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On Some R_1-Properties in Fuzzy Topological Spaces

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Abstract

In this paper, we introduce six R_1-axioms for fuzzy topological spaces (in short, fts). We study their interrelations, goodness and initialities. Besides we recall nine R_0-axioms for fts. A complete answer is given with regard to all possible $(R_1 \Rightarrow R_0)$-type implications for fts.

Keywords: Fuzzy topological space; Fuzzy R_1-axiom; Fuzzy R_0-axiom.

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1. Introduction

The concept of fuzzy sets was first introduced, in 1965, by L. A. Zadeh in his new classical paper [1] as an attempt to mathematically handle those phenomena which are inherently vague, imprecise or fuzzy in nature. He interpreted a fuzzy set on a set X as a mapping from X into the closed unit interval $I = [0, 1]$. Various merits and applications as well as some limitations of fuzzy set theory have since been demonstrated by Zadeh and a large number of subsequent workers.

The advent of fuzzy set theory has also led to the development of some new areas of study in mathematics. It has become a concern and a new tool for the mathematicians working in many different areas of mathematics. These have been generally accomplished by replacing subsets, in various existing mathematical structures, by fuzzy sets. In 1968, Chang [2] did 'fuzzification' of topology by replacing 'subsets' in the definition of fuzzy topology by 'fuzzy sets'. Since then a large body of concepts and results have been growing in this area which has come to be known as "fuzzy topology".

A major deviation in the definition of fuzzy topology was made by Lowen [3, 4]. He gave a modified definition of fuzzy topology by including all constant fuzzy sets in a fuzzy topology.

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The concepts of R_0-type and R_1-type axioms for fts was first introduced by Hutton and Reilly [5] in 1980. In 1990, Ali *et al*. [6] introduced some other definitions of fuzzy R_0-axioms. Srivastava [7], Ali [8, 9], and Azam and Ali [10] also gave some new concepts of R_1-property in fuzzy topology.

In this paper, we introduce six new concepts of R_1-properties of fts each of which is shown as the good extension of the topological R_1-property. We study their interrelations and initialities. In addition, we recall nine concepts of R_0-properties of fts from [6]. In analogy with the well known topological property $(R_1 \Rightarrow R_0)$ we study the relations of this type for fts. It is also shown that, the property $(R_0 \not\Rightarrow R_1)$, in general, is also true for fts.

2. Preliminaries

In this section, we recall some definitions and basic results (which we label as facts) on fuzzy sets and fts. This section is considered as the base and background for the study of subsequent sections.

Definition 2.1 [1]: Let X be a non-empty set and I the unit closed interval [0, 1]. A fuzzy set on X is a function $u : X \to I$. $\forall$ x $\in$ X, u(x) denotes a degree or the grade of membership of x. The set of all fuzzy sets in X is denoted by I^X. Ordinary subsets of X (crisp sets) are also considered as the members of I^X which take the values 0 and 1 only. A crisp set which always takes the value 0 is denoted by 0, similarly a crisp set which always takes the value 1 is denoted by 1.

Definition 2.2 [9]: Let $u \in I^X$. The set $\{x \in X : u(x) > 0\}$ is called the support of u and is denoted by u_o or supp(u). By u^c, we denote the complement of u which is defined as $u^c(x) = 1 - u(x)\ \forall x \in X$.

Definition 2.3 [6]: If $A \subseteq X$, by 1_A we denote the characteristics function A. The characteristics function of a singleton set $\{x\}$ is denoted by 1_x.

Definition 2.4 [9, 11]: A *fuzzy point* x_α in X is a special type of fuzzy set in X with the membership function $x_\alpha(x) = \alpha$ and $x_\alpha(y) = 0$ if $x \neq y$, where $0 < \alpha < 1$ and $x, y \in X$. The fuzzy point x_α is said to have support x and value α. We also write this as $\alpha 1_x$.

