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Numerical Solution of Cahn-Allen Equations by Wavelet Based Lifting Schemes

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Abstract

The Cahn-Allen equation is a reaction-diffusion equation of mathematical physics that describes the process of phase separation in multi-component alloy systems and order-disorder transitions. This paper presented a numerical solution of Cahn-Allen equations by Lifting scheme using different wavelet filter coefficients. The numerical results obtained using this scheme are compared with the exact solution to demonstrate the accuracy and fast convergence in less computational time than the existing scheme. Some problems are taken to show the validity and applicability of the scheme.

Keywords: Orthogonal and Biorthogonal wavelets; Cahn-Allen equation; Lifting scheme.

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1. Introduction

The Cahn-Allen equation is a parabolic partial differential equation that simulates a natural phenomenon in some ways. Numerous physical issues, including crystal formation, image fragmentation, and curvature flow, have been studied using this equation. It has specifically evolved into the fundamental mathematical model of the linking approach used in physical science research on phase transformation and face flexibility. As a result, finding a reliable and precise solution to this issue has gained the interest of numerous researchers. It is difficult to find precise answers or figures for various non-line integers. But now that powerful software and contemporary computers have been developed, it is possible to use analytical or numerical methods to solve these issues [1]. Recently, methods have been utilized to solve partial differential equations, both linear and nonlinear, numerically and analytically, for instance, Hermite polynomials [2], a comprehensive approximation system based on biorthogonal wavelets [3], the Adomian Decomposition Method and Haar Wavelet Method [4] etc.

Wavelets have been employed as a partial differential equation (PDE) solution ever since the 1980s. The analyzed data demonstrates the benefits of this approach to

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establishing unity, odd structures, and transient situations. Galerkin's approach or the collocation method are the foundations of PDE wavelet resolution algorithms.

Some of the works on wavelet-based methods are discrete wavelet transforms (DWT) and full approximation schemes (FAS) [5,6]. The wavelet-based full approximation scheme (WFAS) has proven to be a very efficient and effective approach to many problems related to the fields of computer science and engineering [7]. These techniques can be used as an iterative solution or as a precaution, providing, in many cases, better performance than other highly developed and existing FAS algorithms.

Because of the effectiveness and power of the WFAS, more research has been done to improve it. To accomplish this task, create a flexible orthogonal/biorthogonal discrete wavelet function using the lifting scheme [8]. A wavelet-based lifting scheme was introduced by Sweldens [9], which allows some to improve existing wavelet transform structures. A wavelet-based numerical solution for elasto hydrodynamic lubrication problems with a lift pattern was developed by Shirolasheti *et al*. [10]. The strategy has certain quantitative advantages, such as a reduced number of tasks, which are important in the context of repetitive solutions. Obviously, all efforts to simplify the wavelet PDE solutions are welcome. In PDE, the matrices resulting from the system are dense with a smooth diagonal and smooth off-diagonal. This matrix slip is minimized using a wavelet transform, resulting in an efficient wavelet-based lift scheme.

The lifting scheme is a new way to construct second-generation wavelets, which do not broadcast or expand a single function. The latter refers to wavelets of the first generation or classical methods. The lifting scheme has additional advantages over classical wavelets. This change applies to signals of arbitrary size with proper boundary management. Another feature of the propositional schema is that all objects are in the local domain. This is in contrast to the traditional method, which relies heavily on the frequency domain.

The two major advantages are (i) it leads to a more attractive treatment that is better suited to those who love apps than the basics of mathematics, and (ii) it makes the calculation time more accurate and sometimes increases the calculation speed.

The lifting process starts with a series of well-known filters, after which lifting steps are applied to improve (left) the decay properties of the corresponding wavelet. This process has certain mathematical advantages in the form of a reduced number of critical functions in the context of iterative solutions. In addition, the current work proposes using the scheme to offer a numerical solution to the Cahn-Allen equation.

The current paper is structured as follows: Section 2 introduces the wavelet filter coefficients and lifting scheme. The solution method describes in section 3. In section 4 we provide the numerical results for the problems, and finally, in section 5 the conclusion of the proposed work is given.

