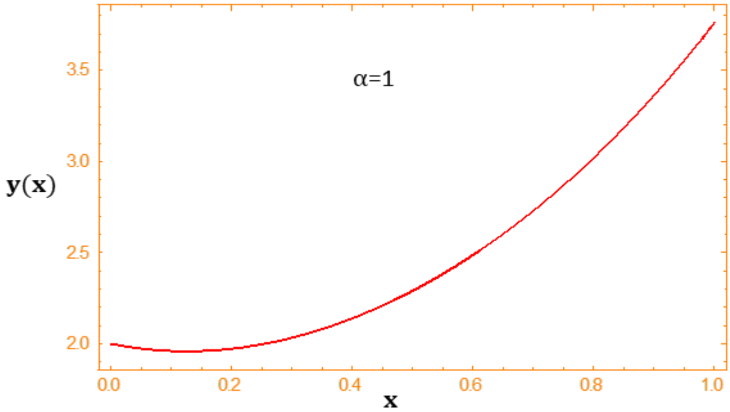

Fig. 6. The solution for "Eq. (27)" for $\alpha = 1$.

Figs. 4, 5 and 6 demonstrate that as the value of α becomes closer to 1, then the graphs of fractional differential equations $\left(\mathfrak{D}_{\mathcal{J}}^{2(0.5)} + \mathfrak{D}_{\mathcal{J}}^{0.5} - 2\right)\mathrm{y(x)} = \mathrm{E}_{0.5}(\mathrm{x}^{0.5})$ and $\left(\mathfrak{D}_{\mathcal{J}}^{2(0.7)} + \mathfrak{D}_{\mathcal{J}}^{0.7} - 2\right)\mathrm{y(x)} = \mathrm{E}_{0.7}(\mathrm{x}^{0.7})$ coincide with the graph of the solution of a classical differential equation $(\mathfrak{D}^2 + \mathfrak{D} - 2)\mathrm{y(x)} = e^x$.

Example 4.3. We consider an inhomogeneous fractional differential equation

$$\left(\mathfrak{D}_{\mathcal{J}}^{\alpha} + 1\right)\mathrm{y(x)} = \cos_{\alpha}(\mathrm{x}^{\alpha}) \tag{28}$$

If $\alpha = 0.5$, then "Eq. (28)" can be written as $\left(\mathfrak{D}_{\mathcal{J}}^{0.5} + 1\right)\mathrm{y(x)} = \cos_{0.5}(\mathrm{x}^{0.5})$

The A.E. is $m + 1 = 0$ and roots are $\mathrm{m} = -1$.

The complementary function is $y_{com}(\mathrm{x}) = \mathrm{C}_1\mathrm{E}_{0.5}(-\mathrm{x}^{0.5})$

The particular integral is $y_{pi} = \frac{1}{\mathfrak{D}_{\mathcal{J}}^{0.5}+1} \cos_{0.5}(\mathrm{x}^{0.5})$

$$= \frac{(\mathfrak{D}_{\mathcal{J}}^{0.5}-1)}{(\mathfrak{D}_{\mathcal{J}}^{0.5}+1)(\mathfrak{D}_{\mathcal{J}}^{0.5}-1)} \cos_{0.5}(\mathrm{x}^{0.5}) = \frac{(\mathfrak{D}_{\mathcal{J}}^{0.5}-1)}{\mathfrak{D}_{\mathcal{J}}^{1}-1} \cos_{0.5}(\mathrm{x}^{0.5})$$

Using "*Theorem 3.2*"

$$y_{pi} = -\frac{\left(\mathfrak{D}_{\mathcal{J}}^{0.5} - 1\right)}{2} \cos_{0.5}(\mathrm{x}^{0.5}) = \frac{1}{2}\left(\cos_{0.5}(\mathrm{x}^{0.5}) + \sin_{0.5}(\mathrm{x}^{0.5})\right)$$

The complete solution of "Eq. (28)" is

$$y(x) = \mathrm{C}_1\mathrm{E}_{0.5}(-\mathrm{x}^{0.5}) + \frac{1}{2}[\cos_{0.5}(\mathrm{x}^{0.5}) + \sin_{0.5}(\mathrm{x}^{0.5})]$$