Definition 2.5 [2]: Let $f : X \to Y$ be a mapping and u a fuzzy set in X. Then the image $f(u)$ is a fuzzy set in Y which is defined as

$$\mathrm{f}(\mathrm{u})(\mathrm{y}) = \begin{cases} \sup\{u(x) : f(x) = y\} & if \;\; f^{-1}(y) \neq \varnothing \\ 0 \;\; if \;\; f^{-1}(y) = \varnothing \end{cases}$$

Definition 2.6 [2]: Let $f : X \to Y$ be a mapping and *u* be a fuzzy set in *Y*. Then the inverse image $f^{-1}(u)$ is the fuzzy set in *X* which is defined as $f^{-1}(u)(x) = u\big(f(x)\big)$ $\forall$ *x* $\in$ *X*.

Definition 2.7 [2]: Chang C. L. defined a fuzzy topological space as follows:

Let *X* be a set. A class *t* of fuzzy sets in *X* is called a fuzzy topology on *X* if *t* satisfies the following conditions:

(i) 0, 1 $\in$ *t*,

(ii) if *u*, *v* $\in$ *t* then $u \wedge v \in t$

and (iii) if $\{u_i : i \in K\}$ is a family of fuzzy sets in t, then $\vee_{i \in K} u_i \in t$.

The pair (*X*, *t*) is then called a fuzzy topological space (in short, fts). The members of *t* are called *t*-open sets (or open sets) and their complements are called *t*-closed set (or closed sets).

Definition 2.8: **L**owen [3] modified the definition of an fts defined by Chang [2] by adding another condition. In the sense of Lowen R. the definition of an fts is as follows:

Let *X* be a set and *t* is a family of fuzzy sets in *X*. Then *t* is called a fuzzy topology on *X* if the following conditions hold:

(i) 0, 1 $\in$ *t*,

(ii) if *u*, *v* $\in$ *t* then $u \wedge v \in t$

(iii) if $\{u_i : i \in K\}$ is a family of fuzzy sets in *t*, then $\vee_{i \in K} u_i \in t$

and (iv) *t* contains all constant fuzzy sets in *X*.

The pair (*X*, *t*) is called an fts.

We shall use the concept of fts due to Lowen R. unless otherwise stated.

Definition 2.9 [6, 9]: Let *u* be a fuzzy set in an fts (*X*, *t*). Then the fuzzy closure $\overline{u}$ and the fuzzy interior u^o of *u* are defined as follows:

$\overline{u} = \inf\left\{\lambda : u \leq \lambda \text{ and } \lambda \in t^c\right\}$.

$u^o = \sup\{\lambda : \lambda \leq u \text{ and } \lambda \in t\}$.

Fact 2.10 [6, 9]: For a fuzzy topological space (X, t) and for $u \in I^X$ **,** the following hold:

(i) $\overline{u} = 1 - u^o$

(ii) For any fuzzy set *u* in *X*, $u^o \le u \le \overline{u}$.

(iii) If u ≤ v, then $\overline{u} \le \overline{v}$ and $u^o \le v^o$.

Definition 2.11 [9]: Let (*X*, *t*) and (*Y*, *s*) be two fts. A function $f : (X, t) \to (Y, s)$ is called

(i) continuous if and only if $f^{-1}(u) \in t$ for each $u \in s$.

(ii) open if and only if $f(u) \in s$ for each $u \in t$.

(iii) closed if and only if $f(u) \in s^c$ for each $u \in t^c$.

Definition 2.12 [9]: Let $\{(X_i, t_i) : i \in K\}$ be a collection of fts. Let $X = \prod_{i \in K} X_i$ be their Cartesian product and $p_i : X \to X_i$ be the projection map. Then the fuzzy topology *t* on *X* generated by $\{p_i^{-1}(u_i) : i \in K, u_i \in t_i\}$ is called the product fuzzy topology on *X* and the pair (*X*, *t*) is called the product fts. It can be verified that $p_i^{-1}(u_i)$, $i \in K$, as defined above, can be expressed as $\prod_{k \in K} \lambda_k$ where $\lambda_k = u_i$ if $k = i$ and $\lambda_k = X_k$ if $k \ne i$.
The product fuzzy topology *t* is also called the coarsest fuzzy topology on *X*.