2. Preliminaries of Wavelet Filter Coefficients and Lifting Scheme

The lifting program begins with a collection of well-known filters; then, lifting measures are used in an effort to improve the decay properties of the corresponding wavelet.
Now, we have discussed the various wavelet filters as follows:

2.1. *Haar wavelet filter coefficients*

We know that low-pass filter coefficients, $[h_0, h_1]^T = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$ and high-pass filter coefficients $[g_0, g_1]^T = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$ play a significant role in decomposition.

2.2. *Daubechies wavelet filter coefficients*

The Daubechies presented scaling functions with very short support. The measurement function ϕ_N is supported $[0, N-1]$, while the corresponding wavelet ψ_N has support in the interval $\left[1-\frac{N}{2}, \frac{N}{2}\right]$.

We have low pass filter coefficients $[\boldsymbol{h_0}, \boldsymbol{h_1}, \boldsymbol{h_2}, \boldsymbol{h_3}]^{\boldsymbol{T}} = \left[\frac{\mathbf{1+\sqrt{3}}}{\mathbf{4\sqrt{2}}}, \frac{\mathbf{3+\sqrt{3}}}{\mathbf{4\sqrt{2}}}, \frac{\mathbf{3-\sqrt{3}}}{\mathbf{4\sqrt{2}}}, \frac{\mathbf{1-\sqrt{3}}}{\mathbf{4\sqrt{2}}}\right]^{\boldsymbol{T}}$ and high pass filter coefficients $[\boldsymbol{g_0}, \boldsymbol{g_1}, \boldsymbol{g_2}, \boldsymbol{g_3}]^{\boldsymbol{T}} = \left[\frac{\mathbf{1-\sqrt{3}}}{\mathbf{4\sqrt{2}}}, -\frac{\mathbf{3-\sqrt{3}}}{\mathbf{4\sqrt{2}}}, \frac{\mathbf{3+\sqrt{3}}}{\mathbf{4\sqrt{2}}}, -\frac{\mathbf{1+\sqrt{3}}}{\mathbf{4\sqrt{2}}}\right]^{\boldsymbol{T}}$

2.3. *Biorthogonal (CDF (2,2)) wavelets*

Let's consider the (5, 3) biorthogonal spline wavelet filter pair; the low pass filter pair are $(\tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ and $(h_{-2}, h_{-1}, h_0, h_1, h_2) = \left(\frac{-1}{4\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{-1}{4\sqrt{2}}\right)$.

But, we have $g_k = (-1)^k \tilde{h}_{1-k}$ and $\tilde{g}_k = (-1)^k h_{1-k}$, the high pass filter pair are $g_0 = \frac{1}{2\sqrt{2}}, g_1 = \frac{-1}{\sqrt{2}}, g_2 = \frac{1}{2\sqrt{2}}$ & $\tilde{g}_{-1} = \frac{1}{4\sqrt{2}}, \tilde{g}_0 = \frac{1}{2\sqrt{2}}, \tilde{g}_1 = \frac{-3}{2\sqrt{2}}, \tilde{g}_2 = \frac{1}{2\sqrt{2}}, \tilde{g}_3 = \frac{1}{4\sqrt{2}}$

Foundations of lifting scheme: Consider the numbers h and g as two neighboring samples in sequence, and this has some connection that we would like to take advantage. A simple line replacement of h and g with a scale of average s and a difference d, i.e., $s = \frac{h+g}{2}$ & $d = g - h$

The theory is that if a and b are closely related, the total expected value of their difference d will be smaller and can be represented by bits of fever. In the case of a = b, the difference is simply zero. We have not lost any information because we can always return h and g to give s and d as: $h = s - \frac{d}{2}$ & $g = s + \frac{d}{2}$

Finally, the wavelet transformation formed by lifting consists of three steps: split. Predict and update as given in Fig. 1 [11].

Split: Splitting the signal into two separate sets of samples.

Predict: If a signal contains a specific structure, we may expect a correlation between the sample and its immediate neighbors, i.e. $d_{j-1} = \text{odd}_{j-1} - P(\text{even}_{j-1})$
Update: Given an even entry, we have predicted that the next odd entry has the same value and stored the difference. We then update our even entry to reflect our knowledge of the signal. i.e. $s_{j-1} = \text{even}_{j-1} + U(d_{j-1})$
A detailed algorithm that uses different wavelets is given in the next section. The general lifting stages of decomposition and signal reconstruction are given in Fig. 2.