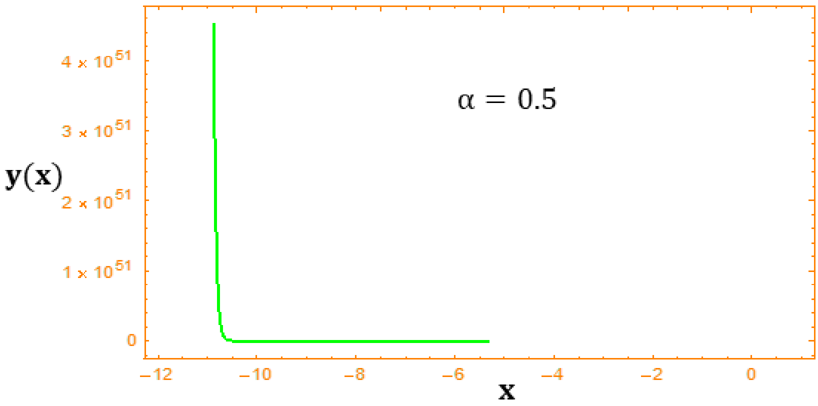

Fig. 7. The solutions for "Eq. (28)" for $\alpha = 0.5$.

If $\alpha = 0.7$, then "Eq. (28)" can be written as $\left(\mathfrak{D}_{\mathcal{J}}^{0.7} + 1\right)\mathrm{y}(\mathrm{x}) = \cos_{0.7}(\mathrm{x}^{0.7})$

The complementary function is $y_{com}(\mathrm{x}) = \mathrm{C}_1\mathrm{E}_{0.7}(-\mathrm{x}^{0.7})$

The particular integral is $y_{pi} = \frac{1}{\mathfrak{D}_{\mathcal{J}}^{0.7}+1}\cos_{0.7}(\mathrm{x}^{0.7})$

$$= \frac{(\mathfrak{D}_{\mathcal{J}}^{0.7}-1)}{(\mathfrak{D}_{\mathcal{J}}^{0.7}+1)(\mathfrak{D}_{\mathcal{J}}^{0.7}-1)}\cos_{0.7}(\mathrm{x}^{0.7}) = \frac{(\mathfrak{D}_{\mathcal{J}}^{0.7}-1)}{\mathfrak{D}_{\mathcal{J}}^{2(0.7)}-1}\cos_{0.7}(\mathrm{x}^{0.7})$$

Using "*Theorem 3.2*"

$$y_{pi} = -\frac{\left(\mathfrak{D}_{\mathcal{J}}^{0.7} - 1\right)}{2}\cos_{0.7}(\mathrm{x}^{0.7}) = \frac{1}{2}[\cos_{0.7}(\mathrm{x}^{0.7}) + \sin_{0.7}(\mathrm{x}^{0.7})\,)$$

The complete solution of "Eq. (28)" for $\alpha = 0.7$ is

$$y(x) = \mathrm{C}_1\mathrm{E}_{0.7}(-\mathrm{x}^{0.7}) + \frac{1}{2}[\cos_{0.7}(\mathrm{x}^{0.7}) + \sin_{0.7}(\mathrm{x}^{0.7})\,)$$

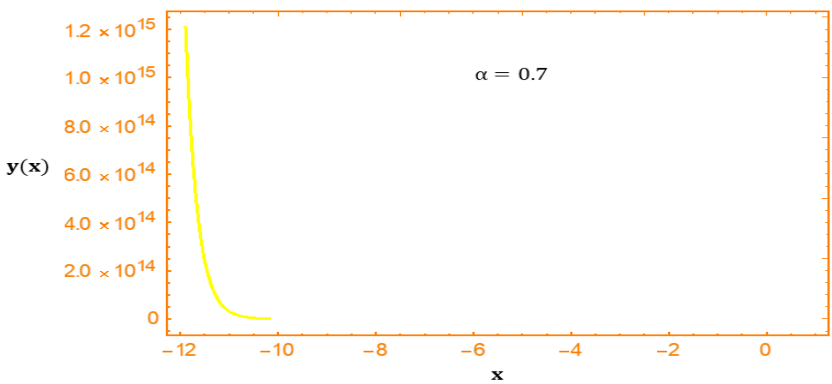

Fig. 8. The solutions for "Eq. (28)" for $\alpha = 0.7$.