Fact 2.13 [9]: For a family $\{(X_i, t_i) : i \in K\}$ of fts and a fuzzy topology *t* on $X = \prod_{i \in K} X_i$, the following are equivalent:

(i) *t* is the product of the fuzzy topologies t_i 's.

(ii) *t* is the smallest fuzzy topology on *X* which makes each projection $p_i : X \to X_i$, $i \in K$ continuous.

(iii) For each fts (*Y*, *s*) the function $f : (X, t) \to (Y, s)$ is continuous if and only if for all $i \in K$, $p_i(f)$ is continuous.

Definition 2.14 [9]: Let $\{f_j : X \to (X_j, t_j); j \in J\}$ be a family of functions from a set *X* to fts (X_j, t_j), $j \in J$. Then the initial fuzzy topology on *X* induced by the family $\{f_j : j \in J\}$, say *t*, is the smallest fuzzy topology on *X*, making each f_j, $j \in J$, continuous. It can be verified that *t* is generated by the family of fuzzy sets $f_j^{-1}(u_j) : u_j \in t_j$ and $j \in J$. For example, the product fuzzy topology is the initial fuzzy topology induced by the family of projections. Similarly, the subspace topology is also the initial fuzzy topology induced by the inclusion map.

Definition 2.15 [9]: A fuzzy topological property FP is said to be an initial property if for each family of functions $\left\{ f_j : X \to \left(X_j, t_j \right); j \in J \right\}$, whenever each fts $\left(X_j, t_j \right); j \in J$, has FP, (X, t) also has FP, t being the initial fuzzy topology on X induced by the family $\left\{ f_j : j \in J \right\}$.

Definition 2.16 [9]: A function f: $X \to \mathbf{R}$ is called *lower semicontinuous* (l.s.c.) if and only if for every $\alpha \in \mathbf{R}$, $\{x \in X : f(x) > \alpha\}$ is an open set. For a topological space (X, T), the l.s.c. fuzzy topology on X associated with T is denoted by $\omega(T)$ and is defined as $\omega(T)$ $= \left\{ u \in I^X : u \text{ is l.s.c.} \right\}$.

Fact 2.17 [9]: Let (X, T) be a topological space. Then

(i) $u \in I^X$ is $\omega(T)$ closed if and only if for all $\alpha \in I$, $u^{-1}[\alpha,1]$ is T-closed.

(ii) $A \subseteq X$ is T-open if and only if 1_A is $\omega(T)$-open.

(iii) $A \subseteq X$ is T-closed if and only if 1_A is $\omega(T)$-closed.

(iv) $\overline{u^{-1}(\alpha,1]} \subseteq \left(\overline{u}\right)^{-1}[\alpha,1]$.

(v) $\overline{\alpha 1_A} = \alpha 1_{\overline{A}}$.

(vi) $\{1_U : U \in T\}$ is a subbase for $\omega(T)$.

(vii) $\{\alpha 1_U : \alpha \in I_0 \text{ and } U \in T\}$ is a base for $\omega(T)$.

Definition 2.18 [2]: Let P be a property of a topological space and FP its fuzzy topological analogue. Then FP is called a *good extension* of P if and only if the statement "(X, T) has P if and only if $(X, \omega(T))$ has FP" holds good for every topological space (X, T).

3. R$_1$- properties

In this section we introduce six R_1-axioms for fts.

Definitions 3.1: We define, for fts (X, t), R_1-properties as follows:

$FR_1(1)$: If $\forall\, x, y \in X, x \neq y$ and $\forall\, \alpha \in I_{0,1}$ $\exists$ $w \in t$ such that either $w(x) > \alpha$ and $w(y) = 0$ or $w(y) > \alpha$ and $w(x) = 0$, then $\exists$ $\mu, \nu \in t$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu = 0$.

