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**Publications**

Polynomial Collocation Methods Based on Successive Integration Technique for Solving Neutral Delay Differential Equations

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Abstract

This paper presents a new approach to using polynomials such as Hermite, Bernoulli, Chebyshev, Fibonacci, and Bessel to solve neutral delay differential equations. The proposed method is based on the truncated polynomial expansion of the function together with collocation points and successive integration techniques. This method reduces the given equation to a system of nonlinear equations with unknown polynomial coefficients which can be easily calculated. The convergence of the proposed method is discussed with several mild conditions. Numerical examples are considered to demonstrate the efficiency of the method. The numerical results reveal that the proposed new approach gives better results than the conventional operational matrix approach of the polynomial collocation method. It demonstrates the reliability and efficiency of this method for solving linear and nonlinear neutral delay differential equations.

*Keywords*: Polynomials; Collocation method; Successive integration technique; Neutral delay differential equations.

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1. Introduction

Delay Differential Equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. The terms involving previous times are called delay terms. The delay terms are classified as constant, state-dependent, and time-dependent. Neutral delay differential equations (NDDEs) are another type of DDEs in which the highest-order derivative of the unknown function occurs with delay terms. DDEs arise in the fields of signal processing, digital images, control systems, epidemiology, chemical kinetics, etc. Some notable applications of DDEs and NDDEs are in chemical kinetics [1], climate model [2], SIR epidemic model [3], iterative survival model of red blood cells [4], immunology model [5] and cell growth model [6].

 DDEs and NDDEs have been studied by many authors and developed various analytical and numerical methods. Some of them are the Adams predictor corrector algorithm [7], Homotopy perturbation method [8], Reproducing kernel Hilbert space method [9], Variational iteration method [10], Elzaki transforms method [11], Rishi transforms method [12], Haar wavelet series method [13], Higher order derivative Runge Kutta method [14], Composite Runge Kutta methods and new one-step techniques [15], Hybrid Multistep block method [16] and Generalized Rational multistep method [17] for solving DDEs and NDDEs.

 The Collocation method based on various polynomials is a powerful technique for solving differential equations. Gulsu *et al.* [18,19] have proposed a collocation method based on Hermite and Chebyshev polynomials for solving DDEs with variable coefficients under mixed conditions. Yiizbas *et al.*[20] have presented the Bessel polynomial operational matrix method for solving NDDEs. Bhrawy *et al.* [21] have proposed a Legendre-Gauss collocation method for solving NDDEs with proportional delay. Tohidi *et al.* [22] have presented the Bernoulli operational matrix for solving DDEs. Koc *et al.* [23] have presented a matrix method based on Fibonacci polynomials for solving DDEs. Ibis *et al.* [24] have applied Hermite polynomials for solving NDDEs with proportional delays.

 The above-mentioned collocation methods using different polynomials are based on operational matrices. In this study, we propose a new approach of using Polynomial Collocation methods based on the Successive Integration Technique for solving linear and nonlinear NDDEs.

 This paper is organized as follows: Section 2 gives the basic definitions of different polynomials. Section 3 provides a description of the method for solving NDDEs. In Section 4, the convergence analysis of the proposed method is discussed. In Section 5, illustrative examples are provided.

**2. Basic Definition of Polynomials**

**2.1. *Hermite polynomial***

The Hermite polynomial of order n is defined on the interval There are different ways to define a Hermite polynomial; one of them is the so-called Rodrigues’ formula.

 (1)

From Eqn. (1) the recurrence relation for the polynomials can be derived as

 (2)

 can be obtained from Eqn. (1) the remaining terms are determined using the recursion relation Eqn. (2).

Thus, we have the following sequence of polynomials:

The order Hermite polynomial has a leading coefficient

**2.2. *Bernoulli polynomial***

The Bernoulli polynomial is named after Jacob Bernoulli, which combines the Bernoulli numbers and binomial coefficients. The generating function for the Bernoulli polynomial of order n is defined by

 (3)

The recursion formula for the Bernoulli polynomial is:

*,*  (4)

can be obtained from Eqn. (3) the remaining terms are determined using the recursion relation Eqn. (4). Thus, we have a few terms of the Bernoulli polynomials as:

**2.3. *Chebyshev polynomial***

The Chebyshev polynomial related to cosine functions on the interval of order n is defined as

 (5)

The recursion relation of the Chebyshev polynomial is:

 (6)

 *and* can be obtained from Eqn. (5). Then the remaining terms are determined from Eqn. (6). Thus, we have the following sequence of polynomials:

**2.4. *Fibonacci polynomial***

The Fibonacci polynomials are a polynomial sequence that can be considered Fibonacci numbers. The Fibonacci polynomials are defined by a recurrence relation

The first few Fibonacci polynomials are:

**2.5. *Bessel polynomial***

The Bessel polynomial is defined by

The recursion equation for the Bessel polynomial is:

The first few Bessel polynomials are:

**3. Description of the Proposed Method**

Consider the nth-order NDDE of the form.

