JSR Publications

Available Online

J. Sci. Res. **4** (2), 327-336 (2012)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

# Some Remarks on Fuzzy $R_0$, $R_1$ and *Regular* Topological Spaces

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Received 12 December 2010, accepted in revised form 18 February 2012

**Abstract**

In this paper, five *regular*-axioms, eighteen $R_1$-axioms and nine $R_0$-axioms for fuzzy topological spaces are recalled. A complete answer is given with regard to all possible $(R_1 \Rightarrow R_0)$-type implications for fuzzy topological spaces. It is also shown that, though the *regular*-axiom implies $R_1$-axiom in 'general topological spaces', this is not true for 'fuzzy topological spaces', in general.

*Keywords*: Fuzzy Topological Space; Fuzzy $R_1$-axiom; Fuzzy $R_0$-axiom; Fuzzy *regular* axiom.



doi: http://dx.doi.org/10.3329/jsr.v4i2.6663 J. Sci. Res. **4** (2), 327-336 (2012)

## 1. Introduction

In 1965, Zadeh [1] defined fuzzy sets with a view to study and formulate mathematically those situations which are imprecise and vaguely defined. Since then, fuzzy set theory has been developed in many directions by many scholars. Chang [2] gave the concept of 'fuzzy topology'. He did the 'fuzzification' of topology by replacing 'subsets' in the definition of topology by 'fuzzy sets'. In 1976, Lowen [3] gave a modified definition of 'fuzzy topology'. Hutton and Reilly [4] introduced the concept of fuzzy $R_0$ and $R_1$ axioms. These studies were further carried out by many researchers [5-13]. In this paper we recall nine $R_0$-axioms from [9], eighteen $R_1$-axioms from [11] and five *regular* axioms from [7, 8] for fuzzy topological spaces (fts, in short). In analogy with the well known topological properties like $(regular \Rightarrow R_1)$ and $(R_1 \Rightarrow R_0)$, we study these types of properties for fts. We give a complete answer with regard to all possible $(R_1 \Rightarrow R_0)$-type implications for fts. It is also shown that, the property $(R_0 \not\Rightarrow R_1)$ is also true for fts; however, the property $(regular \Rightarrow R_1)$ is not true for fts, in general.

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### 1.1 *Preliminaries*

In this section, we recall some definitions on fuzzy sets and fts which will be needed in the sequel.

**Definition-1.1.1.** [1]: Let $X$ be a non-empty set and $I$ the unit closed interval [0, 1]. A fuzzy set is a function $u$: $X \to I$, $\forall\ x \in X$ ; $u(x)$ denotes a degree or the grade of membership of $x$. The set of all fuzzy sets in $X$ is denoted by $I^X$. Ordinary subsets of $X$ (crisp sets) are also considered as the members of $I^X$ which take the values 0 and 1 only. A crisp set which always takes the value 0 is denoted by 0; similarly a crisp set which always takes the value 1 is denoted by 1.

**Definition-1.1.2.** [10]: Let $u$: $X \to I$. Then the set $\{x \in X: u(x) > 0\}$ is called the support of $u$ and is denoted by $u_0$ or supp($u$). Let $A \subseteq X$, then by $1_A$ we denote the characteristic function $A$. The characteristic function of a singleton set $\{x\}$ is denoted by $1_x$.

**Definition-1.1.3.** [10]: Let $u$ be a fuzzy set in $X$. Then by $u^c$, we denote the complement of $u$ which is defined as $u^c(x) = 1 - u(x)\ \forall\ x \in X$.

**Definition-1.1.4.** [1]: Let $u$ and $v$ be two fuzzy sets in $X$. We define

(i) $u = v$ if and only if $u(x) = v(x)\ \forall\ x \in X$.
(ii) $u \subseteq v$ if and only if $u(x) \le v(x)\ \forall\ x \in X$.
(iii) $(u \vee v)(x) = \max\{u(x), v(x)\}\ \forall\ x \in X$.
(iv) $(u \wedge v)(x) = \min\{u(x), v(x)\}\ \forall\ x \in X$.