$FR_1(2)$: If $\forall\, x, y \in X, x \neq y$ and $\forall\, \alpha \in I_{0,1}$ $\exists$ $w \in t$ such that either $w(x) > \alpha$ and $w(y) = 0$ or $w(y) > \alpha$ and $w(x) = 0$, then $\exists$ $\mu, \nu \in t$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \leq 1 - \nu$..

$FR_1(3)$: If $\forall$ $x, y \in X, x \neq y$ and $\forall \alpha \in I_{0,1}$ $\exists$ $w \in t$ such that either $w(x) > \alpha$ and $w(y) = 0$ or $w(y) > \alpha$ and $w(x) = 0$, then $\exists$ $\mu, \nu \in t$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \wedge \nu = 0$.

$FR_1(4)$: If $\forall$ $x, y \in X, x \neq y$ and $\forall \alpha \in I_{0,1}$ $\exists$ $w \in t$ such that either $w(x) > \alpha$ and $w(y) = 0$ or $w(y) > \alpha$ and $w(x) = 0$, then $\exists$ $\mu, \nu \in t$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \leq 1 - \nu$.

$FR_1(5)$: If $\forall$ $x, y \in X, x \neq y$ and $\forall \alpha \in I_{0,1}$ $\exists$ $w \in t$ such that either $w(x) > \alpha$ and $w(y) = 0$ or $w(y) > \alpha$ and $w(x) = 0$, then $\forall$ $\beta, \delta \in I_{0,1}$, $\exists$ $\mu, \nu \in t$ such that $\mu(x) > \beta$, $\nu(y) > \delta$ and $\mu \wedge \nu = 0$.

$FR_1(6)$: If $\forall$ $x, y \in X, x \neq y$ and $\forall \alpha \in I_{0,1}$ $\exists$ $w \in t$ such that either $w(x) > \alpha$ and $w(y) = 0$ or $w(y) > \alpha$ and $w(x) = 0$, then $\exists$ $\mu, \nu \in t$ such that $\mu(x) > 0$, $\nu(y) > 0$ and $\mu \wedge \nu = 0$.

Theorem 3.1: The following implications hold among the R_1-properties mentioned above:

$$\begin{array}{ccccc} FR_1(1) & \Leftrightarrow & FR_1(3) & \Rightarrow & FR_1(5) \\ \Downarrow & & \Downarrow & & \Downarrow \\ FR_1(2) & \Leftrightarrow & FR_1(4) & & FR_1(6) \end{array}$$

Proof:

$FR_1(1) \Rightarrow FR_1(3)$: Let (X, t) be an fts which has the property $FR_1(1)$. Let $x, y \in X, x \neq y$, and $w \in t$ such that $w(x) > \alpha \in I_{0,1}$ and $w(y) = 0$. Then, by the $FR_1(1)$ - property of (X, t), there exist $u, v \in t$ such that $\overline{1_x} \leq u$, $\overline{1_y} \leq v$ and $u \wedge v = 0$. Clearly, $u(x) = 1 = v(y)$ and $u \wedge v = 0$ Hence, (X, t) has the property $FR_1(3)$. Thus $FR_1(1) \Rightarrow FR_1(3)$.