 (7)

with initial conditions

 *for*  (8)

Here is the initial function, and is the delay term.

Let P(t) represent any orthogonal polynomials. For the proposed method, it is assumed that

 (9)

where N is any positive integer,

Here T stands for transpose of the matrix.

The aim is to determine the polynomial coefficients For this, Eqn. (9) is integrated with respect to t from

 (10)

Now, for delay terms

 (11)

Then eqns. (10) and (11) are substituted in (7) and use the collocating points  *,* where  *i = 0, 1…N.* This yields a system of linear or nonlinear equations subject to the linear and nonlinear terms in Eqn. (7). On solving this system of equations, we get the respective polynomial coefficients *’s* from which the solution of the NDDE (7) can be obtained.

**4. Convergency Analysis**

Consider the first-order NDDE of the form.

 (12)

with initial value condition

*.* (13)

Here and are given analytical functions.

 The convergence of the proposed method will be provided under several mild conditions, such as the solution boundedness of the Eqn. (12). Some definitions and lemmas are provided to clarify this section's main convergence theorem.

**4.1. *Definition 4.1*** *[21]*

A function belongs to the Sobolev space , if its weak derivative lies in for all with the norm

where denotes the usual Lebesgue norm

and stands for any finite-dimensional norm in

**4.2. *Lemma 4.1*** *[21]*

For a given function there exists a polynomial of degree N such that

where, is a constant independent of and is the order of smoothness of . Here with the smallest norm is known as the order best polynomial approximation of in the norm of

 Note that if then This implies that converges to at a spectral rate, that is, it would be faster than any given polynomial rate. Moreover, let us denote the set of continuous functions in a linear space on by and the uniform norm in by

Now integrating the Eqn. (12) in the interval and using the initial condition (13), we get

Taking we rewrite the above equation in the following form

 (14)

where

and

 .

In the following Theorem 4.1, we show that the approximate solution expressed in terms of the orthogonal polynomials converges to the exact solution under several mild conditions.

**4.3. *Theorem 4.1***

Let and be the exact and numerical solutions of Eqn. (12). Also, assume the approximations of and be and respectively. Moreover, suppose that. and where

Then, subject to the condition

**Proof:**

Suppose that the unknown functions and are approximated in terms of any orthogonal polynomials. Then, the numerical solution is an approximated polynomial in the form of . We need to find an upper bound for the error between and for Eqn. (14).

According to the assumptions, Eqn. (14) can be written as

 (15)

Now, Eqn. (14) – Eqn. (15) yields

By using the triangle inequality, we get

Since and , the above inequality reduces to

This can be rewritten as

Using the assumptions

 .

in the above inequality, we get

Let us introduce the following notations:

,

.

Then, the above equation becomes

If , then

i.e.,

This is possible because of the smoothness of and , Lemma 4.1 implies

 and

This completes the proof.

**5. Numerical Examples**

In this section, four numerical examples of linear and nonlinear NDDEs are given to demonstrate the accuracy and effectiveness of the proposed collocation method based on successive integration techniques.

**Example 1** [20] Consider the first-order linear NDDE with proportional delay

with initial condition, .

The exact solution is .

The numerical results of the proposed method are compared with the the conventional operational matrix approach using the Bessel polynomial for different values oven in Table 1. The solution graph obtained using the proposed method with N=10 is presented in Fig. 1.