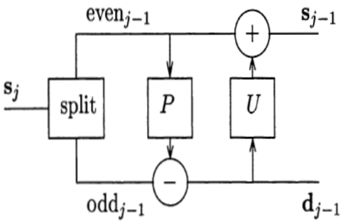


Fig. 1. Steps in lifting scheme.

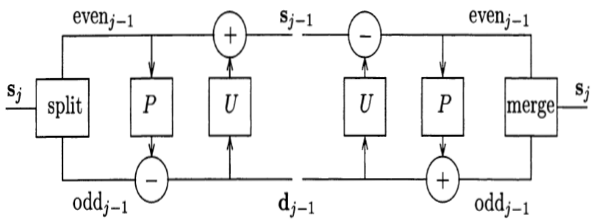


Fig. 2. Lifting wavelet algorithm.

A detailed algorithm that uses different wavelets is given in the next section.

3. Method of Solution

Consider the Cahn-Allen equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1 - u^2),\ 0 \le x \le 1\ \&\ t > 0 \tag{3.1}$$

Where α is any constant.

After discretizing equation (3.1) through the finite difference method (FDM), we get a system of algebraic equations. Through this system, we can write the system as

$$Au = b \tag{3.2}$$

where A is $N \times N$ coefficient matrix, b is $N \times N$ matrix, and u is $N \times N$ matrix to be determined. where $N = 2^J$, N is the number of grid points, and J is the level of resolution.

Solve Eq. (3.2) using the iterative method, we find a approximate solution. The approximate solution contains a specific error, so the sought solution is equal to the sum of the solution and the error. There are many ways to minimize such an error to find an exact solution. Some of them are FAS, WFAS, etc. We now use an advanced method based on different wavelets called a lifting scheme. More recently, lifting schemes have played an important role in signal analysis and image processing in science and technology. But now it reaches statistical measurement [7]. Here, we discuss the algorithm for lifting schemes as follows:

3.1. *Haar wavelet lifting scheme (HWLS)*

Dabeshis and Sveldens showed that every wavelet filter can be transformed into lift steps [8]. More details on the advantages and other important advantages of the proposed method framework can be found in literature [9]. Representation of the Haar wavelet in the form of a nomination presented as;

Decomposition:

Consider an approximate solution $S = u$ as a signal and use the HWLS (finer to coarser) decomposition process such as,

$$\left.\begin{array}{l} d^{(1)} = S_{2j} - S_{2j-1}, \\ s^{(1)} = S_{2j-1} + \frac{1}{2}d^{(1)}, S_1 = \sqrt{2}s^{(1)} \\ \text{and} D = \frac{1}{\sqrt{2}}d^{(1)} \end{array}\right\} \quad (3.3)$$

At this point in the end, we get a new approximation as,

$$S = [S_1 D] \quad (3.4)$$

Reconstruction:

Consider Eq. (3.2) and apply the Here, we get a new approximation process of rebuilding HWLS (coarser to finer) as,

$$\left.\begin{array}{l} d^{(1)} = \sqrt{2}D, \\ s^{(1)} = \frac{1}{\sqrt{2}}S_1, \\ S_{2j-1} = s^{(1)} - \frac{1}{2}d^{(1)} \\ \text{and } S_{2j} = d^{(1)} + S_{2j-1} \end{array}\right\} \quad (3.5)$$

which is the required solution of the given equation.

3.2. Daubechies wavelet lifting scheme (DWLS)

As discussed in section 3.1, we followed the same procedure but used a different wavelet, i.e., Daubechies 4th order wavelet coefficient. The DWLS process is as follows;

Decomposition:

$$\left.\begin{array}{l} s^{(1)} = S_{2j-1} + \sqrt{3}S_{2j}, \\ d^{(1)} = S_{2j} - \frac{\sqrt{3}}{4}s^{(1)} - \left(\frac{\sqrt{3}-2}{4}\right)s_1^{(j-1)}, \\ s^{(2)} = s^{(1)} - d_1^{(j+1)}, \\ S_1 = \frac{\sqrt{3}-1}{\sqrt{2}}s^{(2)} \text{and} \\ D = \frac{\sqrt{3}+1}{\sqrt{2}}d^{(1)} \end{array}\right\} \quad (3.6)$$

Here, we find a new approximation as,

$$S = [S_1 D] \quad (3.7)$$

Reconstruction:

Consider Eq. (3.5), and use the DWLS reconstruction process (coarser to finer) such as,

$$\left.\begin{aligned} &d^{(1)} = \frac{\sqrt{2}}{\sqrt{3}+1} D, \\ &s^{(2)} = \frac{\sqrt{2}}{\sqrt{3}-1} S_1, \\ &s_1^{(j)} = s^{(2)} + d_1^{(j+1)}, \\ &S_{2j} = d^{(1)} + \frac{\sqrt{3}}{4} s_1^{(j)} + \frac{\sqrt{3}-2}{4} s_1^{(j-1)} \text{and} \\ &S_{2j-1} = s^{(1)} - \sqrt{3} S_{2j} \end{aligned}\right\} \tag{3.8}$$

which is the required solution of the given equation.

3.3. *Biorthogonal wavelet lifting scheme (BWLS)*

As discussed in sections 3.1 and 3.2, we follow the same procedure here using another wavelet, i.e., biorthogonal wavelet (CDF (2,2)). The BWLS process is as follows;

Decomposition:

$$\left.\begin{aligned} &d^{(1)} = S_{2j} - \frac{1}{2}\left[S_{2j-1} + S_{2j+2}\right], \\ &s^{(1)} = S_{2j-1} + \frac{1}{4}\left[d_{j-1}^{(1)} + d^{(1)}\right], \\ &D = \frac{1}{\sqrt{2}} d^{(1)}, \\ &S_1 = \sqrt{2} s^{(1)} \end{aligned}\right\} \tag{3.9}$$

In this stage, finally, we get a new signal as,

$$S = [S_1 D] \tag{3.10}$$

Reconstruction:

Consider Eqn. (3.10), then apply the DWLS reconstruction (coarser to finer) procedure as

$$\left.\begin{aligned} &s^{(1)} = \frac{1}{\sqrt{2}} S_1, \\ &d^{(1)} = \sqrt{2} D, \\ &S_{2j-1} = s^{(1)} - \frac{1}{4}\left[d_{j-1}^{(1)} + d^{(1)}\right] \\ &S_{2j} = d^{(1)} + \frac{1}{2}\left[S_{2j-1} + S_{2j+2})\right], \end{aligned}\right\} \tag{3.11}$$

which is the required solution of the given equation.

Coefficients $s_1^{(j)}$ and $d_1^{(j)}$ are the average and detailed coefficients, respectively, of the approximate solution u_a . New methods are tested for some numerical problems, and the results are shown in the next section.

4. Numerical Illustration

In this section, we have used the Lifting scheme for the numerical solution of Cahn-Allen's equations and demonstrated the power and effectiveness of HWLS, DWLS, and BWLS. The error is calculated as, $E_{\max} = \max\left|u_e(x,t) - u_a(x,t)\right|$ where $u_e(x,t)$ and $u_a(x,t)$ are exact and approximate solutions, respectively.

Problem 4.1: Consider the Cahn-Allen equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u^2), 0 \le x \le 1 \& t > 0 \tag{4.1}$$

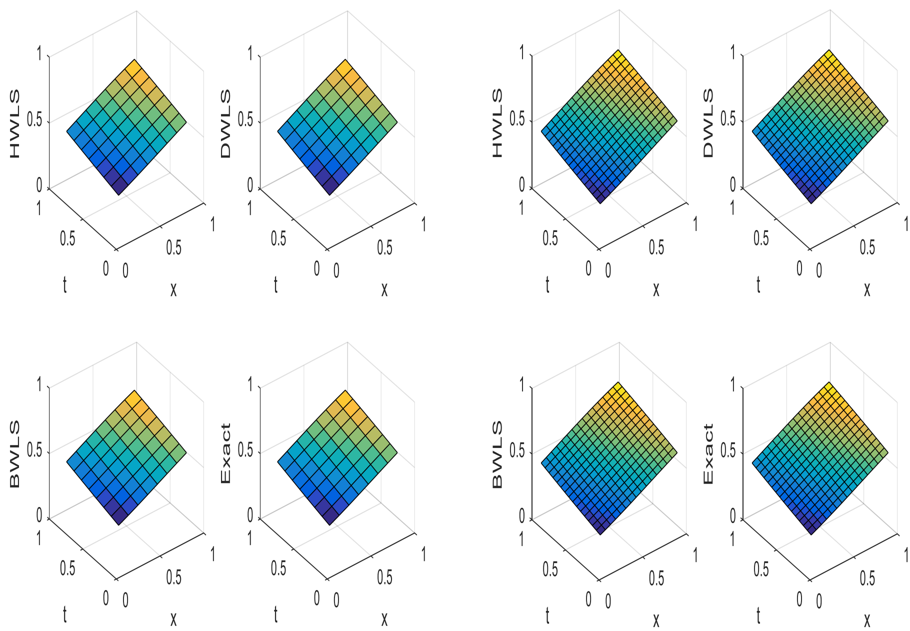
subject to the I.C.: $u(x,0) = \frac{1}{1+e^{\frac{\sqrt{2}}{2}x+1}}$ (4.2)