$FR_1(4) \Rightarrow FR_1(2)$: Consider a $FR_1(4)$-fts (X, t). Let $x, y \in X$, $x \neq y$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) > \alpha$ and $w(y) = 0$. Then by $FR_1(4)$, $\exists$ $u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \leq 1 - v$. Let $z \in X$ and $\beta \in I_{0,1}$ such that $\beta 1_z \nleq u$. This implies that $\beta > u(z)$. Then $u(z) = \delta \in I_{0,1}$. Then $u(z) = \delta \in I_{0,1}$ and $u(y) = 0$ together imply that $\exists$ $\eta, \lambda \in t$ such that $\eta(y) = 1 = \lambda(z)$ and $\lambda \leq 1 - \eta$. Now $1 - \lambda(y) = 1$. Therefore, $\overline{1_y} \leq 1 - \lambda$. Now, $\overline{1_y}(z) \leq 1 - \lambda(z) = 0$ and so $\beta 1_z \nleq \overline{1_y}$. Therefore, $\overline{1_y} \leq u$, which is a contradiction as $u(y) \neq 1$. Therefore $u(z) = 0$. Now, $u(x) = 1 > \alpha, \forall \alpha \in I_{0,1}$ and $u(z) = 0$, together imply that $\exists$ $\eta, \lambda \in t$ such that $\eta(x) = 1 = \lambda(z)$ and $\lambda \leq 1 - \eta$. Now,

$(1-\lambda)(x)=1$. Therefore, $\overline{1_x}\leq 1-\lambda$. But $\overline{1_x}(z)\leq 1-\lambda(z)=0$. Therefore, $\beta 1_z \nleq \overline{1_x}$. Thus we see that, $\beta 1_z \nleq u$ implies $\beta 1_z \nleq \overline{1_x}$. Hence, $\overline{1_x}\leq u$. Similarly we can show that $\overline{1_y}\leq v$. Therefore, (X, t) is R_1^2. Thus $FR_1(4)\Rightarrow FR_1(2)$.

All other proofs are similar. □

Now we give some counter examples to show the non-implications among the fuzzy R_1 properties mentioned above.

Example 3.1 [9]**:** Let X be an infinite set and for any $x, y \in X$, we define u_{xy}, a fuzzy set in X, as follows:
$u_{xy}(x) = 1$, $u_{xy}(y) = 0$ and $u_{xy}(z) = 0.5\ \forall\ z\in X,\ z \neq x, y$. Now consider the fuzzy topology t on X generated by $\{u_{xy}\colon x, y \in X, x \neq y\}\cup$ {constants}. It is clear that, $\overline{1_x}\leq u_{xy}$, $\overline{1_y}\leq u_{yx}$ and $u_{xy}\leq 1-u_{yx}$. Thus, (X, t) is $FR_1(2)$. But (X, t) is not $FR_1(6)$ as $u_{xy}\wedge u_{yx}$ can never be zero. Thus, we see that $FR_1(2)\nRightarrow FR_1(6)$ and therefore, $FR_1(p)\nRightarrow FR_1(q)$; ($p$ = 2, 4 and q = 1, 3, 5, 6).

Example 3.2 [9]: Let $X = I$ and t be the fuzzy topology on X generated by $B = B_1\cup B_2\cup B_3\cup B_4$. Where, $B_1 = \{1_x\colon x\in I_{0,1}\}$, $B_2 = \{u_m\colon m\in \mathbf{N}\}$; u_m is a fuzzy set in X defined by $u_m = 1_{\left[0,\ \frac{1}{m+1}\right]}$,
$B_3 = \{v_{n,F}\colon n\in\mathbf{N}$ and F is a finite crisp subset of $X\}$, Where $v_{n,F}$ is a fuzzy set in X defined by $v_{n,F} = \left(\frac{n}{n+1}\right)1_{\left[\frac{1}{n+1},\ 1\right]-F}$ and B_4 = {constants}.

It can be checked that (X, t) is $FR_1(5)$. But (X, t) is not $FR_1(4)$. For, if we take $x=1$, $y=0$ and $u_1\in B_2$, we see that $u_1(x)=0$ and $u_1(y)=1>\alpha\ \forall\,\alpha\in I_{0,1}$, but there exist no $u, v\in t$ such that $u(x)=1=v(y)$ and $u\leq 1-v$. Thus we see that, $FR_1(5)\nRightarrow FR_1(4)$ and therefore $FR_1(p)\nRightarrow FR_1(q)$; ($p$ = 5, 6 and q = 1, 2, 3, 4).

