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MHD Couette Flow of Viscoelastic Fluid Between Upright Permeable Plates with Energy Dissipation

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Abstract

This study investigates the unsteady, magnetohydrodynamic, incompressible, Couette flow of viscoelastic fluid through equidistant upright plates in the presence of permeable media, continuous heat fluxes, heat source, and energy dissipation. The flow is restricted by the flow between two equidistant upright plates, one moving and the other at rest, and the formation of free convection due to a fixed wall at a time-dependent temperature, while heat flow is constant on the wall moving upwards in its plane. The problem is solved analytically, and the expression for velocity and temperature are derived. The effect of numerous physical parameters on the flow is discussed. The local skin friction is also represented graphically, and the Nusselt number is given in the table.

Keywords: MHD; Viscoelasticity; Heat source; Energy dissipation; Suction.

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1. Introduction

The study of viscoelastic fluids exists in various industrial applications. Perez-Reyes *et al.* [1] studied the application of viscoelastic fluids involving hydrodynamic stability and heat transfer. Also, several authors, Naga Raju *et al.* [2], Yuan *et al.* [3], Dianguo [4], and Bital *et al.* [5] have studied different aspects of viscoelastic fluid. Kumar *et al.* [6] studied Williamson fluid flow around a curved/flat surface with a variable heat source/sink. Akter *et al.* [7] investigated Williamson fluid flow over an exponentially stretching sheet in the existence of non-uniform Heat generation with an inclined magnetic field. Goud [8] explained heat generation/absorption influence on the steady stretched permeable surface in the existence of variable suction/ injection. The consequence of viscous dissipation and boundary wall thickness on steady natural convection Couette flow with variable viscosity and thermal conductivity was carried out by Ajibade and Uma [9]. The combined effect of variable viscosity and variable thermal conductivity on natural convection Couette flow was studied by Ajibade and Tafida [10]. Laminar forced convective

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magnetohydrodynamic (MHD) Couette-Poiseuille flow with viscous and Joule dissipations was studied by Sarkar [11].

The unsteady free convection MHD heat and mass transfer flow with radiation has fascinated several researchers, as it has many applications in engineering and industrial processes. MHD flow has applications in metrology, motion of the Earth's core in solar physics, etc. So, there have been so many concerns among the researchers. Gopal *et al.* [12] and. Shahid *et al.* [13] analyzed numerically the impact of MHD nanofluid under the influence of viscosity. Alim [14] investigated the effect of the Hartmann number on the free convective flow of MHD Fluid in a square cavity with a heated cone of different orientations. Kumar *et al.* [15] studied the influence of suction/ injection on MHD Casson fluid flow over a vertical stretching surface. Fenuga *et al.* [16] presented a mathematical model and solution for an unsteady MHD fourth-grade fluid flow over a vertical plate in a porous medium with magnetic field and suction/injection Effects. By an analytic approach, Kumar [17] studied the radiation effect on MHD flow with an induced magnetic field and Newtonian heating/cooling. MHD boundary layer flow of Casson fluid past an inclined plate in the presence of Soret/Dufour effects, heat source, and first-order chemical reaction was studied by Das [18]. Ganesh and Sridhar [19] analyzed the numerical approach of heat and mass transfer of MHD Casson fluid under radiation over an exponentially permeable stretching sheet with chemical reaction and Hall effect. Recently, Mehta *et al.* [20], Khatun *et al.* [21], Nadeem *et al.* [22], Kumar *et al.* [23], and Krishna *et al.* [24] investigated radiation absorption on MHD. Gani *et al.* [25] studied the MHD flow of a nanofluid for free convection past an inclined plate.

To the best of the author's knowledge, no attempt has been made to explore the impacts of thermal radiation, viscous dissipation, and mass transfer effects on unsteady hydromagnetic flows for viscoelastic fluid in a channel. Motivated by the numerous applications, this work investigates the impacts of viscoelasticity, thermal radiation, viscous dissipation effects, and mass transfer effects on the unsteady hydromagnetic flow of viscoelastic fluid along a channel.

This study is an extension of the work of Mehta *et al.* [20] on viscoelastic fluid incorporating energy dissipation.

2. Mathematical Formulation

Consider the unsteady MHD Couette flow of an incompressible, radiating viscoelastic fluid within a parallel, upright infinite plate channel with a continuous heat source and suction. The distance between the similar plates is d. We choose the plate direction as $\bar{x}$-axis and $\bar{y}$ -axis along perpendicular. The impact of energy dissipation is considered. Initially, both the plates were at rest with uniform temperature T_0 .

Thus, the governing equations are

$$\frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{1}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \upsilon \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \frac{K_0}{\rho}\frac{\partial^3 \bar{u}}{\partial \bar{y}^2 \partial \bar{t}} + g\beta\left(\bar{T} - T_0\right) - \frac{\sigma_e {B_0}^2}{\rho}\bar{u} - \frac{\upsilon}{K}\bar{u}, \tag{2}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \frac{\kappa}{\rho C_p}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \frac{\mu}{\rho C_p}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 - \frac{1}{\rho C_p}\frac{\partial q_r}{\partial \bar{y}} - \frac{Q_0}{\rho C_p}\left(\bar{T} - T_0\right) + \frac{K_0}{\rho C_p}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\sigma_e {B_0}^2}{\rho C_p}\bar{u}^2 \tag{3}$$

With boundary conditions

$$\left.\begin{array}{l} \bar{y} = 0: \bar{u} = U_0\left(1 + \varepsilon e^{\bar{n}\bar{t}}\right), \frac{\partial \bar{T}}{\partial \bar{y}} = -\frac{q}{\kappa} \\ \bar{y} = d: \bar{u} = 0, \bar{T} = T_0 + \frac{qd}{\kappa}\left(1 + \varepsilon e^{\bar{n}\bar{t}}\right) \end{array}\right\} \tag{4}$$

The radiative heat flux in terms of Roseland approximations to taken as

$$q_r = -\frac{4\sigma_e}{3\bar{K}}\frac{\partial \bar{T}^4}{\partial \bar{y}} \tag{5}$$

Expanding $\bar{T}^4$ in a Taylors series about T_0 and neglecting terms of higher orders, we get

$$\bar{T}^4 \cong 4{T_0}^3\bar{T} - 3{T_0}^4 \tag{6}$$

Here, we assume the suction velocity [20], which is normal to the plate as

$$\bar{v} = V_0 \tag{7}$$

Using (5) and (6) in equation (3) we obtain

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \frac{\kappa}{\rho C_p}\frac{\partial^2 \bar{T}}{\partial \bar{y}} + \frac{\mu}{\rho C_p}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 + \frac{16\sigma_e {T_0}^2}{\rho C_p}\frac{\partial q_r}{\partial \bar{y}} - \frac{Q_0}{\rho C_p}\left(\bar{T} - T_0\right) + \frac{K_0}{\rho C_p}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\sigma_e {B_0}^2}{\rho C_p}\bar{u}^2 \tag{8}$$

Introducing the following non-dimensional quantities

$$y = \frac{\bar{y}}{d}, t = \frac{\bar{t}\upsilon}{d^2}, u = \frac{\bar{u}}{U_0}, \theta = \frac{\bar{T} - T_0}{\frac{qd}{\kappa}}, n = \frac{d^2\bar{n}}{\upsilon}, Gr = \frac{g\beta d^3 q}{U_0 \kappa \upsilon}, Ha^2 = \frac{\sigma_e {B_0}^2 d^2}{\rho \upsilon}$$

$$, K^2 = \frac{d^2}{\bar{K}}, S = \frac{Q_0 d^2}{\upsilon \rho C_P}, \ Pr = \frac{\upsilon \rho C_p}{\kappa}, R = \frac{K\bar{K}}{4\sigma_e {T_0}^3}, \ Su = \frac{V_0 d}{\upsilon}, K_1 = \frac{K_0}{\rho d^2}, Ec = \frac{{U_0}^2}{qdC_P}.$$