If $\alpha = 1$, then "Eq. (28)" can be written as $(\mathfrak{D} + 1)\mathrm{y}(\mathrm{x}) = \cos x$

The complementary function is $y_{com}(\mathrm{x}) = \mathrm{C}_1 e^{-x}$

The particular integral is $y_{pi} = \frac{1}{\mathfrak{D}+1} \cos x$

$$y_{pi} = \frac{(\mathfrak{D}-1)}{(\mathfrak{D}^2-1)} \cos x = \frac{(\mathfrak{D}-1)}{-2} \cos x = \frac{1}{2}[\cos x + \cos x]$$

The complete solution of "Eq. (28)" for $\alpha = 1$ is

$$y(x) = \mathrm{C}_1 e^{-x} + \frac{1}{2}[\cos x + \cos x]$$

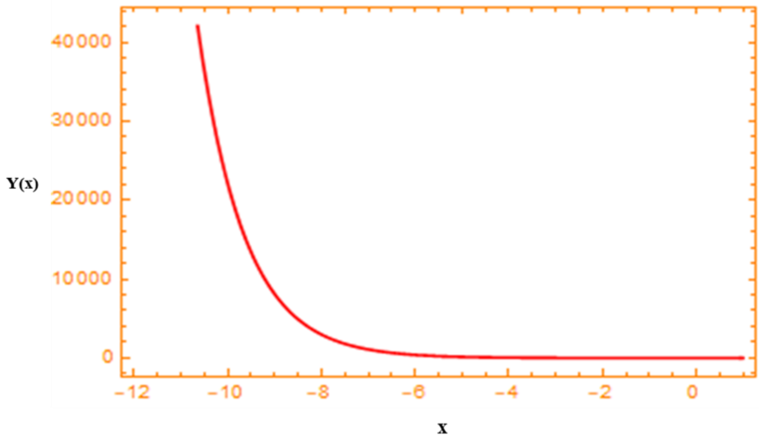

Fig. 9. The solutions for "Eq. (28)" for $\alpha = 1$.

Figs. 7, 8 and 9 demonstrate that as the value of α becomes closer to 1, then the graphs of fractional differential equations $(\mathfrak{D}_{\mathcal{J}}^{0.5} + 1)\mathrm{y}(\mathrm{x}) = \cos_{0.5}(\mathrm{x}^{0.5})$ and $(\mathfrak{D}_{\mathcal{J}}^{0.7} + 1)\mathrm{y}(\mathrm{x}) = \cos_{0.7}(\mathrm{x}^{0.7})$ coincide with the graph of the solution of a classical differential equation $(\mathfrak{D} + 1)\mathrm{y}(\mathrm{x}) = \cos x$ . .

Example 4.4. We consider an inhomogeneous fractional differential equation

$$(\mathfrak{D}_{\mathcal{J}}^{2\alpha} + 2\,\mathfrak{D}_{\mathcal{J}}^{\alpha} + 1)\mathrm{y}(\mathrm{x}) = \mathrm{E}_\alpha(-\mathrm{x}^\alpha)\cos_\alpha(\mathrm{x}^\alpha) \tag{29}$$

Let $\alpha = 0.5$, then the A.E. is $m^2 + 2m + 1 = 0$ and roots are $\mathrm{m} = -1, -1$.