**Definition-1.1.5.** [1]: For a family of fuzzy sets $\{u_i : i \in J\}$ in $X$. We define

(i) $\bigcup_{i \in J} u_i(x) = \sup\{u_i(x)\}\ \forall\ x \in X$. (ii) $\bigcap_{i \in J} u_i(x) = \inf\{u_i(x)\}\ \forall\ x \in X$.

**Definition-1.1.6.** [14]: A fuzzy point $x_\alpha$ in $X$ is a special type of fuzzy set in $X$ with the membership function $x_\alpha(x) = \alpha$ and $x_\alpha(y) = 0$ if $x \neq y$, where $0 < \alpha < 1$ and $x, y \in X$. The fuzzy point $x_\alpha$ is said to have support $x$ and value $\alpha$. We also write this as $\alpha 1_x$.

**Definition-1.1.7.** [14]: Let $\alpha 1_x$ be a fuzzy point in $X$ and $u \in I^X$. Then $\alpha 1_x \in u$ if and only if $\alpha \le u(x)$.

**Definition-1.1.8.** [10]: Let $f$: $X \to Y$ be a mapping and $u \in I^X$. Then the image $f(u)$ is a fuzzy set in $Y$ which is defined as

$$f(u)(y)=\begin{cases}\sup\{u(x):\ f(x)=y\} & \text{if } \ f^{-1}(y)\neq\varnothing\\ 0 \ \text{ if } \ f^{-1}(y)=\varnothing\end{cases}$$

**Definition-1.1.9.** [10]: Let *f*: $X \to Y$ be a mapping and *u* be a fuzzy set in *Y*. Then the inverse image $f^{-1}(u)$ is a fuzzy set in *X* which is defined by $f^{-1}(u)(x)=u(f(x)) \ \ \forall \ \ x\in X$.

**Definition-1.1.10.** [2]: Chang [2] defined an fts as follows:

Let *X* be a set. A class *t* of fuzzy sets in *X* is called a fuzzy topology on *X* if *t* satisfies the following conditions:

(i) $0, 1 \in t$,
(ii) if $u, v \in t$ then $u \wedge v \in t$ and
(iii) if $\{u_i : i\in K\}$ is a family of fuzzy sets in *t*, then $\underset{i\in K}{\vee}(u_i)\in t$ .

The pair (*X*, *t*) is then called an fts. The members of *t* are called *t*-open sets (or open sets) and their complements are called *t*-closed set (or closed sets).

**Definition-1.1.11.** [3]: Lowen [3] modified the definition of an fts defined by Chang [2] by adding another condition. In the sense of R. Lowen [3], the definition of an fts is as follows:

Let *X* be a set and *t* a family of fuzzy sets in *X*. Then *t* is called a fuzzy topology of *X* if the following conditions hold:

(i) $0, 1 \in t$,
(ii) if $u, v \in t$ then $u \wedge v \in t$,
(iii) if $\{u_i : i\in K\}$ is a family of fuzzy sets in *t*, then $\underset{i\in K}{\vee}(u_i)\in t$ and
(iv) *t* contains all constant fuzzy sets in X.

The pair (*X*, *t*) is called an fts. Throughout this work, we use the concept of fts due to Lowen [3].

**Definition-1.1.12.** [10]: Let *u* be a fuzzy set in an fts (*X*, *t*). Then the fuzzy closure $\overline{u}$ and the fuzzy interior $u^o$ of *u* are defined as follows: $\overline{u}=\inf\left\{\lambda : u\le\lambda \ \ and \ \ \lambda\in t^c\right\}$, $u^o=\sup\{\lambda : \lambda\le u \ \ and \ \ \lambda\in t\}$.

**Definition-1.1.13.** [2]: Let $f:(X,\ t)\to(Y,\ s)$ be a mapping between fts. Then *f* is called

(i) fuzzy continuous if and only if $f^{-1}(u)\in t$ for each $u\in s$.
(ii) fuzzy open if and only if $f(u)\in s$ for each $u\in t$ .

(iii) fuzzy closed if and only if $f(u) \in s^c$ for each $u \in t^c$.

## 2. Fuzzy $R_0$ topological spaces

In this section, we recall nine $R_0$ -axioms of fts from [9].