and B.C.s:

$$\left.\begin{aligned} u(0,t) &= \frac{1}{1+e^{\frac{3}{2}t+1}} \\ u(1,t) &= \frac{1}{1+e^{\frac{\sqrt{2}}{2}\left(1+\frac{3\sqrt{2}}{2}t\right)+1}} \end{aligned}\right\} \tag{4.3}$$

Which has the exact solution $u(x,t) = \frac{1}{1+e^{\frac{\sqrt{2}}{2}\left(x+\frac{3\sqrt{2}}{2}t\right)+1}}$ [12].

Using the methods described in section 3, we find the numerical solutions and, in comparison with the exact solutions, are presented in Fig. 3. The maximum absolute errors with CPU time of the methods are presented in Table 1.



a) $N \times N = 8 \times 8$ b) $N \times N = 16 \times 16$

Fig. 3. Comparison of numerical solutions with the exact solution of problem 4.1. for a) $N \times N = 8 \times 8$ and b) $N \times N = 16 \times 16$.

Table 1. Maximum error and CPU time (in seconds) for the methods of problem 4.1.

$N \times N$	Method	E_{max}	Setup time	Running time	Total time
4×4	FDM	2.6118e-03	6.9161	0.0019	6.9180
	HWLS	2.6118e-03	0.0010	0.0029	0.0039
	DWLS	2.6118e-03	0.0003	0.0095	0.0098
	BWLS	2.6118e-03	0.0003	0.0040	0.0043
16×16	FDM	5.5119e-04	7.0204	0.0023	7.0227
	HWLS	5.5119e-04	0.0012	0.0029	0.0041
	DWLS	5.5119e-04	0.0003	0.0043	0.0046
	BWLS	5.5119e-04	0.0005	0.0027	0.0032
64×64	FDM	1.6530e-04	8.1284	0.0039	8.1323
	HWLS	1.6530e-04	0.0009	0.0032	0.0041
	DWLS	1.6530e-04	0.0003	0.0096	0.0099
	BWLS	1.6530e-04	0.0004	0.0043	0.0047

Problem 4.2: Consider, another Cahn-Allen equation with different conditions

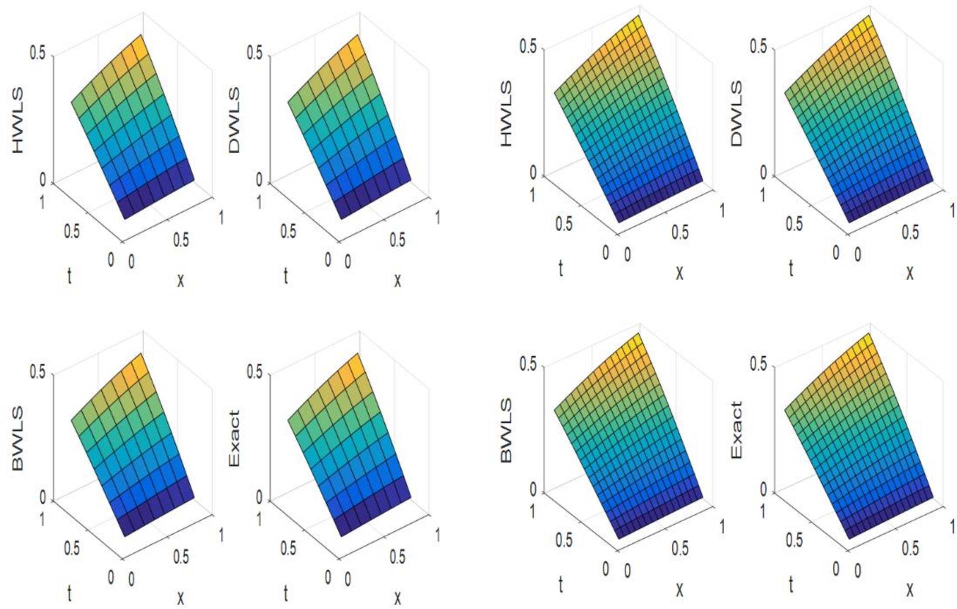
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u^2), 0 \leq x \leq 1 \& t > 0 \tag{4.4}$$

