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Short Communication

A Note on Pair Sum Graphs

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Abstract

Let G be a (p,q) graph. An injective map $f: V(G) \rightarrow \{\pm 1, \pm 2, ..., \pm p\}$ is called a pair sum labeling if the induced edge function, $f_e: E(G) \rightarrow Z - \{0\}$ defined by $f_e(uv)=f(u)+f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{q/2}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{(q-1)/2}\} \cup \{k_{(q+1)/2}\}$ according as q is even or odd. Here we prove that every graph is a subgraph of a connected pair sum graph. Also we investigate the pair sum labeling of some graphs which are obtained from cycles. Finally we enumerate all pair sum graphs of order ≤ 5.

Keywords: Cycle; Path; Bistar; Complete graph; Complete bipartite graph; Triangular snake.

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1. Introduction

The graphs considered here will be finite, undirected and simple. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. p and q denote, respectively the number of vertices and edges of G and are called the order and size of G. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup (G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The graph $C_n \hat{O} K_{1,m}$ is obtained from C_n and $K_{1,m}$ by identifying any vertex of C_n and the central vertex of $K_{1,m}$. Terms and terminology are used here in the sense of Harary [1], Chartand [2] and Gallian [3]. The notion of pair sum labeling has been introduced in [4]. In [4] we have investigated the pair sum labeling behavior of complete graph, cycle, path, bistar etc. In this note we show that every graph is a subgraph of a pair sum graph. Also we determine all pair sum graphs of order ≤ 5.

2. Pair sum labeling

Definition 2.1: Let G be a (p, q) graph. A one - one map $f: V(G) \rightarrow \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling if the induced edge function $f_e: E(G) \rightarrow Z - \{0\}$ defined by

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$f_e(uv)=f(u)+f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{q/2}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{(q-1)/2}\} \cup \{k_{(q+1)/2}\}$ according as q is even or odd. A graph with a pair sum labeling defined on it is called a pair sum graph.

We now give some basic results on pair sum graphs.

Theorem 2.2: Every graph is a subgraph of a connected pair sum graph.

Proof: Let G be a graph with n vertices. Take two copies of the complete graph K_n. Let u_i $(1 \le i \le n)$ be the vertices of the first copy of K_n and v_i $(1 \le i \le n)$ the vertices in the second copy of K_n. Let $m=2^n-2n$. We construct the new graph G^* as follows.

Let $V(K_{1,m}) = \{w, w_i : 1 \le i \le m\}$ and $E(K_{1,m}) = \{ww_i : 1 \le i \le m\}$.The graph G^* is obtained from $K_n \cup K_n \cup K_{1,m}$ by joining the vertex w to u_1 and v_2. Obviously G is a subgraph of G^*. Assign the label $2^{i+1}$to u_i $(1 \le i \le n)$ and -2^{i+1} to v_i $(1 \le i \le n)$. Label the vertices $w_1, w_2, \ldots, w_{m/2}$ by $1, 3, 5, \ldots, m-1$ and $w_{(m/2)+1}, w_{(m/2)+2}, \ldots, w_m$ by $-5, -7, \ldots, -(m+3)$. Finally assign the integer 2 to w. Clearly this vertex labeling is a pair sum labeling of G^*.

Theorem 2.3: If G is a pair sum graph then $G \cup mK_1$ is also a pair sum graph.

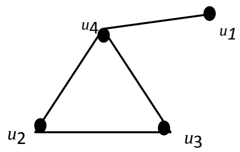
Proof: Let f be a pair sum labeling of G. Let $V(G) = \{u_i : 1 \le i \le p\}$ and $V(mK_1) = \{v_i : 1 \le i \le m\}$.

Define $h: G \cup mK_1 \rightarrow \{\pm 1, \pm 2, \ldots, \pm (p+m)\}$ by $h(u_i) = f(u_i)$ and $h(v_i) = p+i$ $(1 \le i \le m)$. Then clearly h is a pair sum labeling of $G \cup mK_1$.

3. On pair sum graphs of order ≤ 5.

Here we enumerate all pair sum graphs of order ≤ 5. For enumeration of pair sum graphs we need the following results.