**Definitions-2.1.** [9]: We define, for fts $(X, t)$, $R_0$-axioms as follows:

$R_0^1$ : For every pair $x, y \in X$, $x \neq y$, $\overline{1_y}(x) = 0 \Rightarrow \overline{1_x}(y) = 0$

$R_0^2$ : For every pair

$x, y \in X$, $x \neq y$, $\left(\forall \alpha \in I_0, \overline{\alpha 1_x}(y) = \alpha\right) \Leftrightarrow \left(\overline{\beta 1_y}(x) = \beta, \forall \beta \in I_0\right)$

$R_0^3$ : $\forall \lambda \in t$, $\forall x \in X$ and $\forall \alpha < \lambda(x)$, $\overline{\alpha 1_x} \leq \lambda$

$R_0^4$ : $\forall \lambda \in t$, $\forall x \in X$ and $\forall \alpha \leq \lambda(x)$, $\overline{\alpha 1_x} \leq \lambda$

$R_0^5$ : For every pair $x, y \in X$, $x \neq y$, $\overline{1_x}(y) = 1 \Rightarrow \overline{1_y}(x) = 1$

$R_0^6$ : For every pair $x, y \in X$, $x \neq y$, $\overline{1_x}(y) = \overline{1_y}(x)$

$R_0^7$ : For every pair $x, y \in X$, $x \neq y$, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0, 1\}$

$R_0^8$ : For every pair $x, y \in X$, $x \neq y$ and $\forall \alpha \in I_0$, $\overline{\alpha 1_x}(y) = \alpha \Rightarrow \overline{\alpha 1_y}(x) = \alpha$

$R_0^9$ : For every pair $x, y \in X$, $x \neq y$ and $\forall \alpha \in I_0$, $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$

**Theorem-2.1** [9]: The accompanying diagram (Fig. 1) illustrates the interrelations among the $R_0$ -properties mentioned in the section 2:

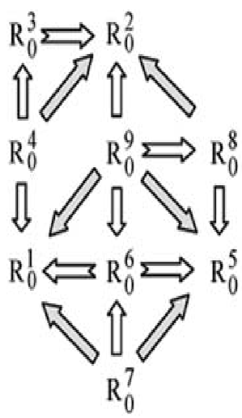


Fig. 1. Interrelations among the $R_0$ -properties [9].

For proof see [9]. □

## 3. Fuzzy $R_1$-topological spaces

In this section, we recall eighteen definitions of fuzzy $R_1$-topological spaces from [11].

**Definitions-3.1** [11]**:** An fts $(X, t)$ is said to have the property

1. **P1,** if $\forall\ x, y \in X, x \neq y$, $\exists\ w \in t$ such that $w(x) \neq w(y)$.
2. **P2,** if $\forall\ x, y \in X, x \neq y$, $\exists\ w \in t$ such that either $w(x) = 0 < w(y)$ or $w(x) > 0 = w(y)$.
3. **P3,** if $\forall\ x, y \in X, x \neq y$, $\exists\ w \in t$ such that either $w(x) = 1,\ w(y) = 0$ or $w(x) = 0,$ $w(y) = 1$.
4. **Q1**, if $\forall\ x, y \in X, x \neq y$, $\exists\ u, v \in t$ such that $\overline{1_x} \leq u, \overline{1_y} \leq v$ and $u \wedge v = 0$.
5. **Q2**, if $\forall\ x, y \in X, x \neq y$, $\exists\ u, v \in t$ such that $\overline{1_x} \leq u, \overline{1_y} \leq v$ and $u \leq 1 - v$.
6. **Q3**, if $\forall\ x, y \in X, x \neq y$, $\exists\ u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \wedge v = 0$.
7. **Q4**, if $\forall\ x, y \in X, x \neq y$, $\exists\ u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \leq 1 - v$.
8. **Q5**, if $\forall\ x, y \in X, x \neq y$ and $\forall\ \alpha, \beta \in I_{0,1}$, $\exists\ u, v \in t$ such that $u(x) > \alpha$ and $v(y) > \beta$ and $u \wedge v = 0$.
9. **Q6**, if $\forall\ x, y \in X, x \neq y$, $\exists\ u, v \in t$ such that $u(x) > 0$, $v(y) > 0$ and $u \wedge v = 0$.