In equation (2) and (8) we get

$$\frac{\partial^2 u}{\partial y^2} - Su\ \frac{\partial u}{\partial y} - K_1\frac{\partial^3 u}{\partial y^2 \partial t} - \frac{\partial u}{\partial t} - (Ha^2 + K^2)u = -Gr\theta \tag{9}$$

$$\left(\frac{3R+4}{3RPr}\right)\frac{\partial^2 \theta}{\partial y^2} - Su\ \frac{\partial \theta}{\partial y} - \frac{\partial \theta}{\partial t} - S\theta + Ec\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 + SuK_1Ec\ \frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = -Ha^2 Ec\ u^2 \tag{10}$$

The non-dimensional boundary conditions are given by

$$\left.\begin{array}{l} \bar{y} = 0: \bar{u} = (1 + \varepsilon e^{nt}), \frac{\partial \theta}{\partial y} = -1; \\ \bar{y} = 1: \bar{u} = 0, \theta = (1 + \varepsilon e^{nt}) \end{array}\right\} \tag{11}$$

3. Method of Solution

For solving equations (9) and (10) with the boundary condition (11), we assume

$$\left.\begin{aligned} u(y,t) &= u_0(y) + \varepsilon e^{nt}(y) + o(\varepsilon^2); \\ \theta(y,t) &= \theta_0(y) + \varepsilon e^{nt}(y) + o(\varepsilon^2) \end{aligned}\right\} \tag{12}$$

Using (12),(9), and (10), and comparing harmonic and non-harmonic terms by neglecting. ε^2, we obtain

$$\frac{\partial^2 u_0}{\partial y^2} - Su\,\frac{\partial u_0}{\partial y} - (Ha^2 + K^2)u_0 = -Gr\theta_0, \tag{13}$$

$$(1 - K_1 n)\frac{\partial^2 u_1}{\partial y^2} - Su\,\frac{\partial u_1}{\partial y} - (Ha^2 + K^2 + n)u_1 = -Gr\theta_1, \tag{14}$$

$$\left(\frac{3R+4}{3PrR}\right)\frac{\partial^2 \theta_0}{\partial y^2} - Su\,\frac{\partial \theta_0}{\partial y} - S\theta_0 + Ec\left(\frac{\partial u_0}{\partial y}\right)^2 + SuK_1Ec\left(\frac{\partial u_0}{\partial y}\right)\left(\frac{\partial^2 u_0}{\partial y^2}\right) = -Ha^2 Ec\, {u_0}^2 \tag{15}$$

$$\left(\frac{3R+4}{3PrR}\right)\frac{\partial^2 \theta_1}{\partial y^2} - Su\,\frac{\partial \theta_1}{\partial y} - (S+n)\theta_1 + 2Ec\left(\frac{\partial u_0}{\partial y}\right)\left(\frac{\partial u_1}{\partial y}\right) +$$

$$SuK_1Ec\left(\frac{\partial u_1}{\partial y}\right)\left(\frac{\partial^2 u_0}{\partial y^2}\right) = -2u_0u_1Ha^2Ec\ . \tag{16}$$

Reduced restrictions are

$$\left.\begin{aligned} y = 0&: u_0 = 1, \frac{\partial \theta_0}{\partial y} = -1; u_1 = 1, \frac{\partial \theta_1}{\partial y} = 0; \\ y = 1&: u_0 = 0, \theta_0 = 1; u_1 = 0, \theta_1 = 0 \end{aligned}\right\} \tag{17}$$

Since$Ec << 1$ for incompressible fluid, we assume

$$\left.\begin{aligned} u_0 &= u_{00} + Ecu_{01}; u_1 = u_{10} + Ecu_{11}; \\ \theta_0 &= \theta_{00} + Ec\theta_{01}; \theta_1 = \theta_{10} + Ec\theta_{11} \end{aligned}\right\} \tag{18}$$

Substituting (18) into the equations (12) to (16) and equating the co-efficient of the same powers of *Ec* and neglecting terms of o(Ec^2) ,we get the following equations

Zeroth order equations:

$$\frac{\partial^2 u_{00}}{\partial y^2} - Su\,\frac{\partial u_{00}}{\partial y} - (Ha^2 + K^2)u_{00} = -Gr\theta_{00}, \tag{19}$$

$$(1 - K_1 n)\frac{\partial^2 u_{10}}{\partial y^2} - Su\,\frac{\partial u_{10}}{\partial y} - (Ha^2 + K^2 + n)u_{10} = -Gr\theta_{10}\ , \tag{20}$$

$$\left(\frac{3R+4}{3PrR}\right)\frac{\partial^2 \theta_{00}}{\partial y^2} - Su\,\frac{\partial \theta_{00}}{\partial y} - S\theta_{00} = 0, \tag{21}$$

$$\left(\frac{3R+4}{3PrR}\right)\frac{\partial^2 \theta_{10}}{\partial y^2} - Su\,\frac{\partial \theta_{10}}{\partial y} - S\theta_{10} = 0, \tag{22}$$

First-order equations:

$$\frac{\partial^2 u_{01}}{\partial y^2} - Su\,\frac{\partial u_{01}}{\partial y} - (Ha^2 + K^2)u_{01} = -Gr\theta_{01}, \tag{23}$$

$$(1 - K_1 n)\frac{\partial^2 u_{11}}{\partial y^2} - Su\,\frac{\partial u_{11}}{\partial y} - (Ha^2 + K^2 + n)u_{11} = -Gr\theta_{11}, \tag{24}$$

$$\left(\frac{3R+4}{3PrR}\right)\frac{\partial^2\theta_{01}}{\partial y^2}-Su\ \frac{\partial\theta_{01}}{\partial y}-S\theta_{01}+\left(\frac{\partial u_{00}}{\partial y}\right)^2+SuK_1\left(\frac{\partial u_{00}}{\partial y}\right)\left(\frac{\partial^2 u_{00}}{\partial y^2}\right)=-Ha^2{u_{00}}^2, \quad (25)$$

$$\left(\frac{3R+4}{3PrR}\right)\frac{\partial^2\theta_{11}}{\partial y^2}-Su\ \frac{\partial\theta_{11}}{\partial y}-S\theta_{11}+2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u_{10}}{\partial y}\right)+SuK_1\left(\frac{\partial u_{10}}{\partial y}\right)\left(\frac{\partial^2 u_{10}}{\partial y^2}\right)=$$
$$-2Ha^2u_{00}u_{10}. \quad (26)$$

The boundary conditions (17) become

$$\left.\begin{aligned}&y=0: u_{00}=1, u_{10}=-1, \frac{\partial\theta_{00}}{\partial y}=-1, \frac{\partial\theta_{10}}{\partial y}=0, u_{01}=0, u_{11}=0, \frac{\partial\theta_{01}}{\partial y}=0, \frac{\partial\theta_{11}}{\partial y}=0;\\&y=1: u_{00}=0, u_{10}=0, \theta_{00}=1, \theta_{10}=1, u_{01}=0, u_{11}=0, \theta_{01}=0, \theta_{11}=0\end{aligned}\right\}(27)$$

Solving the equations (19) - (26) with condition (27), we get

$$u_{00}=(A_5+A_7-A_8)e^{-r_5y}-(A_6-A_9+A_{10})e^{-r_6y}-A_{11}e^{-r_1y}+A_{12}e^{-r_2y},$$

$$u_{10}=(1-A_{13}+A_{14}-A_{15})e^{-r_7y}+(A_{13}-A_{16}+A_{17})e^{-r_8y}+A_{18}e^{-r_3y}+A_{19}e^{-r_4y},$$

$$\theta_{00}=A_1e^{-r_1y}-A_2e^{-r_2y},$$

$$\theta_{10}=A_3e^{-r_3y}-A_4e^{-r_4y},$$

$$u_{01}=A_{20}e^{-r_3y}-A_{21}e^{-r_4y}+(A_{22}-A_{23})e^{-r_5y}+(A_{24}-A_{25})e^{-r_6y},$$

$$u_{11}=A_{26}e^{-r_3y}-A_{27}e^{-r_4y}+(A_{28}-A_{29})e^{-r_7y}+(A_{30}-A_{31})e^{-r_8y},$$

$$\theta_{01}=C_1e^{-r_1y}+C_2e^{-r_2y}+D_1e^{-2r_1y}+D_2e^{-2r_2y}+D_3e^{-2r_5y}+D_4e^{-2r_6y}+D_5e^{-(r_5+r_6)y}+D_6e^{-(r_1+r_5)y}+D_7e^{-(r_2+r_5)y}+D_8e^{-(r_1+r_6)}+D_9e^{-(r_2+r_6)y}+D_{10}e^{-(r_1+r_2)y},$$