Theorem 3.1 [4]**:** Any path is a pair sum graph.
Theorem 3.2 [4]**:** Any cycle is a pair sum graph.
Theorem 3.3 [4]**:** The complete graph K_n is a pair sum graph iff $n \le 4$.
Theorem 3.4 [4]**:** The complete bipartite graphs $K_{1,n}$ and $K_{2,n}$ are pair sum graphs.
Theorem 3.5 [4]**:** The bistar $B_{m,n}$ is a pair sum graph.
Theorem 3.6 [4]**:** If G is a pair sum graph with pair sum labeling f, then $\Sigma_{u \in V(G)} d(u) f(u) = 0$ if size of G is even.
Theorem 3.7 [4]**:** In a pair sum graph, x and $-x$ are not labels of adjacent vertices.
Theorem 3.8 [5]**:** Any triangular snake is a pair sum graph.

Theorem 3.9: The graph 

is not a pair sum graph.

Proof: Let $f(u_1) = x_1$, $f(u_2) = x_2$, $f(u_3) = x_3$ and $f(u_4) = x_4$.

By Theorem 3.6, $x_1+2x_2+2x_3+3x_4 = 0$

=>$x_1+x_4 = -2[x_2+x_3+x_4] \rightarrow (1)$
=>$2| x_1+x_4$
=>$x_1+x_4 = 2k \rightarrow (2)$.

Case (i): $k = 1$
=>$x_1+x_4 = 2 \rightarrow (3)$
=>$-x_1+x_2+x_3 = -3 \rightarrow (4)$

Let (l, m, n) be the ordered 3- tuples which satisfy the equation (4). Using theorem 3.7 and definition 2.1, we have the following ordered 3- tuples.

(i) (1, 2,-4) (ii) (1,-4, 2) (iii) (2, 3,-4) (iv) (2,-4, 3) (v) (-2,-1,-4) (vi) (-2,-4,-1) (vii) (3, 1,-1) (viii) (3,-1, 1) (ix) (3, 2,-2) (x) (3,-2, 2) (xi) (3,4,-4) (xii) (3,-4,4) (xiii) (-3, -2,-4) (xiv) (-3,-4,-2) (xv) (4, -1,2) (xvi) (4,2,-1) (xvii) (4,-2,3) (xviii) (4,3,-2).

Sub case (i) $(x_1, x_2, x_3) = (1, 2,-4)$
By (3), $x_4 = 1$, which is not possible.

Sub case (ii) $(x_1, x_2, x_3) = (1,-4, 2)$
By (3), $x_4 = 1$, which is an impossibility.

Sub case (iii) $(x_1, x_2, x_3) = (2, 3,-4)$
By (3), $x_4 = 0$, which is a contradiction to definition 2.1.

Sub case (iv) $(x_1, x_2, x_3) = (2,-4, 3)$
By (3), $x_4 = 0$, which is again contradiction to the definition.

Sub case (v) $(x_1, x_2, x_3) = (-2,-1, -4)$
By (3), $x_4 = 4$, then the label of the edge u_4u_3 is 0, a contradiction.

Sub case (vi) $(x_1, x_2, x_3) = (-2,-4, -1)$
By (3), $x_4 = 4$, which is not possible as in sub case (v).

Sub case (vii) $(x_1, x_2, x_3) = (3, 1, -1)$
By (3), $x_4 = -1$, which is not possible, since -1 act as two edge labels.

Sub case (viii) $(x_1, x_2, x_3) = (3,-1, 1)$
By (3), $x_4 = -1$, which is not possible as in sub case (vii).

Sub case (ix) $(x_1, x_2, x_3) \in \{(3, 2, -2), (3,-2, 2), (3, 4,-4), (3,-4, 4)\}$
By theorem 3.6, which is not possible.

Sub case (x) $(x_1, x_2, x_3) \in \{(-3,-2, -4), (-3,-1,-2)\}$
By (3), $x_4 = 5$, which is not possible.