**Definitions-3.2** [11]: An fts $(X, t)$ is called an

1. $FR_1(i)$-fts, if $(X, t)$ has **P1** $\Rightarrow (X, t)$ has **Q1.**
2. $FR_1(ii)$-fts, if $(X, t)$ has **P1** $\Rightarrow (X, t)$ has **Q2.**
3. $FR_1(iii)$-fts, if $(X, t)$ has **P1** $\Rightarrow (X, t)$ has **Q3.**
4. $FR_1(iv)$-fts, if $(X, t)$ has **P1** $\Rightarrow (X, t)$ has **Q4.**
5. $FR_1(v)$-fts, if $(X, t)$ has **P1** $\Rightarrow (X, t)$ has **Q5.**
6. $FR_1(vi)$-fts, if $(X, t)$ has **P1** $\Rightarrow (X, t)$ has **Q6.**
7. $FR_1(vii)$-fts, if $(X, t)$ has **P2** $\Rightarrow (X, t)$ has **Q1.**
8. $FR_1(viii)$-fts, if $(X, t)$ has **P2** $\Rightarrow (X, t)$ has **Q2.**
9. $FR_1(ix)$-fts, if $(X, t)$ has **P2** $\Rightarrow (X, t)$ has **Q3.**
10. $FR_1(x)$-fts, if $(X, t)$ has **P2** $\Rightarrow (X, t)$ has **Q4.**
11. $FR_1(xi)$-fts, if $(X, t)$ has **P2** $\Rightarrow (X, t)$ has **Q5.**
12. $FR_1(xii)$-fts, if $(X, t)$ has **P2** $\Rightarrow (X, t)$ has **Q6.**
13. $FR_1(xiii)$-fts, if $(X, t)$ has **P3** $\Rightarrow (X, t)$ has **Q1.**
14. $FR_1(xiv)$-fts, if $(X, t)$ has **P3** $\Rightarrow (X, t)$ has **Q2.**
15. $FR_1(xv)$-fts, if $(X, t)$ has **P3** $\Rightarrow (X, t)$ has **Q3.**

16. $FR_1(xvi)$-fts, if $(X, t)$ has **P3** $\Rightarrow$ $(X, t)$ has **Q4.**
17. $FR_1(xvii)$-fts, if $(X, t)$ has **P3** $\Rightarrow$ $(X, t)$ has **Q5.**
18. $FR_1(xviii)$-fts, if $(X, t)$ has **P3** $\Rightarrow$ $(X, t)$ has **Q6.**

**Theorem-3.3** [11]: The accompanying diagram (Fig. 2) illustrates the interrelations among the $FR_1$-propertits mentioned in Section 3:

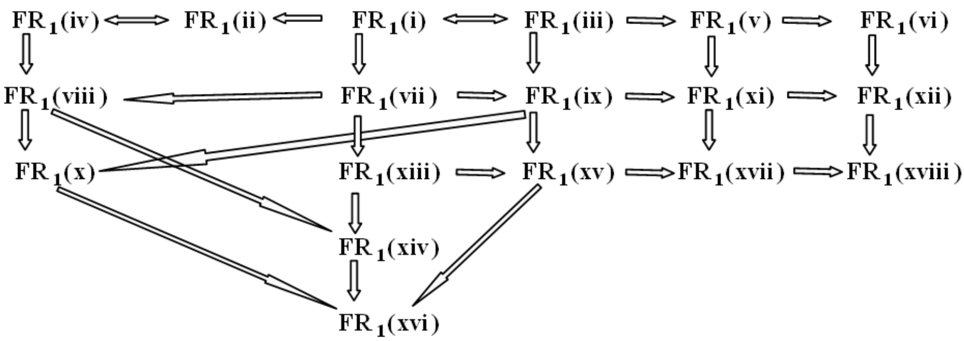


Fig. 2. Interrelations among the $R_1$ -properties [11].

For proof see [11]. □

## 4. Relations between fuzzy $R_0$ and $R_1$-axioms

In this section, we give a complete answer with regard to all possible $(R_1 \Rightarrow R_0)$-type implications for fts.