The complementary function is $y_{com}(\mathrm{x}) = \mathrm{C}_1\mathrm{E}_{0.5}(-\mathrm{x}^{0.5}) + \mathrm{x}^{0.5}\mathrm{C}_2\mathrm{E}_{0.5}(-\mathrm{x}^{0.5})$

The particular integral is

$$y_{pi} = \frac{1}{\mathfrak{D}_{\mathcal{J}}^{2\alpha} + 2\,\mathfrak{D}_{\mathcal{J}}^{\alpha} + 1}\mathrm{E}_{0.5}(-\mathrm{x}^{0.5})\cos_{0.5}(\mathrm{x}^{0.5}) = \frac{1}{(\mathfrak{D}_{\mathcal{J}}^{\alpha} + 1)^2}\mathrm{E}_{0.5}(-\mathrm{x}^{0.5})\cos_{0.5}(\mathrm{x}^{0.5})$$

Using "*Theorem 3.4*"

$$y_{pi} = \mathrm{E}_{0.5}(-\mathrm{x}^{0.5})\frac{1}{\mathfrak{D}_{\mathcal{J}}^{2\alpha}} \cos_{0.5}(\mathrm{x}^{0.5}) = -\,\mathrm{E}_{0.5}(-\mathrm{x}^{0.5})\cos_{0.5}(\mathrm{x}^{0.5})$$

The complete solution of "Eq. (29)" is

$$y(x) = \mathrm{C}_1\mathrm{E}_{0.5}(-\mathrm{x}^{0.5}) + \mathrm{x}^{0.5}\mathrm{C}_2\mathrm{E}_{0.5}(-\mathrm{x}^{0.5}) - \mathrm{E}_{0.5}(-\mathrm{x}^{0.5})\cos_{0.5}(\mathrm{x}^{0.5})$$

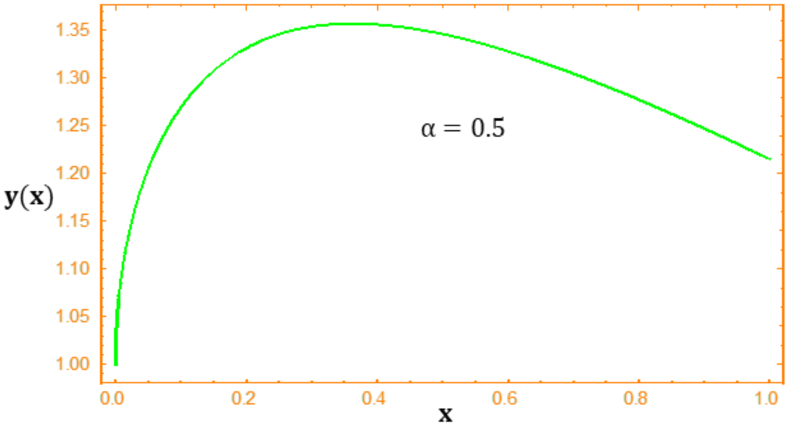

Fig. 10. The solutions for "Eq. (29)" for $\alpha = 0.5$.

If $\alpha = 0.7$,then the C.F. is $y_{com}(\mathrm{x}) = \mathrm{C_1 E_{0.7}(-x^{0.7}) + x^{0.7} C_2 E_{0.7}(-x^{0.7})}$
The particular integral is

$$y_{pi} = \frac{1}{\mathfrak{D}_{\mathcal{J}}^{2\alpha} + 2\,\mathfrak{D}_{\mathcal{J}}^{\alpha} + 1}\mathrm{E_{0.7}(-x^{0.7})\cos_{0.7}(x^{0.7})} = \frac{1}{\left(\mathfrak{D}_{\mathcal{J}}^{\alpha} + 1\right)^2}\mathrm{E_{0.7}(-x^{0.7})\cos_{0.5}(x^{0.7})}$$

Using "*Theorem 3.4*"

$$y_{pi} = \mathrm{E_{0.7}(-x^{0.7})}\frac{1}{\mathfrak{D}_{\mathcal{J}}^{2\alpha}}\ \mathrm{\cos_{0.7}(x^{0.7})} = -\mathrm{E_{0.7}(-x^{0.7})\cos_{0.7}(x^{0.7})}\ 4$$

The complete solution of "Eq. (29)" is

$$y(x) = \mathrm{C_1 E_{0.7}(-x^{0.7}) + x^{0.7} C_2 E_{0.7}(-x^{0.7}) - E_{0.7}(-x^{0.7})\cos_{0.7}(x^{0.7})}$$