Example 3.3 [10]: Let $X = \{x, y\}$ and $t = \left\langle \left\{\frac{1}{2}1_x, \frac{1}{2}1_y\right\} \cup \{\text{constants}\}\right\rangle$. Then (X, t) is an fts and it is $FR_1(6)$, But it is not $FR_1(5)$. For, if we take $\beta, \delta \in I_{0,1}$ such that $\beta > 0.5$ and $\delta > 0.5$, we see that there exist no $u, v \in t$ such that $u(x) > \beta$, $v(y) > \delta$ and $u \wedge v = 0$. Thus we see that, $FR_1(6)\nRightarrow FR_1(5)$.

Theorem 3.2: All $FR_1(k)$; $(1 \le k \le 6)$ are good extensions of the topological R_1-property. That is, $(X, \mathcal{T})$ is an R_1-space, if and only if $(X, \omega(\mathcal{T}))$ satisfies $FR_1(k)$; $(1 \le k \le 12)$.

Note: By theorem 3.1, we have only to prove the following:

(a) If $(X, \mathcal{T})$ is an R_1-space, then $(X, \omega(\mathcal{T}))$ satisfies $FR_1(1)$.

(b) If $(X, \omega(\mathcal{T}))$ satisfies $FR_1(k)$; $(k \in \{4, 6\})$, then $(X, \mathcal{T})$ is an R_1-space.

Proof:

(a) Suppose $(X, \mathcal{T})$ is an R_1- topological space. Let $x, y \in X$, $x \ne y$, and $\alpha \in I_{0,1}$, and $w \in t$ such that $w(x) > \alpha$ and $w(y) = 0$. Now $w^{-1}(\alpha, 1] \in \omega(\mathcal{T})$ such that $x \in w^{-1}(\alpha, 1]$ and $y \notin w^{-1}(\alpha, 1]$. This implies that $x \notin \overline{\{y\}}$ in $\mathcal{T}$. Hence there exist $\mathcal{U}, \mathcal{V} \in \mathcal{T}$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$ and $\mathcal{U} \cup \mathcal{V} = \emptyset$. Since an R_1-topological space is also an R_0- topological space, $\overline{\{x\}} \subseteq \mathcal{U}$ and $\overline{\{y\}} \subseteq \mathcal{V}$. Also we know that, $1_{\overline{\{x\}}} = \overline{1_x}$ and $\overline{1_y} = 1_{\overline{\{y\}}}$. Therefore, $\overline{1_x} \le 1_{\mathcal{U}}$ and $\overline{1_y} \le 1_{\mathcal{V}}$. Moreover, $1_{\mathcal{U}} \wedge 1_{\mathcal{V}} = 0$. Hence $(X, \omega(\mathcal{T}))$ satisfies $FR_1(1)$.

(b) Suppose $(X, \mathcal{W}(\mathcal{T}))$ satisfies $FR_1(4)$. Let $x, y \in X$ such that $x \notin \overline{\{y\}}$ in $\mathcal{T}$. Then $\exists$ $w \in \mathcal{T}$ such that $x \in w$ and $y \notin w$. Now $1_w \in \omega(\mathcal{T})$ such that $1_w(y) = 0$ and $1_w(x) = 1 > \alpha\ \forall\ \alpha \in I_{0,1}$. Therefore $\exists$ $\mu, \nu \in \omega(\mathcal{T})$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \le 1 - \nu$. Take $U = \mu^{-1}\left(\frac{1}{2}, 1\right]$ and $V = \nu^{-1}\left(\frac{1}{2}, 1\right]$. Clearly, $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Therefore, $(X, \mathcal{T})$ is an R_1-topological space.