**Theorem-4.1:** The following relations hold between the fuzzy $R_0$ -axioms and fuzzy $R_1$ -axioms:

(a) $FR_1(xvi) \Rightarrow R_0^1$, and so $FR_1(k) \Rightarrow R_0^1$, where $k \in \{i - iv, vii - x, xiii - xvi\}$.

(b) $FR_1(xiii) \not\Rightarrow R_0^5$, and so $FR_1(k) \not\Rightarrow R_0^m$, where $k \in \{xiii, xiv, ...., xviii\}$ and $m \in \{5, 6, ...., 9\}$.

(c) $FR_1(v) \Rightarrow R_0^8$, and so $FR_1(k) \Rightarrow R_0^m$ where $k \in \{i, iii, v\}$ and $m \in \{2,5,8\}$.

(d) $FR_1(vi) \Rightarrow R_0^2$, and so $FR_1(k) \Rightarrow R_0^2$ where $k \in \{i, iii, v, vi\}$.

(e) $FR_1(vi) \not\Rightarrow R_0^8$, and so $FR_1(k) \not\Rightarrow R_0^m$, where $k \in \{vi, xii, xviii\}$ and $m \in \{8, 9\}$.

(f) $FR_1(vi) \not\Rightarrow R_0^3$, and so $FR_1(k) \not\Rightarrow R_0^m$, where $k \in \{vi, xii, xviii\}$ and $m \in \{3, 4\}$.

(g) $FR_1(iv) \Rightarrow R_0^4$, and so $FR_1(k) \Rightarrow R_0^m$ where $k \in \{i - iv\}$ and $m \in \{1, 2, 3, 4\}$.

(h) $R_0^m \not\Rightarrow FR_1(k)$, where $k \in \{i, ii, \ldots\ldots, xviii\}$ and $m \in \{1, 2, \ldots\ldots, 9\}$.

**Proof (a):** Let (*X*, *t*) be an $FR_1(xvi)$-fts and $x, y \in X$, $x \neq y$ such that $\overline{1_y}(x) = 0$. Therefore, $\exists\ \lambda \in t^c$ such that $\lambda(y) = 1$ and $\lambda(x) = 0$. Take $w = 1 - \lambda$. Now $w \in t$ such that $w(x) = 1$ and $w(y) = 0$. Since, (*X*, *t*) is an $FR_1(xvi)$-fts, $\exists\ u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \leq 1 - v$. Put, $\kappa = 1 - v \in t^c$. Now $\kappa(y) = 0$ and $\kappa(x) = 1$. Consequently, $\overline{1_x}(y) = 0$ .Hence (X, t) is $R_0^1$. □

**Proof (b):**
**Example-1:** Consider a fuzzy topological space (*X*, *t*), where *X* ={*x*, *y*}, *u*(*x*) = 0.5, *u*(*y*) = 0 and *t* = < {*u*} ∪ {constants} >. Clearly, (*X*, *t*) is $FR_1(xiii)$ but it is not $R_0^5$. For $\overline{1_x}(y) = 1$ but $\overline{1_y}(x) < 1$. □

**Proof (c):** Let (*X*, *t*) be an $FR_1(v)$-fts. Let $x, y \in X$, $x \neq y$, $\alpha \in I_0$ such that $\overline{\alpha 1_x}(y) < \alpha$. This implies that there exists $m \in t^c$ such that *m*(*x*) = α and *m*(*y*) < α. Let *w* = 1 – *m* ∈ *t*. Then *w*(*x*) ≠ *w*(*y*). Since (*X*, *t*) is an $FR_1(v)$-fts, there exist $u, v \in t$ such that $u(x) > \alpha_1$, $v(y) > \alpha_2$, and $u \wedge v = 0\ \forall\ \alpha_1, \alpha_2 \in I_{0,1}$. Choose $\alpha_1, \alpha_2$ in such a way that $\alpha = \alpha_2$ and $\alpha_1 > 1 - \alpha$. Now $\alpha 1_y < v \leq 1 - u$. Therefore, $\overline{\alpha 1_y} \leq \overline{1 - u} = 1 - u$ and so $\overline{\alpha 1_y}(x) \leq 1 - u(x) < 1 - \alpha_1 < \alpha$. Hence, (*X*, *t*) is $R_0^8$. [Note [9]:
$(\forall \alpha \in I_0,\ \overline{\alpha 1_x}(y) = \alpha \Rightarrow \overline{\alpha 1_y}(x) = \alpha) \Leftrightarrow (\forall \alpha \in I_0,\ \overline{\alpha 1_x}(y) < \alpha \Rightarrow \overline{\alpha 1_y}(x) < \alpha)$] □