$$\theta_{11}=C_3e^{-r_3y}+C_4e^{-r_4y}+E_1e^{-2r_3y}+E_2e^{-2r_4y}+E_3e^{-2r_7y}+E_4e^{-2r_8y}+E_5e^{-(r_5+r_7)y}+E_6e^{-(r_5+r_8)y}+E_7e^{-(r_2+r_5)y}+E_8e^{-(r_4+r_5)y}+E_9e^{-(r_6+r_8)y}+E_{10}e^{-(r_3+r_6)y}+E_{11}e^{-(r_4+r_6)y}+E_{12}e^{-(r_1+r_7)y}+E_{13}e^{-(r_1+r_8)y}+E_{14}e^{-(r_1+r_3)y}+E_{15}e^{-(r_1+r_4)y}+E_{16}e^{-(r_2+r_7)y}+E_{17}e^{-(r_2+r_8)y}+E_{18}e^{-(r_2+r_3)y}+E_{19}e^{-(r_2+r_4)y}+E_{20}e^{-(r_7+r_8)y}+E_{21}e^{-(r_3+r_7)y}+E_{22}e^{-(r_4+r_7)y}+E_{23}e^{-(r_3+r_8)y}+E_{24}e^{-(r_3+r_4)y}+E_{25}e^{-(r_4+r_8)y}.$$

Thus, the expressions for velocity and temperature are given by

$$u(y,t)=u_{00}+Ec\,u_{01}+\varepsilon e^{nt}(u_{10}+Ecu_{11}), \quad (28)$$

$$\theta(y,t)=\theta_{00}+Ec\,\theta_{01}+\varepsilon e^{nt}(\theta_{10}+Ec\,\theta_{11}). \quad (29)$$

The coefficient skin friction for the plate $y=0$is

$$\tau=-\left[\frac{\partial u}{\partial y}-K_1\frac{\partial^2 u}{\partial y\partial t}\right]_{y=0}$$

The Nusselt number for the plate $y=0$is

$$Nu = -\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = -\left(\frac{d\theta_{00}}{dy} + Ec\,\frac{d\theta_{01}}{dy} + \varepsilon e^{nt}\left(\frac{d\theta_{10}}{dy} + Ec\,\frac{d\theta_{11}}{dy}\right)\right).$$

4. Results and Discussion

Here, it is worth to mention that the present result coincided with that of Mehta *et al*. [20] when $\mathrm{K}_1 = 0$ and $\mathrm{Ec} = 0$.

For computational purposes, we have assigned the values to the parameters as Gr=0.01, Su =0.1, Ha=1, K=0.2, R=1, Pr=0.1, S=2, Ec=1.

Figs. 1-6 represents the velocity profile for various Hartmann numbers (Ha), suction parameter (Su), Grashof number for heat transfer (Gr), Eckert number (Ec), permeability parameter (K), and Prandtl number (Pr), respectively. Figure 1 shows that the increase in Ha from $\mathrm{Ha} = 1$ to $\mathrm{Ha} = 3$ through $\mathrm{Ha} = 2$ reduces$u$. Thus, the fluid velocity reduces due to the Hartmann number. This is due to the Lorentz force, which acts in the opposite direction of fluid flow. Figure 2 shows that the increase in Su form $\mathrm{Su} = 0.1$ to $\mathrm{Su} = 0.5$ through $\mathrm{Su} = 0.3$ increases the fluid velocity. Figure 3 shows that the enhancement in Gr increases the fluid velocity. This is because the increasing Grashof number increases the buoyancy force of fluid and thus increases the fluid velocity. Figure 4 shows that the Eckert number reduces the fluid velocity. Figure 5 shows that the radiation parameter reduces the fluid velocity. Fig. 6 portrays that the increase in Pr increases the velocity profile.

Figs. 7-14 represent the temperature profile for various Hartmann numbers (Ha), suction parameter (Su), Grashof number (Gr), Eckert number (Ec), permeability parameter (K), Prandtl number (Pr), radiation parameter (R), and heat source parameter (S), respectively. Figure 7 shows that the temperature inside the channel rises as Ha rises. Figure 8 shows that the increment in the suction parameter reduces the fluid temperature. Thus, the additional heat sink agent reduces the temperature of the fluid inside the channel. The impact of the Grashof number (Fig. 9) and the Eckert number (Fig. 10) shows that the increase in the Grashof number and Eckert number increases the temperature of the fluid inside the channel. Eckert number is the relationship between flow kinetic energy and heat enthalpy difference. As the Eckert number rises, so does the kinetic energy, and the temperature of the fluid rises.

Also, it is seen that the fluid temperature reduces due to an increase in permeability parameter (Fig. 11), Prandtl number (Fig. 12), and radiation parameter (Fig. 13). From Fig. 14, it is seen that the increase in from $\mathrm{S} = 2$ to $\mathrm{S} = 6$ through $\mathrm{S} = 4$ raises θ. Thus, the additional heat source parameter raises the temperature inside the channel.

Non-dimensional shear drag is represented as skin friction. Table 1 shows the computed values for skin friction at the surface $(y = 0)$. It shows that the increase in $\mathrm{Gr}, \mathrm{Ha}, \mathrm{K},$ and Pr increases the skin friction coefficient, while the opposite effect is seen for the parameters $\mathrm{Su}, \mathrm{Ec}, \mathrm{R},$ and S.

Table 2 shows the computed values for the Nusselt number at the surface ($y = 0$). It shows that the Nusselt number decreases due to the influence of Gr, Su, Ha, Ec, and Pr, while the opposite effect is seen for the parameter S.

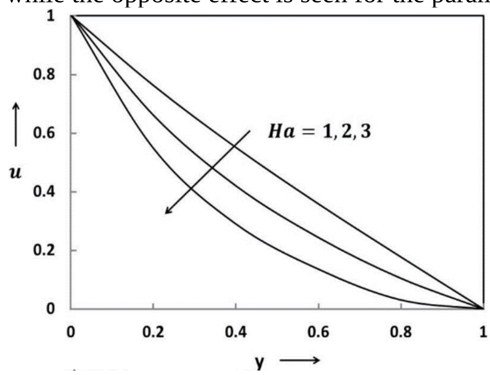

Fig. 1. Velocity profile for Ha.

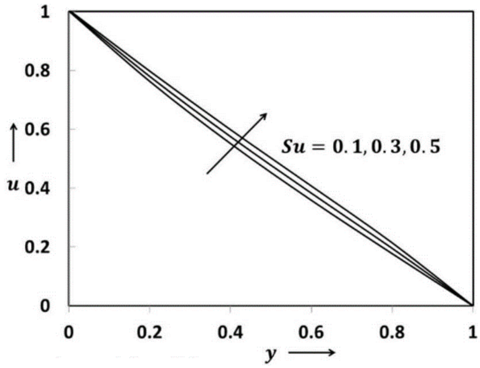

Fig. 2. Velocity profile for Su.

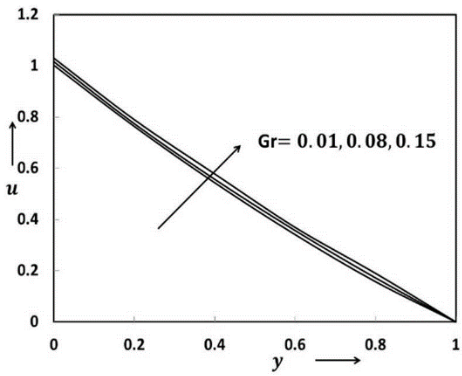

Fig. 3. Velocity profile for Gr.

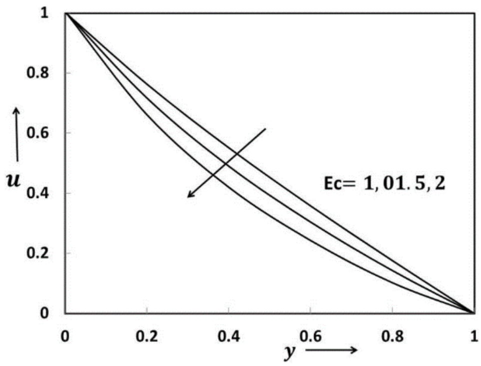

Fig. 4. Velocity profile for Ec.