Sub case (xi) $(x_1, x_2, x_3) \in \{(4,-1, 2), (4, 2,-1)\}$
By (3), $x_4 = -2$ and in this case, the label of u_4u_3 is zero which is a contradiction to definition 2.1.

Sub case (xii) $(x_1, x_2, x_3) \in \{(4,-2, 3), (4, 3,-2)\}$
By (3), $x_4 = -2$.Then two edges receive 2 as an edge label.

In all the above sub cases, the given graph is not a pair sum graph.

Case (ii): $k = 2$
By (2), $x_1+x_4 = 4 \rightarrow (5)$
=>$x_1+x_2+x_3 = -6 \rightarrow (6)$

In this case, we have the following ordered 3- tuples.

(i) (1, -2,-3) (ii) (1,-3,-2) (iii) (1,-4,-1) (iv) (1,-1,-4) (v) (-1,-3,-4) (vi) (-1,-4,-3) (vii) (2, -1,-3) (viii) (2,-3, -1) (ix) (3, -1,-2) (x) (3,-2, -1) (xi) (3,1,-4) (xii) (3,-4,1) (xiii) (4, 1, -3) (xiv) (4,-3, 1).

Sub case (i) $(x_1, x_2, x_3) \in \{ (1, -2,-3) , (1,-3,-2) \}$

By (5), $x_4 = 3$. In this case the edge label u_4u_3 becomes zero which is a contradiction.

Sub case (ii) $(x_1, x_2, x_3) \in \{(1, -4,-1) , (1,-1,-4)\}$

By (5), $x_4 = 3$, which is a contradiction.

Sub case (iii) $(x_1, x_2, x_3) \in \{(-1, -3,-4) , (-1,-4,-3)\}$

By (5), $x_4 = 5$, which is not possible.

Sub case (iv) $(x_1, x_2, x_3) \in \{(2, -1,-3) , (2,-3,-1)\}$

By (5), $x_4 = 2$, which is not possible.

Sub case (v) $(x_1, x_2, x_3) \in \{(3, -1,-2) , (3,-2,-1)\}$

By (5), $x_4 = 1$ and this implies that zero appears as an edge label, a contradiction.

Sub case (vi) $(x_1, x_2, x_3) \in \{(3, 1,-4) , (3,-4, 1)\}$

By (5), $x_4 = 1$, a contradiction.

Sub case (vii) $(x_1, x_2, x_3) \in \{(4, 1, -3) , (4, -3, 1)\}$

By (5), $x_4 = 0$, a contradiction.

In all the above sub cases, the given graph is not a pair sum graph.

Case (iii): $k = 3$

By (2), $x_1+x_4 = 6 \rightarrow (7)$

$\Rightarrow x+x_2+x_3 = -9 \rightarrow (8)$

Here we have the following ordered 3- tuples.

(i) (2, -4,-3) (ii) (2,-3, -4) (iii) (3,-4,-2) (iv) (3,-2, -4) (v) (4,-2,-3) (vi) (-4,-3,-2) (vii) (4, -4,-1) (viii) (4,-1, -4)

Sub case (i) $(x_1, x_2, x_3) \in \{(2, -4,-3) , (2,-3,-4)\}$

By (7), $x_4 = 4$, which is a contradiction since, zero appear as an edge label.

Sub case (ii) $(x_1, x_2, x_3) \in \{(3, -4,-2) , (3,-2,-4)\}$

By (7), $x_4 = 2$, which is a contradiction.

Sub case (iii) $(x_1, x_2, x_3) \in \{(4, -2,-3) , (4,-3,-2)\}$

By (7), $x_4 = 2$ and since the edge label is zero, we have a contradiction.

Sub case (iv) $(x_1, x_2, x_3) \in \{(4, -4,-1) , (4,-1,-4)\}$

By (7), $x_4 = 2$. In this case, all the edge labels are different, a contradiction.

Case (iv): $k = -1,-2,-3$.

Similar arguments as in case (i) to (iii) we get a contradiction.

Case (v): $k > 3$.

Then $x_1+x_4 \geq 8$, which is not possible.