**Proof (d):** Let (*X*, *t*) be an $FR_1(vi)$-fts. Let $x, y \in X$, $x \neq y$ and $w \in t$ such that *w*(*x*) > *w*(*y*). Then, by $FR_1(vi)$ there exist $u, v \in t$ such that *u*(*x*) > 0, *v*(*y*) > 0 and $u \wedge v = 0$. Clearly, *v*(*y*) > *v*(*x*). Hence, (*X*, *t*) is $R_0^2$. [Note [9]: {An fts $(X, t)$ is $R_0^2$ } ⇔ { ∀ *x*, *y*∈ *X*, *x* ≠ *y*, if ∃ a *t*-open set λ such that λ(y) < λ(x) then ∃ a t-open set μ such that μ(x) < μ(y).} □

**Proof (e):**
**Example-2:** Consider an fts (*X*, *t*) where *X* = {*x*, *y*}, $t =< \{u_1, u_2, u_3, u_4\} > \cup \{\text{constants}\}$, $u_1(x) = u_1(y) = u_2(x) = 0.6$, $u_2(y) = 0.7$, $u_3(x) = u_4(y) = 0$, $u_3(y) = 0.8$ and $u_4(x) = 0.4$. It can be checked that (*X*, *t*) is $FR_1(vi)$. Let $m_k = 1 - u_k$, $k = 1, 2, 3, 4$. Now $m_1(x) = 0.4 = m_2(x)$, $m_3(x) = 1$, $m_4(x) = 0.6$, $m_1(y) = 0.4$, $m_2(y) = 0.3$, $m_3(y) = 0.2$

and $m_4(y) = 1$. Take $\alpha$ = 0.4. Then $\overline{\alpha 1_x}(y) = 0.2 < \alpha$. But $\overline{\alpha 1_y}(x) = 0.4 = \alpha$. Therefore, $(X, t)$ is not $R_0^8$. □

**Proof (f):**
**Example-3:** Consider an fts $(X, t)$ where $X = \{x, y\}$, $u(x) = 0.6$, $u(y) = 0 = v(x)$ and $v(y) = 0.4$. Clearly, $(X, t)$ is $FR_1(vi)$. Let $\alpha$ = 0.5. Now $\alpha < u(x)$. It can be checked that $\overline{\alpha 1_x}(y) = \alpha > u(y)$. Therefore, $\overline{\alpha 1_x}(y) \nleq u$. Hence, $(X, t)$ is not $R_0^3$. □

**Proof (g):** Let $(X, t)$ be an $FR_1(iv)$-fts. Let $x \in X$, $\lambda \in t$ and $\alpha \in I_1$ such that $\alpha \leq \lambda(x)$. Suppose $\overline{\alpha 1_x} \nleq \lambda$. This implies that there exist $y \in X$, $x \neq y$ such that $\overline{\alpha 1_x}(y) > \lambda(y)$. Thus $\lambda(x) \neq \lambda(y)$. Hence there exist $p, q \in t$ such that $p(x) = 1 = q(y)$ and $p \leq 1 - q$. Put $m = 1 - p$ and $n = 1 - q$. Now $m, n \in t^c$ such that $m(x) = 0 = n(y)$ and $m(y) = 1 = n(x)$. Therefore, $\overline{\alpha 1_x}(y) \leq \overline{1_x}(y) \leq n(y) = 0$, which is a contradiction. Therefore, $\overline{\alpha 1_x} \leq \lambda$. Hence $(X, t)$ is $R_0^4$. □