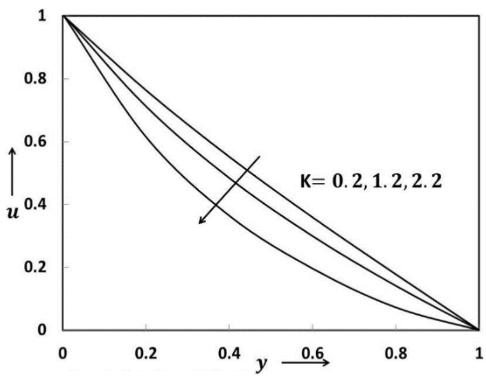

Fig. 5. Velocity profile for K.

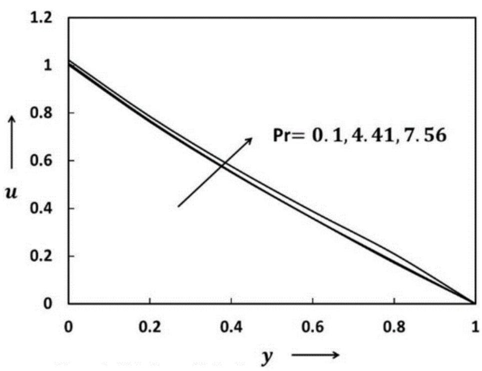

Fig. 6. Velocity profile for Pr.

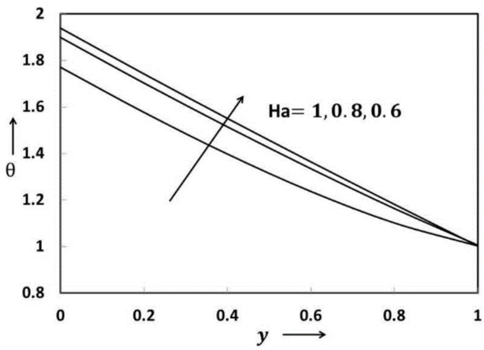

Fig. 7. Temperature profile for Ha.

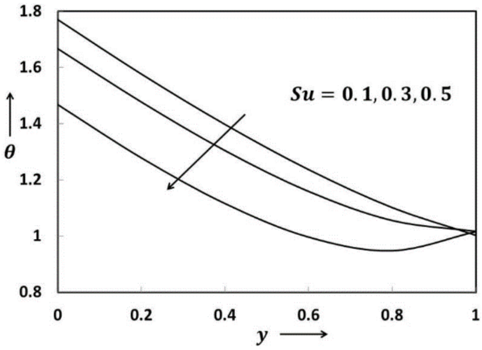

Fig. 8. Temperature profile for Su.

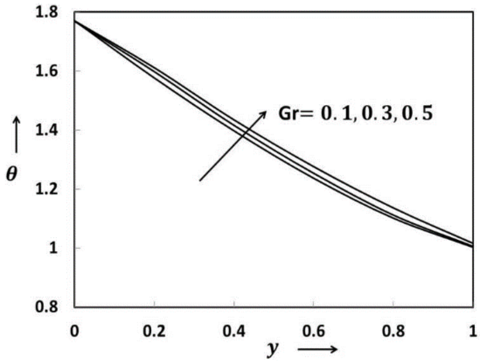

Fig. 9. Temperature profile for Gr.

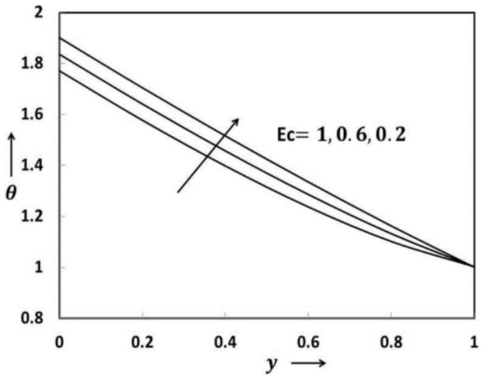

Fig. 10. Temperature profile for Ec.

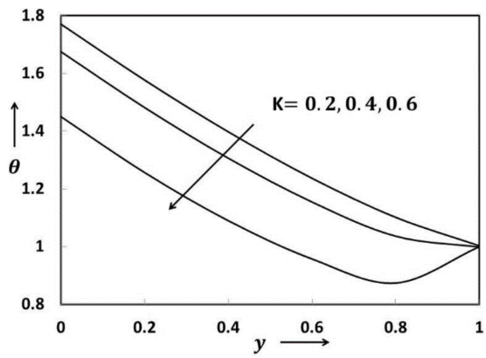

Fig. 11. Temperature profile for K.

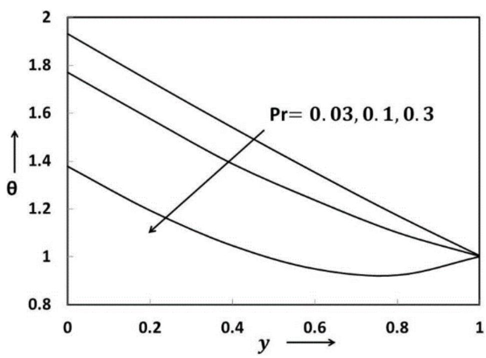

Fig. 12. Temperature profile for Pr.

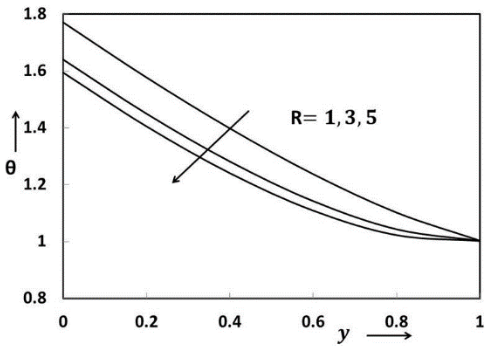

Fig. 13. Temperature profile for R.

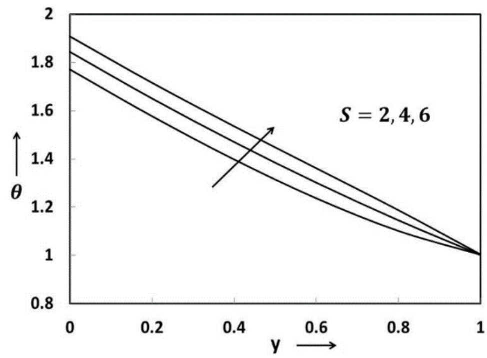

Fig. 14. Temperature profile for S.

Table 1. Skin friction coefficient at the plate $y = 0$.

Gr	Ha	K	Ec	Pr	R	S	Su	Cf_0
0.01	1	0.2	1	0.1	1	2	0.2	1.2762
0.08	1	0.2	1	0.1	1	2	0.2	1.2795
0.15	1	0.2	1	0.1	1	2	0.2	1.2827
0.01	2	0.2	1	0.1	1	2	0.2	2.1867
0.01	3	0.2	1	0.1	1	2	0.2	3.2191
0.01	1	0.2	1	0.1	1	2	0.2	1.6664
0.01	1	1.2	1	0.1	1	2	0.2	2.4321
0.01	1	2.2	1.5	0.1	1	2	0.2	1.2703
0.01	1	0.2	2	0.1	1	2	0.2	1.2645
0.01	1	0.2	1	0.1	1	2	0.2	1.2798
0.01	1	0.2	1	0.1	1	2	0.2	1.2925
0.01	1	0.2	1	0.1	2	2	0.2	1.2744
0.01	1	0.2	1	0.1	3	2	0.2	1.2734
0.01	1	0.2	1	0.1	1	4	0.2	1.2725
0.01	1	0.2	1	0.1	1	6	0.2	1.2687
0.01	1	0.2	1	0.1	1	0.1	0.3	1.1812
0.01	1	0.2	1	0.1	1	0.1	0.5	1.0922

Table 2. Nusselt number for the plate $y = 0$.