Theorem 3.10: $C_3 \hat{O} K_{1,m}$ is a pair sum graph if, $m > 1$

Proof:

Case (i): $m = 1$

By theorem 3.9, $C_3 \hat{O} K_{1, m}$ is not a pair sum graph.

Case (ii): $m > 1$

Let C_3 be the cycle $u_1 u_2 u_3 u_1$. Let $V(K_{1,m}) = \{v, v_i: 1 \le i \le n\}$ and $E(K_{1,m}) = \{vv_i: 1 \le i \le m\}$. Without loss of generality we assume v is identified to u_1.
Define $f(u) = 1$, $f(u_2) = -2$, $f(u_3) = 3$, $f(v_i) = -4-i$ $1 \le i \le m/2$ if m is even and $1 \le i \le (m+1)/2$ if m is odd, $f(v_{(m/2)+i}) = 3+i$ $1 \le i \le m/2$ if m is even and $1 \le i \le (m-1)/2$ if m is odd. Clearly f is a pair sum labeling of $C_3 \hat{O} K_{1,m}$.

Theorem 3.11: $C_4 \hat{O} K_{1,m}$ is a pair sum graph for all m.

Proof:

Without loss of generality we assume u is identified to u_1.

Define $f(u) = -1$
$f(u_2) = 2$
$f(u_3) = 1$
$f(u_4) = -2$
$f(v_i) = -2-i$ $1 \le i \le m/2$ if m is even and $1 \le i \le (m+1)/2$ if m is odd
$f(u_{(m/2)+i}) = 4+i$ $1 \le i \le m/2$ if m is even and $1 \le i \le (m-1)/2$ if m is odd

Then f is a pair sum labeling. Therefore $C_4 \hat{O} K_{1,m}$ is a pair sum graph.

Theorem 3.12: Let C_n: $u_1 u_2 \ldots u_n u_1$ be the cycle where $n \equiv 0,1,2 \pmod 4$. Let G be the graph with $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{u_1u_3\}$. Then G is a pair sum graph.

Proof:

Case (i): $n = 4m+2$
$f(u_i) = 2(m+1)-i$ $1 \le i \le 2m+1$
$f(u_{2m+1+i}) = -(2(m+1)-i)$ $1 \le i \le 2m+1$

Case (ii): $n = 4m$
$f(u_1) = 2m+1$
$f(u_{i+1}) = 2m-i$ $1 \le i \le 2m-1$
$f(u_{2m+1}) = -(2m+1)$
$f(u_{2m+1+i}) = -(2m-i)$ $1 \le i \le 2m-1$

Case (iii): $n = 4m+1$
$f(u_1) = -4$
$f(u_{1+i}) = i$ $1 \le i \le 2m+1$
$f(u_{n-2i+2}) = -(2i-1)$ $1 \le i \le m$
$f(u_{n-2i+1}) = -(4+2i)$ $1 \le i \le m-1$

Then f is a pair sum labeling. Therefore G is a pair sum graph.

Theorem 3.13: Let C_n: $u_1 u_2 \ldots u_n u_1$ be the cycle. Let G be the graph with $V(G) = V(C_n) \cup \{v_1, v_2\}$ and $E(G) = E(C_n) \cup \{u_1v_1, u_nv_2\}$. Then G is a pair sum graph.

Proof:

Case (i): n is odd, $n = 2m+1$
$f(v_1) = m+2$

$$f(v_2) = \begin{cases} -m & \text{when } m \text{ is odd} \\ -(m+4) & \text{when } m \text{ is even} \end{cases}$$

$f(u_i) = m+2-i$ $\quad 1 \leq i \leq m+1$
$f(u_{m+2i}) = -2-2i$ $\quad 1 \leq i \leq m/2$ if m is even and $1 \leq i \leq (m+1)/2$ if m is odd
$f(u_{m+2i+1}) = 1-2i$ $\quad 1 \leq i \leq m/2$ if m is even and $1 \leq i \leq (m-1)/2$ if m is odd