**Proof (h):**
**Example-4** [13]**:** Let $X$ be an infinite set. For $x, y \in X$, we define $U_{xy} \in I^X$ as follows:

$$U_{xy}(z) = \begin{cases} 0 & \text{if} \quad z \in \{x, y\} \\ 1 & \text{if} \quad z \notin \{x, y\} \end{cases}$$

Let $t$ be the fuzzy topology on $X$ generated by $\{U_{xy} : x, y \in X\}$. It can be checked that if $x \neq y$, $\overline{1_x}(y) = 0$. Therefore, $(X, t)$ is $R_0^4$, $R_0^7$ and $R_0^9$. But $(X, t)$ is neither $FR_1(xvi)$ nor $FR_1(xviii)$ as there exist no $u, v \in$ t such that $u \leq 1 - v$. Therefore, $(X, t)$ is not $FR_1(k)$, $k \in \{i, ii, ......., xviii\}$ □

## 5. Fuzzy *regular* axioms

In this section, we recall five definitions of fuzzy regular axioms from [7, 8], and we show that, the well known topological property $(regular \Rightarrow R_1)$ is not true, in general, for fts.

**Definition-5.1:** An fts $(X, t)$ is called

(a) $FR(i)$ if and only if $\alpha \in I_0$, $\lambda \in t^c$, $x \in X$ and $\alpha \leq 1 - \lambda(x)$ imply that there exist $u, v \in t$ such that $\alpha \leq u(x)$, $\lambda \leq v$ and $u \leq 1 - v$.
(b) $FR(ii)$ if and only if $\alpha \in I_0$, $\lambda \in t^c$, $x \in X$ and $\alpha \lesssim 1 - \lambda(x)$ imply that there exist $u, v \in t$ such that $\alpha \lesssim u(x)$, $\lambda \leq v$ and $u \leq 1 - v$.

(c) $FR(iii)$ if and only if each $u \in t$ is a supremum of $u_j, j \in J$, where $\forall\, j,\ u_j \in t$ and $\overline{u_j} \leq u$.

(d) $FR(iv)$ if and only if $\lambda \in t^c$, $x \in X$ and $\lambda(x) = 0$ imply that there exist $u, v \in t$ such that $u(x) = 1$, $\lambda \leq v$ and $u \leq 1 - v$.

(e) $FR(v)$ if and only if $\lambda \in t^c$, $x \in X$ and $1 - \lambda(x) > 0$ imply that there exist $u, v \in t$ such that $u(x) > 0$, $\lambda \leq v$ and $u \leq 1 - v$.

Note-1 [7, 8]: Let $x \in X$ and $\lambda$ be a fuzzy set in $X$. Then for $\alpha \in I_0$, "$\alpha \lesssim \lambda(x)$" means $\alpha < \lambda(x)$ if $\alpha \neq 1$ and $\lambda(x) = 1$ if $\alpha = 1$.

Note-2 [7, 8]: The following implications exist among $FR(i)$, $FR(ii)$, …., $FR(v)$:

$$\begin{array}{ccccccc} FR(i) & \Rightarrow & FR(ii) & \Rightarrow & FR(iii) & \Rightarrow & FR(v) \\ & & \Downarrow & & & & \\ & & FR(iv) & & & & \end{array}$$

For proof see [7, 8, 10]. □

**Example-5 :** Let $X = \{x, y, z\}$. For every pair $x, y \in X$ we define $U_{xy} \in I^X$ as follows: $U_{xy}(x) = 1$, $U_{xy}(y) = 0$ and $U_{xy}(z) = 0.5$. Let $t$ be the fuzzy topology on $X$ generated by $\langle U_{xy} \in I^X : x, y \in X \rangle$. Now it can be easily verified that $(X, t)$ is $FR(i)$. But $(X, t)$ is neither $FR_1(xvi)$ nor $FR_1(xviii)$, since there exist no $u, v \in t$ such that $u \wedge v = 0$. Therefore, $FR(k) \not\Rightarrow FR_1(m)$, $k \in \{i, ii, \ldots.., v\}$ and $m \in \{i, ii, \ldots.., xviii\}$. Thus we see that the property $(regular \Rightarrow R_1)$ is not true, in general, for fts. □

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