Gr	Su	Ha	Ec	Pr	S	Nu_1
0.01	0.1	1	1	0.1	2	-0.7933
0.08	0.1	1	1	0.1	2	-0.8302
0.15	0.1	1	1	0.1	2	-0.8820
0.01	0.3	1	1	0.1	2	-1.1881
0.01	0.5	1	1	0.1	2	-1.9022
0.01	0.1	2	1	0.1	2	-27.9739
0.01	0.1	3	1	0.1	2	-430.2964
0.01	0.1	1	1.5	0.1	2	-1.0466
0.01	0.1	1	2	0.1	2	-1.3000
0.01	0.1	1	1	0.3	2	-1.8852
0.01	0.1	1	1	4.41	2	-4.4744
0.01	0.1	1	1	0.1	4	-0.4936
0.01	0.1	1	1	0.1	6	-0.1987

5. Conclusion

The above study explores the unsteady, magnetohydrodynamic, incompressible Couette flow of viscoelastic fluid across equidistant upright plates in permeable media, continuous heat fluxes, a heat source, and energy dissipation. From the above study, we have that the fluid velocity inside the channel rises due to the influence of the suction parameter, Grashof number, and Prandtl number, while the opposite effect is seen for Hartmann number, Eckert number, and permeability parameter. The fluid temperature inside the channel rises due to the influence of the Grashof number, Eckert number, and heat source parameter, while the opposite behavior is observed for increasing suction parameter, permeability parameter, Prandtl number, and radiation parameter. Also, it is observed that the skin friction coefficient increases due to the increment in Grashof number, Hartman number, permeability parameter, and Prandtl number, while the suction parameter, Eckert number, radiation parameter, and heat source parameter reduce the skin friction. Moreover, the Nusselt number reduces due to the impact of the Grashof number, suction parameter, Hartmann number, Eckert number, and Prandtl number, while the suction parameter increases the Nusselt number.

Nomenclatures

$\boldsymbol{B_0}$ Magnetic field strength
$\boldsymbol{C_p}$ Specific heat at constant pressure
Ec Eckart number
Gr Grashof number
K Permeability parameter
Pr Prandtle number
q_r Radiative heat flux
Ha Hartmann number
Q_0 Heat absorption
R Radiation parameter
S Heat source parameter
K_1 Viscoelastic parameter
S_u Suction
T Time
$\overline{u}$ Dimensional velocity along$\overline{x}$-axis
$\overline{v}$ Dimensional velocity along $\overline{y}$-axis
u Non-dimensional velocity along x-axis
v Non-dimensional velocity along y-axis
y Coordinate axis normal to the plate
N_u Nusselt number
C_f Skin friction

Greek symbols

κ Thermal conductivity
β Co-efficient of thermal expansion
θ Temperature
ρ Fluid density
σ_e Electric conductivity
υ Kinematic viscosity
μ Co-efficient of viscosity

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Appendix

$$A=\frac{3R+4}{3RPr},\ r_1=\frac{-Su-\sqrt{Su^2+4AS}}{2A},\ r_2=\frac{-Su+\sqrt{Su^2+4AS}}{2A},\ r_3=\frac{-Su-\sqrt{Su^2+4A(S+n)}}{2A},$$

$$r_4=\frac{-Su+\sqrt{Su^2+4A(S+n)}}{2A},\ r_5=\frac{-Su-\sqrt{Su^2+4(Ha^2+K^2)}}{2},\ r_6=\frac{-Su+\sqrt{Su^2+4(Ha^2+K^2)}}{2},$$

$$r_7=\frac{-Su-\sqrt{Su^2+4(1-k_1n)(Ha^2+K^2+n)}}{2(1-k_1n)},\ \ r_8=\frac{-Su+\sqrt{Su^2+4(1-k_1n)(Ha^2+K^2+n)}}{2(1-k_1n)},$$

$$A_1=\frac{r_2-e^{-r_2}}{r_2e^{-r_1}-r_1e^{-r_2}},A_2=\frac{r_1-e^{-r_1}}{r_2e^{-r_1}-r_1e^{-r_2}},A_3=\frac{r_4}{r_4e^{-r_3}-r_3e^{-r_4}},$$

$$A_4=\frac{r_3}{r_4e^{-r_3}-r_3e^{-r_4}},A_5=\frac{e^{-r_6}}{e^{-r_6}-e^{-r_5}},A_6=\frac{e^{-r_5}}{e^{-r_6}-e^{-r_5}},$$

$$A_7=\frac{A_1Gr(e^{-r_6}-e^{-r_1})}{(e^{-r_6}-e^{-r_5})\{r_1{}^2+Sur_1-(Ha^2+K^2)\}},A_8=\frac{A_2Gr(e^{-r_2}-e^{-r_6})}{(e^{-r_6}-e^{-r_5})\{r_2{}^2+Sur_2-(Ha^2+K^2)\}},$$

$$A_9=\frac{A_2Gr(e^{-r_6}-e^{-r_2})}{(e^{-r_6}-e^{-r_5})\{r_2{}^2+Sur_2-(Ha^2+K^2)\}},$$

$$A_{10}=\frac{A_1Gr(e^{-r_1}-e^{-r_6})}{(e^{-r_1}-e^{-r_6})\{r_1{}^2+Sur_1-(Ha^2+K^2)\}},$$

$$A_{11}=\frac{A_1Gr}{r_1{}^2+Sur_1-(Ha^2+K^2)},A_{12}=\frac{A_2Gr}{r_2{}^2+Sur_2-(Ha^2+K^2)},$$

$$A_{13}=\frac{e^{-r_7}}{e^{-r_7}-e^{-r_8}},A_{14}=\frac{\{A_3(e^{-r_3}-e^{-r_7})-A_4\}Gr}{(1-k_1n)r_4{}^2+Sur_4-(Ha^2+K^2+n)},$$

$$A_{15}=\frac{\{A_4(e^{-r_3}-e^{-r_7})-A_4\}Gr}{(1-k_1n)r_3{}^2+Sur_3-(Ha^2+K^2+n)},$$

$$A_{16}=\frac{A_3Gr(e^{-r_3}-e^{-r_7})}{(e^{-r_7}-e^{-r_8})\{(1-k_1n)r_4{}^2+Sur_4-(Ha^2+K^2+n)\}},$$

$$A_{17}=\frac{GrA_4(e^{-r_4}-e^{-r_7})}{(e^{-r_7}-e^{-r_8})\{(1-k_1n)r_3{}^2+Sur_3-(Ha^2+K^2+n)\}},$$

$$A_{18}=\frac{A_3Gr}{(1-k_1n)r_3{}^2+Sur_3-(Ha^2+K^2+n)},A_{19}=\frac{A_4Gr}{(1-k_1n)r_4{}^2+Sur_4-(Ha^2+K^2+n)},$$

$$A_{20}=\frac{r_1A_1Gr}{r_1{}^2+Sur_1-(Ha^2+K^2)},A_{21}=\frac{r_2A_2Gr}{r_2{}^2+Sur_2-(Ha^2+K^2)},$$

$$A_{22}=\frac{r_2A_2Gr(e^{-r_6}-e^{-r_2})}{(e^{-r_6}-e^{-r_5})\{r_2{}^2+Sur_2-(Ha^2+K^2)\}},A_{23}=\frac{r_1A_1Gr(e^{-r_6}-e^{-r_1})}{(e^{-r_6}-e^{-r_5})\{r_1{}^2+Sur_1-(Ha^2+K^2)\}}$$

$$A_{24}=\frac{r_1A_1Gr(e^{-r_5}-e^{-r_1})}{(e^{-r_6}-e^{-r_5})\{r_1{}^2+Sur_1-(Ha^2+K^2)\}},A_{25}=\frac{r_2A_2Gr(e^{-r_5}-e^{-r_2})}{(e^{-r_6}-e^{-r_5})\{r_2{}^2+Sur_2-(Ha^2+K^2)\}},$$

$$A_{26}=\frac{r_3A_3Gr}{(1-k_1n)r_3{}^2+Sur_3-(Ha^2+K^2+n)},A_{27}=\frac{r_4A_4Gr}{(1-k_1n)r_4{}^2+Sur_4-(Ha^2+K^2+n)},$$