subject to the I.C.: $u(x,0) = \frac{sinh\left(\frac{x}{\sqrt{2}}\right)}{1+cosh\left(\frac{x}{\sqrt{2}}\right)}$ (4.5)

and B.C.s:
$$\left.\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= \frac{e^{\frac{1}{\sqrt{2}}} - e^{-\frac{1}{\sqrt{2}}}}{e^{\frac{1}{\sqrt{2}}} + e^{-\frac{1}{\sqrt{2}}} + 2e^{-\left(\frac{3}{2}\right)t}} \end{aligned}\right\} \tag{4.6}$$

Which has the exact solution $u(x,t) == \frac{e^{\frac{x}{\sqrt{2}}} - e^{-\frac{x}{\sqrt{2}}}}{e^{\frac{x}{\sqrt{2}}} + e^{-\frac{x}{\sqrt{2}}} + 2e^{-\left(\frac{3}{2}\right)t}}$ [13].

By applying the methods explained in Section 3, we get the numerical solutions and compared them with the exact solutions presented in Fig. 4. The maximum absolute errors with the CPU time of the methods are presented in Table 2.



b) $N \times N = 8 \times 8$ b) $N \times N = 16 \times 16$

Fig. 4. Comparison of numerical solutions with the exact solution of problem 4.2. for a) $N \times N = 8 \times 8$ and b) $N \times N = 16 \times 16$.

Table 2. Maximum error and CPU time (in seconds) for the methods of problem 4.2.

$N \times N$	Method	E_{max}	Setup time	Running time	Total time
1	2	3	4	5	6
4×4	FDM	1.1747e-03	3.7246	0.0019	3.7265
	HWLS	1.1747e-03	0.0010	0.0029	0.0039
	DWLS	1.1747e-03	0.0003	0.0098	0.0101
	BWLS	1.1747e-03	0.0003	0.0040	0.0043
8×8	FDM	6.1610e-04	5.1221	0.0569	5.1790
	HWLS	6.1610e-04	0.0093	0.0646	0.0739
	DWLS	6.1610e-04	0.0075	0.2419	0.2494
	BWLS	6.1610e-04	0.0076	0.0934	0.1010
16×16	FDM	3.2847e-04	3.5575	0.0023	3.5598
	HWLS	3.2847e-04	0.0010	0.0029	0.0039
	DWLS	3.2847e-04	0.0003	0.0096	0.0099
	BWLS	3.2847e-04	0.0004	0.0040	0.0044
32×32	FDM	1.7954e-04	5.3060	0.0028	5.3088
	HWLS	1.7954e-04	0.0010	0.0040	0.0050
	DWLS	1.7954e-04	0.0003	0.0094	0.0097
	BWLS	1.7954e-04	0.0003	0.0041	0.0044
64×64	FDM	1.0187e-04	7.2244	0.0040	7.2284
	HWLS	1.0187e-04	0.0010	0.0030	0.0040
	DWLS	1.0187e-04	0.0003	0.0097	0.0100
	BWLS	1.0187e-04	0.0004	0.0044	0.0048

5. Conclusion

In this paper, we find the numerical solution of Cahn-Allen equations using different wavelet filters by Lifting schemes. We observe that

- The numerical solutions obtained by the different Lifting schemes agree with the exact solution.
- The convergence of the presented schemes, i.e., the error decreases as the level of resolution N increases.
- In addition, the calculations involved in the lifting schemes are simpler, more straightforward, and lower in calculation costs compared to the classical method, i.e., FDM.

The Lifting schemes introduced in particular by HWLS and BWLS are therefore very effective for solving nonlinear partial differential equations.

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