Table 1. Absolute Errors for Example 1.

|  |  |
| --- | --- |
| Time t | Bessel polynomial collocation method |
| N = 3 | N = 6 | N = 10 |
| Matrix Approach | Successive Integration Technique | Matrix Approach | Successive Integration Technique | Matrix Approach | Successive Integration Technique |
| 0.2 | 1.14 e-3 | 2.82 e-04 | 3.66 e-7 | 5.30 e-08 | 7.54 e-13 | 2.17 e-13 |
| 0.4 | 1.56 e-3 | 3.69 e-04 | 2.12 e-7 | 3.23 e-08 | 4.72 e-13 | 2.21 e-13 |
| 0.6 | 3.26 e-4 | 1.61 e-04 | 1.36 e-7 | 2.13 e-08 | 2.47 e-13 | 2.76 e-14 |
| 0.8 | 4.20 e-4 | 8.27 e-05 | 1.90 e-7 | 1.65 e-08 | 4.81 e-14 | 4.13 e-14 |
| 1.0 | 7.82 e-3 | 1.97 e-04 | 3.74 e-6 | 1.16 e-08 | 1.67e-11 | 5.96 e-13 |



Fig. 1. Solution Graph for Example 1.

**Example 2** [19] Consider the second-order linear NDDE with constant delay and variable coefficient

with initial condition and .

The exact solution is .

The numerical results of the proposed method are compared with the conventional operational matrix approach by using the Chebyshev polynomial for different values of N are given in Table 2. The solution graph obtained using the proposed method with N = 8 is presented in Fig. 2.





Fig. 2. Solution Graph for Example 2.

Table 2. Absolute Errors for Example 2.

|  |  |
| --- | --- |
| Time T | Chebyshev Polynomial Collocation Method |
| N = 6 | N = 7 |
| Matrix Approach | Successive Integration Technique | Matrix Approach | Successive Integration Technique |
| -0.2 | 0.30 e-4 | 1.49 e-7 | 0.80 e-5 | 7.21 e-7 |
| -0.4 | 0.95 e-4 | 2.69 e-6 | 0.42 e-4 | 2.90 e-6 |
| -0.6 | 0.16 e-3 | 9.51 e-6 | 0.10 e-3 | 6.30 e-6 |
| -0.8 | 0.21 e-3 | 2.06 e-5 | 0.18 e-3 | 1.05 e-5 |
| -1.0 | 0.26 e-3 | 3.53 e-5 | 0.28e-3 | 1.57e-5 |

**Example 3** [25] Consider the following nonlinear NDDE

with initial condition and

The exact solution is

The absolute errors are determined by using the proposed method based on five polynomials, namely Hermite, Bernoulli, Chebyshev, Fibonacci, and Bessel, with different values of N. The numerical results at t = 1 are presented in Table 3.

Table 3. Absolute Errors for Example 3.

|  |  |  |  |
| --- | --- | --- | --- |
| Polynomials | N = 3 | N = 5 | N = 7 |
| Bessel | 1.17e-03 | 4.30e-05 | 1.66e-05 |
| Bernoulli | 6.08e-05 | 4.11e-06 | 1.62e-07 |
| Chebyshev | 1.24e-03 | 2.38e-05 | 3.45e-06 |
| Hermite | 6.89e-04 | 6.98e-06 | 3.71e-07 |
| Fibonacci | 9.03e-04 | 6.39e-06 | 4.16e-06 |

**Example 4** [25]

Consider the following nonlinear state-dependent NDDE

with initial condition

The exact solution is

The absolute errors are determined by using the proposed method based on five polynomials, namely Hermite, Bernoulli, Chebyshev, Fibonacci, and Bessel, with different values of N. The numerical results at t = 1 are presented in Table 4.

Table 4. Absolute Errors for Example 4.

|  |  |  |  |
| --- | --- | --- | --- |
| Polynomials | N = 3 | N = 5 | N = 7 |
| Bessel | 6.27e-04 | 1.08e-06 | 1.01e-08 |
| Bernoulli | 6.26e-04 | 1.07e-06 | 1.78e-08 |
| Chebyshev | 6.27e-04 | 1.08e-06 | 3.16e-09 |
| Hermite | 6.27e-04 | 1.08e-06 | 3.43e-09 |
| Fibonacci | 6.27e-04 | 1.09e-06 | 3.07e-09 |

**6. Conclusion**

In this paper, a new application of the Polynomial collocation method based on successive integration techniques is presented for solving neutral delay differential equations. The convergence analysis of the presented method has been discussed. Numerical examples of linear and nonlinear neutral delay differential equations are considered to demonstrate the efficiency of the proposed method. It is evident that the proposed polynomial collocation method based on successive integration techniques gives better results than the conventional operational matrix approach. The proposed method is computationally simple but gives results with good accuracy. Also, it is observed that accuracy increases as N increases. Hence, it is concluded that the proposed method is suitable for solving neutral delay differential equations.

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