$A_{28} = \frac{r_3 A_3 Gr(e^{-r_8} - e^{-r_3})}{\{(1-k_1 n)r_3{}^2 + Sur_3 - (Ha^2 + K^2 + n)\}(e^{-r_8} - e^{-r_7})}$,

$A_{29} = \frac{r_4 A_4 (e^{-r_7} - e^{-r_4})}{\{(1-k_1 n)r_4{}^2 + Sur_4 - (Ha^2 + K^2 + n)\}(e^{-r_8} - e^{-r_7})}$,

$A_{30} = \frac{r_3 A_3 Gr(e^{-r_7} - e^{-r_3})}{\{(1-k_1 n)r_3{}^2 + Sur_3 - (Ha^2 + K^2 + n)\}(e^{-r_8} - e^{-r_7})}$, $A_{31} = \frac{r_4 A_4 Gr(e^{-r_7} - e^{-r_4})}{\{(1-k_1 n)r_4{}^2 + Sur_4 - (Ha^2 + K^2 + n)\}(e^{-r_8} - e^{-r_7})}$,

$B_1 = A_{11}{}^2$, $B_2 = A_{12}{}^2$, $B_3 = (A_5 + A_7 - A_8)^2$, $B_4 = -(A_6 - A_9 + A_{10})^2$,

$B_5 = -(A_5 + A_7 - A_8)(A_6 - A_9 + A_{10})$, $B_6 = -2A_{11}(A_5 + A_7 - A_8)$,

$B_7 = 2A_{12}(A_5 + A_7 - A_8)$, $B_8 = 2A_{11}(A_6 - A_9 + A_{10})$, $B_9 = 2A_{12}(A_6 - A_9 + A_{10})$,

$B_{10} = -2A_{11}A_{12}(A_6 - A_9 + A_{10})$, $B_{11} = -A_{11}{}^2 r_1{}^2$, $B_{12} = -A_{12} r_2{}^3$, $B_{13} = -r_5{}^3(A_5 + A_7 - A_8)^2$,

$B_{14} = -r_6{}^3(A_6 - A_9 + A_{10})^2$, $B_{15} = r_5 r_6 (r_5 + r_6)(A_5 + A_7 - A_8)(A_6 - A_9 + A_{10})$,

$B_{16} = -A_{11} r_1 r_6 (r_1 + r_6)(A_6 - A_9 + A_{10})$, $B_{17} = A_{12} r_6 r_2 (r_2 + r_6)(A_6 - A_9 + A_{10})$,

$B_{18} = A_{11} r_1 r_5 (r_1 + r_5)(A_5 + A_7 - A_8)$, $B_{19} = -A_{12} r_2 r_5 (r_2 + r_5)(A_7 - A_8 + A_5)$,

$B_{20} = A_{12} A_{11} r_1 r_2 (r_1 + r_2)$, $B_{21} = A_{11}{}^2 r_1{}^2$, $B_{22} = A_{12}{}^2 r_2{}^2$, $B_{23} = r_5{}^2(A_5 + A_7 - A_8)$,

$B_{24} = r_6{}^2(A_6 - A_9 + A_{10})^2$, $B_{25} = -r_5 r_6 (A_5 + A_7 - A_8)(A_6 - A_9 + A_{10})$,

$B_{26} = 2A_{11} r_1 r_6 (A_6 - A_9 + A_{10})$, $B_{27} = -2A_{12} r_2 r_6 (A_6 - A_9 + A_{10})$,

$B_{27} = -2A_{12} r_2 r_6 (A_6 - A_9 + A_{10})$, $B_{28} = -2A_{11} r_1 r_5 (A_5 + A_7 - A_8)$, $B_{30} = -2A_{11} A_{12} r_1 r_2$,

$$C_1 = D_{21} + D_{22} + D_{23} + D_{24} + D_{25} + D_{26} + D_{27} + D_{28} + D_{29} + D_{30}$$

$C_2 = D_{11} + D_{12} + D_{13} + D_{14} + D_{15} + D_{16} + D_{17} + D_{18} + D_{19} + D_{20}$,

$C_3 = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8 + Z_9 + Z_{10} + Z_{11} + Z_{12} + Z_{13} + Z_{14} + Z_{15} + Z_{16} + Z_{16} \ Z_{17} + Z_{18} + Z_{19} + Z_{20} + Z_{21} + Z_{22} + Z_{23} + Z_{24} + Z_{25}$

$C_4 = Z_{26} + Z_{27} + Z_{28} + Z_{29} + Z_{30} + Z_{31} + Z_{32} + Z_{33} + Z_{34} + Z_{35} + Z_{36} + Z_{37} + Z_{38} + Z_{39} + Z_{40} + Z_{41} + Z_{42} + Z_{43} + Z_{44} + Z_{45} + Z_{46} + Z_{47} + Z_{48} + Z_{49} + Z_{50}$,

$D_1 = -\frac{Ha^2 B_1 + B_{21} + SuK_1 B_{11}}{A(4r_1{}^2 + 2Sur_1 - S)}$, $D_2 = \frac{Ha^2 B_2 + B_{22} + SuK_1 B_{12}}{A(4r_2{}^2 + 2Sur_2 - S)}$, $D_3 = \frac{Ha^2 B_3 + B_{23} + SuK_1 B_{13}}{A(4r_5{}^2 + 2Sur_5 - S)}$,

$D_4 = \frac{Ha^2 B_4 + B_{24} + SuK_1 B_{14}}{A(4r_6{}^2 + 2Sur_6 - S)}$, $D_5 = \frac{Ha^2 B_5 + B_{25} + SuK_1 B_{15}}{A((r_5 + r_6)^2 + Su(r_5 + r_6) - S)}$

$D_6 = \frac{Ha^2 B_6 + B_{28} + SuK_1 B_{18}}{A((r_1 + r_5)^2 + Su(r_1 + r_5) - S)}$, $D_7 = \frac{Ha^2 B_7 + B_{29} + SuK_1 B_{19}}{A((r_2 + r_5)^2 + Su(r_2 + r_5) - S)}$,

$D_8 = \frac{Ha^2B_8+B_{26}+SuK_1B_{16}}{A((r_1+r_6)^2+Su(r_1+r_6)-S)}$, $D_9 = \frac{Ha^2B_9+B_{27}+SuK_1B_{17}}{A((r_2+r_6)^2+Su(r_2+r_6)-S)}$,

$D_{10} = \frac{Ha^2B_{10}+B_{30}+SuK_1B_{20}}{A((r_1+r_2)^2+Su(r_1+r_2)-S)}$, $D_{11} = \frac{r_1D_1e^{-r_1}(e^{-r_1}-2)}{r_2e^{-r_1}-r_1e^{-r_2}}$, $D_{12} = \frac{D_2\left(r_1e^{-2r_2}-2r_2e^{-r_1}\right)}{r_2e^{-r_1}-r_1e^{-r_2}}$,

$D_{13} = \frac{D_3\left(r_1e^{-2r_5}-2r_5e^{-r_1}\right)}{r_2e^{-r_1}-r_1e^{-r_2}}$, $D_{14} = \frac{D_4\left(r_1e^{-2r_6}-2r_6e^{-r_1}\right)}{r_2e^{-r_1}-r_1e^{-r_2}}$,

$D_{15} = \frac{D_5\left(r_1e^{-(r_5+r_6)}-(r_5+r_6)e^{-r_1}\right)}{r_2e^{-r_1}-r_1e^{-r_2}}$, $D_{16} = \frac{D_6\left(r_1e^{-(r_5+r_1)}-(r_5+r_1)e^{-r_1}\right)}{r_2e^{-r_1}-r_1e^{-r_2}}$,

$D_{17} = \frac{D_7\left(r_1e^{-(r_5+r_2)}-(r_5+r_2)e^{-r_1}\right)}{r_2e^{-r_1}-r_1e^{-r_2}}$, $D_{18} = \frac{D_8\left(r_1e^{-(r_1+r_6)}-(r_1+r_6)e^{-r_1}\right)}{r_2e^{-r_1}-r_1e^{-r_2}}$,