Again, suppose $(X, \omega(\mathcal{T}))$ satisfies $FR_1(6)$. Let $x, y \in X$ such that $x \notin \overline{\{y\}}$ in $\mathcal{T}$. Then $\exists$ $w \in \mathcal{T}$ such that $x \in w$ and $y \notin w$. Now $1_w \in \omega(\mathcal{T})$ such that $1_w(y) = 0$ and $1_w(x) = 1 > \alpha\ \forall\ \alpha \in I_{0,1}$. Therefore $\exists$ $\mu, \nu \in \omega(\mathcal{T})$ such that $\mu(x) > 0$, $\nu(y) > 0$ and $\mu \wedge \nu = 0$. Now, $x \in \mu^{-1}(0, 1] \in \mathcal{T}$, $y \in \nu^{-1}(0, 1] \in \mathcal{T}$ such that $\mu^{-1}(0, 1] \cap \nu^{-1}(0, 1] = \emptyset$. Therefore, $(X, \mathcal{T})$ is an R_1-topological space. □

Theorem 3.3: The properties $FR_1(k)$, $(1 \le k \le 6)$ are initial, i.e., if $(f_j: X \to (X_j, t_j))$ is a source in fts where all (X_j, t_j) are $FR_1(k)$, then the initial fuzzy topology t on X is also $FR_1(k)$.

Proof:

Let $\{(X_j, t_j): j\in J\}$ be a family of $FR_1(1)$-fts, $\{f_j: X\to(X_j, t_j); j\in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j\in J\}$. Let $x, y \in X$, $x \neq y$, $\alpha\in I_{0,1}$ and $w\in t$ such that $w(x) > \alpha$ and $w(y) = 0$. Since $w\in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p\in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{ f_{p_k}^{-1} w_{p_k} : 1\le k\le n\right\}$. As w(x) > α and w(y) = 0 , we can find some k such that $1 \le k \le n$, say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) > \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$. This implies that $w_{p_{k'}} f_{p_{k'}}(x) > \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is $FR_1(1)$, $\exists$ $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\overline{1_{f_{p_{k'}}(x)}} \le \mu_{p_{k'}}$, $\overline{1_{f_{p_{k'}}(y)}} \le \nu_{p_{k'}}$ and $\mu_{p_{k'}} \wedge \nu_{p_{k'}} = 0$. Also since $f_{p_{k'}}$ is continuous, we have $f_{p_{k'}}\left(\overline{1_x}\right) \le \overline{1_{f_{p_{k'}}(x)}}$. Now put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Then $\mu, \nu \in t$ such that $\overline{1_x} \le \mu$, $\overline{1_y} \le \nu$ and $\mu \wedge \nu = 0$. Hence (X, t) is $FR_1(1)$. Thus we see that $FR_1(1)$ is an initial property.

All other proofs are similar. □

Corollary-3.4: Since initiality implies productivity and heredity all the $FR_1(k)$ properties; $(k = 1, 2, ..., 6)$ are productive and hereditary.

4. R_0-properties

We recall from [6], nine definitions of the R_0-axioms of an fts used in the sequel:

Definitions-4.1 [6]: We define, for fts (X, t), R_0-properties as follows:

R_0^1 : For every pair $x, y\in X$, $x \neq y$, $\overline{1_y}(x) = 0 \Rightarrow \overline{1_x}(y) = 0$

R_0^2 : For every pair $x, y\in X$, $x \neq y$, $\left(\forall\alpha\in I_0 : \overline{\alpha 1_x}(y) = \alpha \Leftrightarrow \forall\beta\in I_0 : \overline{\beta 1_y}(x) = \beta\right)$

R_0^3 : $\forall\lambda\in t$, $\forall x\in X$ and $\forall\alpha < \lambda(x)$, $\overline{\alpha 1_x} \le \lambda$

R_0^4 : $\forall\lambda\in t$, $\forall x\in X$ and $\forall\alpha \le \lambda(x)$, $\overline{\alpha 1_x} \le \lambda$

R_0^5 : For every pair $x, y\in X$, $x \neq y$, $\overline{1_x}(y) = 1 \Rightarrow \overline{1_y}(x) = 1$