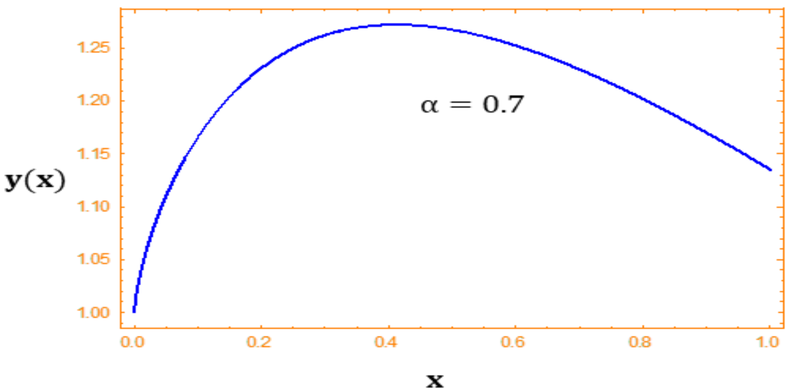

Fig. 11. The solutions for "Eq. (29)" for $\alpha = 0.7$.

If $\alpha = 1$,then the C.F. is $y_{com}(\mathrm{x}) = \mathrm{C_1}\, e^{-x} + \mathrm{x}\, e^{-x}$

The particular integral is $y_{pi} = \frac{1}{\mathfrak{D}^2+2\mathfrak{D}+1}e^{-x}\cos x = \frac{1}{(\mathfrak{D}+1)^2}\ e^{-x}\cos x$

Using "*Theorem 3.4*"

$$y_{pi} = e^{-x}\frac{1}{\mathfrak{D}^2} \cos x = -e^{-x}\cos x$$

The complete solution of "Eq. (29)" is

$$y(x) = C_1 e^{-x} + x\, C_2 e^{-x} - e^{-x}\cos x$$

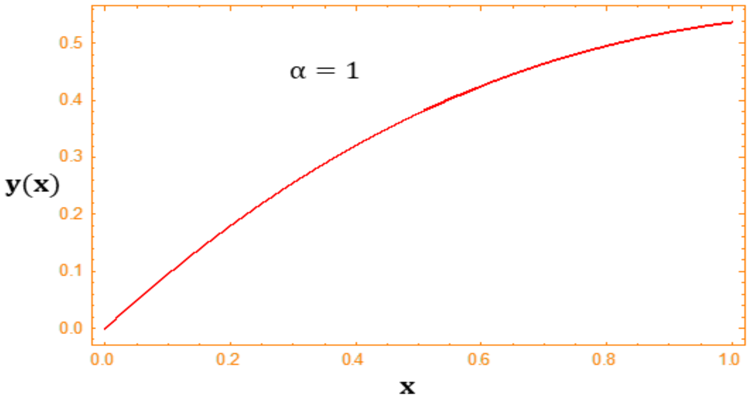

Fig. 12. The solutions for "Eq. (29)" for $\alpha = 1$.

Figs. 10 to 12 demonstrate that as the value of α becomes closer to 1, then the graphs of fractional differential equations $\left(\mathfrak{D}_{\mathcal{J}}^{2(0.5)} + 2\,\mathfrak{D}_{\mathcal{J}}^{0.5} + 1\right)y(x) = E_{0.5}(-x^{0.5})\cos_{0.5}(x^{0.5})$ and $\left(\mathfrak{D}_{\mathcal{J}}^{2(0.7)} + 2\,\mathfrak{D}_{\mathcal{J}}^{0.7} + 1\right)y(x)\ E_{0.7}(-x^{0.7})\cos_{0.7}(x^{0.7})$ coincide with the graph of the solution of a classical differential equation $(\mathfrak{D}^2 + 2\mathfrak{D} + 1)y(x) = e^{x}\cos x$.

5. Conclusion

In this research paper, an analytical method to solve linear inhomogeneous fractional differential equations with constant coefficients is developed using jumarie fractional derivative. This method is based on finding complementary functions and particular integrals of fractional differential equations and gives an association with the method to solve ordinary classical differential equations of integer order. This method is easier and more accurate.

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