$D_{19} = \frac{D_9\left(r_1e^{-(r_2+r_6)}-(r_2+r_6)e^{-r_1}\right)}{r_2e^{-r_1}-r_1e^{-r_2}}$, $D_{20} = \frac{D_{10}\left(r_1e^{-(r_1+r_2)}-(r_1+r_2)e^{-r_1}\right)}{r_2e^{-r_1}-r_1e^{-r_2}}$,

$D_{21} = \frac{D_1\left(r_2e^{-2r_1}-r_1e^{-r_2}\right)}{r_1e^{-r_2}-r_2e^{-r_1}}$, $D_{22} = \frac{D_2r_2e^{-r_2}(e^{-r_2}-2)}{r_1e^{-r_2}-r_2e^{-r_1}}$,

$D_{23} = \frac{D_3\left(r_2e^{-2r_5}-2r_5e^{-r_2}\right)}{r_1e^{-r_2}-r_2e^{-r_1}}$, $D_{24} = \frac{D_4\left(r_2e^{-2r_6}-2r_6e^{-r_2}\right)}{r_1e^{-r_2}-r_2e^{-r_1}}$,

$D_{25} = \frac{D_5\left(r_2e^{-(r_5+r_6)}-(r_5+r_6)e^{-r_2}\right)}{r_1e^{-r_2}-r_2e^{-r_1}}$,

$D_{26} = \frac{D_6\left(r_2e^{-(r_5+r_1)}-(r_5+r_1)e^{-r_2}\right)}{r_1e^{-r_2}-r_2e^{-r_1}}$,

$D_{27} = \frac{D_7\left(r_2e^{-(r_2+r_5)}-(r_2+r_5)e^{-r_2}\right)}{r_1e^{-r_2}-r_2e^{-r_1}}$, $D_{28} = \frac{D_8\left(r_2e^{-(r_1+r_6)}-(r_1+r_6)e^{-r_2}\right)}{r_1e^{-r_2}-r_2e^{-r_1}}$,

$D_{29} = \frac{D_9\left(r_2e^{-(r_2+r_6)}-(r_2+r_6)e^{-r_2}\right)}{r_1e^{-r_2}-r_2e^{-r_1}}$, $D_{30} = \frac{D_{10}\left(r_2e^{-(r_1+r_2)}-(r_1+r_2)e^{-r_2}\right)}{r_1e^{-r_2}-r_2e^{-r_1}}$,

$E_1 = -\frac{SuK_1Q_{33}}{A(4r_3{}^2+2Sur_3-(S+n))}$, $E_2 = -\frac{SuK_1Q_{34}}{A(4r_4{}^2+2Sur_4-(S+n))}$,

$E_3 = -\frac{SuK_1Q_{35}}{A\left(4r_7{}^2+2Sur_7-(S+n)\right)}$, $E_4 = -\frac{SuK_1Q_{36}}{A\left(4r_8{}^2+2Sur_8-(S+n)\right)}$,

$E_5 = -\frac{2\left(Ha^2Q_1+Q_{17}\right)}{A((r_5+r_7)^2+Su(r_5+r_7)-(S+n))}$, $E_6 = -\frac{2\left(Ha^2Q_2+Q_{18}\right)}{A((r_5+r_8)^2+Su(r_5+r_8)-(S+n))}$,

$E_7 = -\frac{2Ha^2Q_3}{A((r_2+r_5)^2+Su(r_2+r_5)-(S+n))}$, $E_8 = -\frac{2\left(Ha^2Q_4+Q_{20}\right)}{A((r_4+r_5)^2+Su(r_4+r_5)-(S+n))}$,

$E_9 = -\frac{2\left(Ha^2Q_6+Q_{22}\right)}{A((r_6+r_8)^2+Su(r_6+r_8)-(S+n))}$, $E_{10} = -\frac{2\left(Ha^2Q_7+Q_{23}\right)}{A((r_3+r_6)^2+Su(r_3+r_6)-(S+n))}$,

$E_{11} = -\frac{2\left(Ha^2Q_8+Q_{24}\right)}{A((r_4+r_6)^2+Su(r_4+r_6)-(S+n))}$, $E_{12} = -\frac{2\left(Ha^2Q_9+Q_{25}\right)}{A((r_1+r_7)^2+Su(r_1+r_7)-(S+n))}$,

$E_{13} = -\frac{2\left(Ha^2Q_{10}+Q_{26}\right)}{A((r_1+r_8)^2+Su(r_1+r_8)-(S+n))}$, $E_{14} = -\frac{2\left(Ha^2Q_{11}+Q_{27}\right)}{A((r_1+r_3)^2+Su(r_1+r_3)-(S+n))}$,

$E_{15} = -\frac{2(Ha^2Q_{12}+Q_{28})}{A((r_1+r_4)^2+Su(r_1+r_5)-(S+n))}$, $E_{16} = -\frac{2(Ha^2Q_{13}+Q_{29})}{A((r_2+r_7)^2+Su(r_2+r_7)-(S+n))}$,

$E_{17} = -\frac{2(Ha^2Q_{14}+Q_{30})}{A((r_2+r_8)^2+Su(r_2+r_8)-(S+n))}$, $E_{18} = -\frac{2(Ha^2Q_{15}+Q_{31})}{A((r_2+r_3)^2+Su(r_2+r_3)-(S+n))}$,

$E_{19} = -\frac{2(Ha^2Q_{16}+Q_{32})}{A((r_2+r_4)^2+Su(r_2+r_4)-(S+n))}$, $E_{20} = -\frac{SuK_1Q_{37}}{A((r_7+r_8)^2+Su(r_7+r_8)-(S+n))}$,

$E_{21} = -\frac{SuK_1Q_{38}}{A((r_3+r_7)^2+Su(r_3+r_7)-(S+n))}$, $E_{22} = -\frac{SuK_1Q_{39}}{A((r_4+r_7)^2+Su(r_4+r_7)-(S+n))}$,

$E_{23} = -\frac{SuK_1Q_{40}}{A((r_3+r_8)^2+Su(r_3+r_8)-(S+n))}$, $E_{24} = -\frac{SuK_1Q_{41}}{A((r_3+r_4)^2+Su(r_3+r_4)-(S+n))}$,

$E_{25} = -\frac{SuK_1Q_{42}}{A((r_4+r_8)^2+Su(r_4+r_8)-(S+n))}$,

$Q_1 = (A_5 + A_7 - A_8)(1 - A_{13} + A_{14} - A_{15})$, $Q_2 = (A_5 + A_7 - A_8)(A_{13} - A_{16} + A_{17})$, $Q_3 = A_{18}(A_5 + A_7 - A_8)$, $Q_4 = A_{19}(A_5 + A_7 - A_8)$, $Q_5 = -(A_6 - A_9 + A_{10})(1 - A_{13} + A_{14} - A_{15})$, $Q_6 = -(A_6 - A_9 + A_{10})(A_{13} - A_{16} + A_{17})$, $Q_7 = -A_{18}(A_6 - A_9 + A_{10})$, $Q_8 = -A_{19}(A_6 - A_9 + A_{10})$, $Q_9 = -A_{11}(1 - A_{13} + A_{14} - A_{15})$, $Q_{10} = -A_{11}(A_{13} - A_{16} + A_{17})$, $Q_{11} = -A_{11}A_{18}$, $Q_{12} = -A_{11}A_{19}$, $Q_{13} = -A_{12}(1 - A_{13} + A_{14} - A_{15})$, $Q_{14} = -A_{12}(A_{13} - A_{16} + A_{17})$, $Q_{15} = A_{12}A_{18}$, $Q_{16} = A_{12}A_{19}$, $Q_{17} = r_5r_7(A_5 + A_7 + -A_8)(1 - A_{13} + A_{14} - A_{15})$, $Q_{18} = r_5r_8(A_5 + A_7 - A_{10})(A_{13} - A_{16} + A_{17})$, $Q_{18} = r_5r_8(A_5 + A_7 - A_{10})(A_{13} - A_{16} + A_{17})$, $Q_{20} = r_4A_{19}r_5$, $Q_{21} = -r_6r_7(A_6\_A_9 + -A_8)(1 - A_{13} + A_{14} - A_{15})$, $Q_{22} = -r_6r_8(A_6\_A_9 + -A_{10})(A_{13} - A_{16} + A_{17})$, $Q_{23} = -r_3A_{18}r_6(A_6 - A_9 + A_{10})$,
$Q_{24} = -r_4r_6A_{19}(A_6 - A_9 + A_{10})$, $Q_{25} = -r_3rA_{11}r_7(1 - A_{13} + A_{14} - A_{15})$,
$Q_{26} = -r_1A_{11}r_8(A_{13} - A_{16} + A_{17})$, $Q_{27} = -A_{11}A_{18}r_1r_3$, $Q_{28} = -A_{11}A_{19}r_1r_4$,
$Q_{29} = A_{12}r_2r_7$, $Q_{30} = -A_{12}r_8r_2(A_{13} - A_{16} + A_{17})$, $Q_{31} = r_2A_{18}A_{12}r_3$, $Q_{32} = A_{19}A_{12}$,
$Q_{33} = -{A_{18}}^2{r_3}^3$, $Q_{34} = -{A_{19}}^2{r_4}^3$, $Q_{35} = -{A_7}^2(1 - A_{13} + A_{14} - A_{15})^2$, $Q_{36} = -{A_8}^2(A_{13} - A_{16} + A_{17})^2$, $Q_{37} = -r_7r_8(r_7 + r_8)(A_{13} - A_{16} + A_{17})(1 - A_{13} + A_{14} - A_{15})$, $Q_{38} = -A_{18}r_3r_7(1 - A_{13} + A_{14} - A_{15})$, $Q_{39} = -A_{19}r_4r_7(r_4 + r_7)(1 - A_{13} + A_{14} - A_{18})$, $Q_{40} = -A_{18}r_3r_8(r_3 + r_8)(A_{13} - A_{16} + A_{17})$, $Q_{41} = -A_{18}A_{19}r_3r_4$,