Case (ii): n is even

Sub case (a) $n = 4m+2$

$f(v_1) = 4m+2$
$f(v_2) = -(2m+2)$
$f(u_i) = i$ $\quad 1 \leq i \leq 2m+1$
$f(u_{2m+1+i}) = -i$ $\quad 1 \leq i \leq 2m+1$

Sub case (b) $n = 4m$

$f(v_1) = 4m+2$
$f(v_2) = -(2m+2)$
$f(u_i) = i$ $\quad 1 \leq i \leq 2m-1$
$f(u_{2m}) = 2m+1$
$f(u_{2m+i}) = -i$ $\quad 1 \leq i \leq 2m-1$
$f(u_{4m}) = -(2m+1)$

Then f is a pair sum labeling. Therefore G is a pair sum graph.

Theorem 3.14: K_2+mK_1 is a pair sum graph for all values of m.

Proof:

Let f be a pair sum labeling of $K_{2,m}$ as in theorem 3.4.
Define $h: V(K_2+mK_1) \rightarrow \{\pm 1, \pm 2, \ldots, \pm (m+2)\}$ by $h(u) = f(u)$ for all $u \in V(K_2+mK_1)$.
Clearly $h_e(E(K_2+mK_1)) = f_e(E(K_{2,m})) \cup \{-m-3\}$. Therefore h is a pair sum labeling.

Theorem 3.15: The following graphs are pair sum.

(a). K_1, $2K_1$, $3K_1$, $4K_1$, $5K_1$, P_2, $P_2 \cup K_1$, P_3, P_4, $P_2 \cup 2K_1$, $P_3 \cup K_1$, $2P_2$, $P_2 \cup 3K_1$, $P_3 \cup 2K_1$, $2P_2 \cup K_1$, $P_4 \cup K_1$, $P_3 \cup P_2$, P_5.

(b). C_3, $C_3 \cup 2K_1$, $C_4 \cup K_1$, $C_3 \cup P_2$, C_4, C_5, $C_3 \cup K_1$.

Proof: Graphs in case (a) are pair sum by theorems 3.1and 2.3. In case (b) graphs are pair sum from theorems 3.2 and 2.3.

Theorem 3.16: The following graphs are pair sum.

(a). $K_{1,3}$, $K_{2,3}$, $K_{1,4}$, $K_{1,3} \cup K_1$

(b). K_2+3K_1

Proof: (a) Follows from theorems 3.4 and 2.3.
(b) Follows from 3.14.

Theorem 3.17: K_4 and $K_4 \cup K_1$ are pair sum graphs.

Proof:

Follows from theorems 3.2 and 2.3.

Theorem 3.18: Graphs given below in Fig. 1 are pair sum.

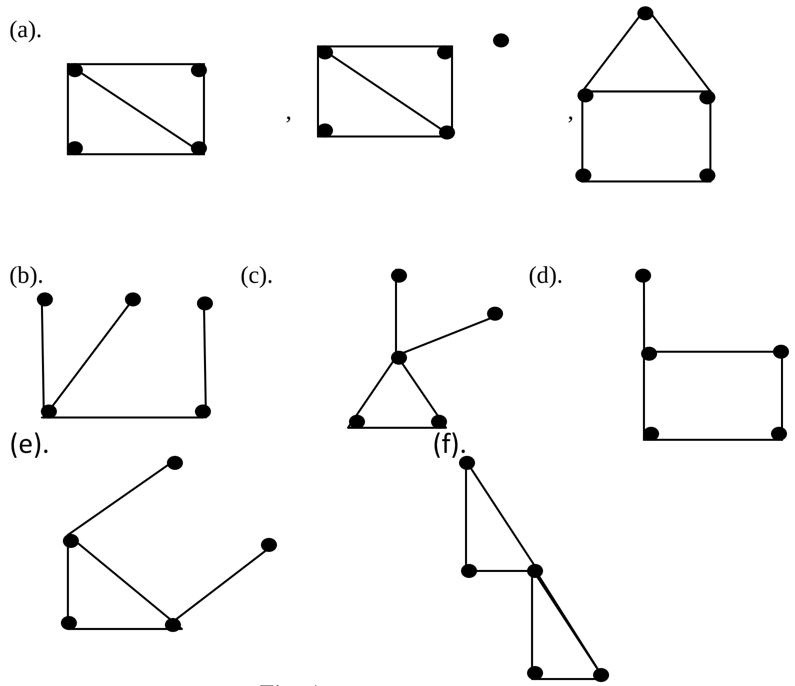


Fig. 1

Proof:

Using theorem 3.12 and theorem 2.3 case (a) graphs are pair sum graph. In case (b) graph is isomorphic to $B_{2,1}$ and hence follows from 3.5. Graphs in case (c) and case (d) are pair sum by theorem 3.10 and theorem 3.11 respectively. Pair sum labeling of graph (e) follows from theorem 3.13.Graph in case (f) is a pair sum by theorem 3.8.

Theorem 3.19: The following graphs (Fig. 2) are not pair sum.

Fig. 2

Proof: Follows from theorems 3.9 and 3.3 respectively.

Finally we display a pair sum labeling of remaining nine graphs (Fig. 3).

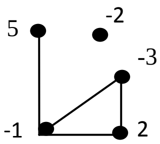
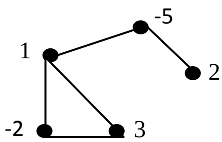
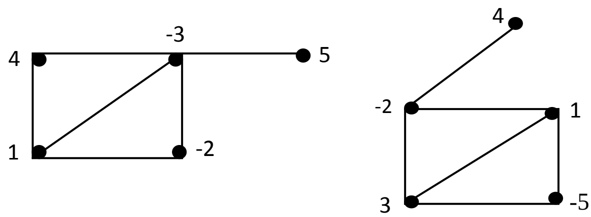
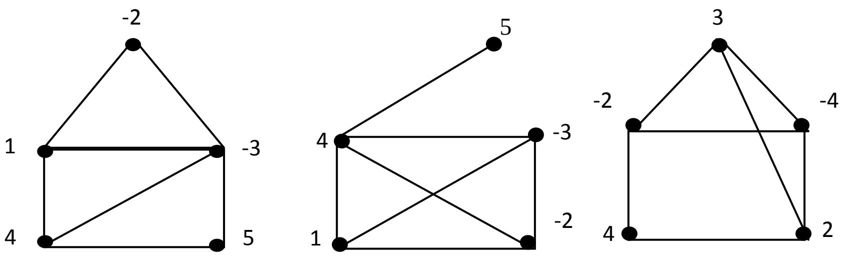
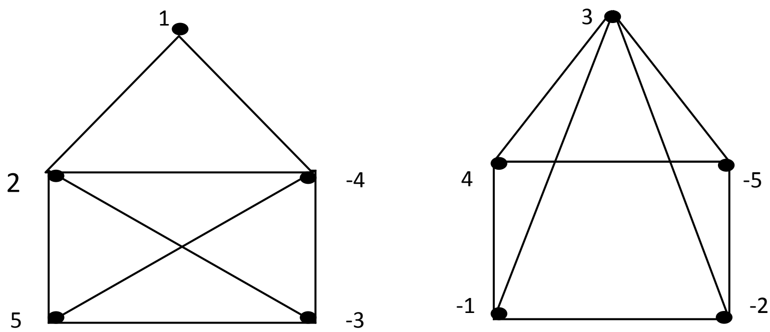
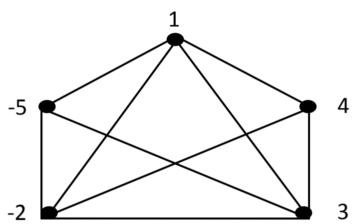
Size	Pair sum labeling
4	
5	
6	
7	
8	
9	

Fig. 3

3. Remark

The following table (Table 1) gives the number of graphs of order ≤ 5 which are pair sum and not pair sum.

Table 1.

Order	Pair sum	Not pair sum
1	1	0
2	2	0
3	4	0
4	10	1
5	33	1

4. Conclusion

Out of 52 graphs, 50 graphs are pair sum and 2 graphs are not pair sum.

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