$Q_{42} = -A_{19}r_4r_8(r_4 + r_8)(A_{13} - A_{16} + A_{17})$,

$Z_1 = \frac{E_1(r_4e^{-2r_3}-2r_3e^{-r_4})}{r_3e^{-r_4}-r_4e^{-r_3}}$, $Z_2 = \frac{E_2r_4e^{-r_4}(r_4e^{-r_4}-2)}{r_3e^{-r_4}-r_4e^{-r_3}}$, $Z_3 = \frac{E_3(r_4e^{-2r_7}-r_7e^{-r_4})}{r_3e^{-r_4}-r_4e^{-r_3}}$,

$Z_4 = \frac{E_4(r_4e^{-2r_8}-r_8e^{-r_4})}{r_3e^{-r_4}-r_4e^{-r_3}}$, $Z_5 = \frac{E_5(r_4e^{-(r_3+r_7)}-(r_5+r_7)e^{-r_4})}{r_3e^{-r_4}-r_4e^{-r_3}}$,

$$Z_6 = \frac{E_6\left(r_4 e^{-(r_5+r_8)} - (r_5+r_8)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},\ Z_7 = \frac{E_7\left(r_4 e^{-(r_2+r_5)} - (r_2+r_5)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},$$

$$Z_8 = \frac{E_8\left(r_4 e^{-(r_4+r_5)} - (r_4+r_5)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},\ Z_9 = \frac{E_9\left(r_4 e^{-(r_6+r_8)} - (r_6+r_8)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},$$

$$Z_{10} = \frac{E_{10}\left(r_4 e^{-(r_3+r_6)} - (r_3+r_6)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},\ Z_{11} = \frac{E_{11}\left(r_4 e^{-(r_4+r_6)} - (r_4+r_6)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},$$

$$Z_{12} = \frac{E_{12}\left(r_4 e^{-(r_1+r_7)} - (r_1+r_7)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},\ Z_{13} = \frac{E_{13}\left(r_4 e^{-(r_1+r_8)} - (r_1+r_8)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},$$

$$Z_{14} = \frac{E_5\left(r_4 e^{-(r_3+r_1)} - (r_3+r_1)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},\ Z_{15} = \frac{E_{15}\left(r_4 e^{-(r_1+r_4)} - (r_1+r_4)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},$$

$$Z_{16} = \frac{E_{16}\left(r_4 e^{-(r_2+r_7)} - (r\quad +r_7)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},\ Z_{17} = \frac{E_{17}\left(r_4 e^{-(r_2+r_8)} - (r_2+r_8)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},$$

$$Z_{18} = \frac{E_{18}\left(r_4 e^{-(r_2+r_3)} - (r_2+r_3)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},\ Z_{19} = \frac{E_{19}\left(r_4 e^{-(r_2+r_4)} - (r_2+r_4)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},$$

$$Z_{20} = \frac{E_{20}\left(r_4 e^{-(r_8+r_7)} - (r_8+r_7)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},\ Z_{21} = \frac{E_{21}\left(r_4 e^{-(r_3+r_7)} - (r_3+r_7)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},$$

$$Z_{22} = \frac{E_{22}\left(r_4 e^{-(r_4+r_7)} - (r_4+r_7)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},\ Z_{23} = \frac{E_{23}\left(r_4 e^{-(r_3+r_6)} - (r_3+r_6)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},$$

$$Z_{24} = \frac{E_{24}\left(r_4 e^{-(r_3+r_4)} - (r_3+r_4)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},\ Z_{25} = \frac{E_{25}\left(r_4 e^{-(r_4+r_8)} - (r_4+r_8)e^{-r_4}\right)}{r_3 e^{-r_4} - r_4 e^{-r_3}},$$

$$Z_{26} = \frac{E_1 r_3 e^{-r_3}(r_4 e^{-r_3} - 2)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{27} = \frac{E_2\left(r_3 e^{-2r_4} - 2r_4 e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{28} = \frac{E_3\left(r_3 e^{-2r_7} - 2r_7 e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{29} = \frac{E_4\left(r_3 e^{-2r_8} - 2r_8 e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{30} = \frac{E_5\left(r_3 e^{-(r_5+r_7)} - (r_5+r_7)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{31} = \frac{E_6\left(r_3 e^{-(r_3+r_8)} - (r_3+r_8)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{32} = \frac{E_7\left(r_3 e^{-(r_2+r_5)} - (r_5+r_2)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{33} = \frac{E_8\left(r_3 e^{-(r_4+r_5)} - (r_4+r_5)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{34} = \frac{E_9\left(r_3 e^{-(r_6+r_8)} - (r_6+r_8)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{35} = \frac{E_{10}\left(r_3 e^{-(r_3+r_6)} - (r_3+r_6)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{36} = \frac{E_{11}\left(r_3 e^{-(r_4+r_6)} - (r_4+r_6)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{37} = \frac{E_{12}\left(r_3 e^{-(r_1+r_7)} - (r_1+r_7)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{38} = \frac{E_{13}\left(r_3 e^{-(r_1+r_8)} - (r_1+r_8)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{39} = \frac{E_{14}\left(r_3 e^{-(r_1+r_3)} - (r_1+r_3)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{40} = \frac{E_{15}\left(r_3 e^{-(r_1+r_4)} - (r_1+r_4)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{41} = \frac{E_{16}\left(r_3 e^{-(r_2+r_7)} - (r_2+r_7)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{42} = \frac{E_{17}\left(r_3 e^{-(r_2+r_8)} - (r_2+r_8)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{43} = \frac{E_{18}\left(r_3 e^{-(r_2+r_3)} - (r_2+r_3)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{44} = \frac{E_{19}\left(r_3 e^{-(r_2+r_4)} - (r\quad +r_4)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{45} = \frac{E_{20}\left(r_3 e^{-(r_7+r_8)} - (r_7+r_8)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{46} = \frac{E_{21}\left(r_3 e^{-(r_3+r_7)} - (r_3+r_7)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{47} = \frac{E_{22}\left(r_3 e^{-(r_4+r_7)} - (r_4+r_7)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{48} = \frac{E_{23}\left(r_3 e^{-(r_3+r_6)} - (r_3+r_6)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},$$

$$Z_{49} = \frac{E_{24}\left(r_3 e^{-(r_3+r_4)} - (r_3+r_4)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}},\ Z_{50} = \frac{E_{25}\left(r_3 e^{-(r_4+r_8)} - (r_4+r_8)e^{-r_3}\right)}{r_4 e^{-r_3} - r_3 e^{-r_4}}.$$