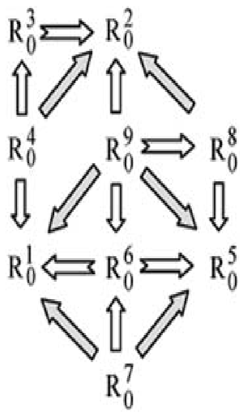
R_0^6 : For every pair $x, y\in X$, $x \neq y$, $\overline{1_x}(y) = \overline{1_y}(x)$

R_0^7 : For every pair $x, y\in X$, $x \neq y$, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0, 1\}$

R_0^8 : For every pair $x, y \in X$, $x \neq y$, and $\forall \alpha \in I_0$, $\overline{\alpha 1_x}(y) = \alpha \Rightarrow \overline{\alpha 1_y}(x) = \alpha$

R_0^9 : For every pair $x, y \in X$, $x \neq y$, and $\forall \alpha \in I_0$, $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$

Theorem 4.1 [6]: Between the fuzzy R_0-porperties, mentioned above, there exist the following implications:



Theorem 4.2: The following relations hold between the fuzzy R_1-axioms and fuzzy R_0-axioms discussed above:

(a) $R_1^1 \not\Rightarrow R_0^2$, and so $R_1^k \not\Rightarrow R_0^m$, where $k \in \{1,2,3,4,5,6\}$ and $m \in \{2,3,4,8,9\}$.

(b) $R_1^4 \Rightarrow R_0^1$, and so $R_1^k \Rightarrow R_0^1$, where $k \in \{1,2,3,4\}$.

(c) $R_1^6 \Rightarrow R_0^5$, and so $R_1^k \Rightarrow R_0^5$, where $k \in \{1,3,5,6\}$.

(d) $R_1^5 \not\Rightarrow R_0^1$, and so $R_1^k \not\Rightarrow R_0^m$, where $k \in \{5,6\}$ and $m \in \{1,4,6,7,9\}$.

(e) $R_1^4 \Rightarrow R_0^7$, and so $R_1^k \Rightarrow R_0^m$, where $k \in \{1,2,3,4\}$ and $m \in \{1,5,6,7\}$.

(f) $R_0^m \not\Rightarrow FR_1(k)$, where $k \in \{i - xviii\}$ and $m \in \{1-9\}$.

Proof (a):

Example 4.1: Consider an fts (X,t) where $X = \{x, y\}$, $u(x) = 0.1$, $u(y) = 0.2$ and $t = < \{u\} \cup \{\text{constants}\} >$. It can be verified that (X,t) is $FR_1(1)$ but it is not R_0^2.

Proof (b): Consider an $FR_1(4)$-fts (X,t). Let $x, y \in X$, $x \neq y$ such that $\overline{1_x}(y) = 0$. Put $w = 1 - \overline{1_x}$. Now $w \in t$ such that $w(x) = 0$ and $w(y) = 1 > \alpha \in I_{0,1}$. Since (X, t) is

$FR_1(4)$, $\exists\ u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \le 1 - v$. Put $\lambda = 1 - u \in t^c$. Clearly, $\overline{1_y} \le \lambda$ and so $\overline{1_y}(x) = 0$. Hence (X, t) is R_0^1.

Proof (c): Consider an $FR_1(6)$-fts (X, t). Let $x, y \in X$, $x \ne y$ such that $\overline{1_x}(y) = 1$.
Suppose $\overline{1_y}(x) < 1$.
Put $w = 1 - \overline{1_y}$. Now $w \in t$ such that $w(x) > 0$ and $w(y) = 1$. Then for some $\alpha \in I_{0,1}$, $w(x) > \alpha$. Since (X, t) is $FR_1(6)$, $\exists\ u, v \in t$ such that $u(x) > 0, v(y) > 0$ and $u \wedge v = 0$. $\lambda = 1 - v \in t^c$. Clearly, $\overline{1_x} \le \lambda$. Now, $\overline{1_x}(y) \le \lambda(y) < 1$, which is a contradiction. Therefore, $\overline{1_y}(x) = 1$ and so (X, t) is R_0^5.

Proof (d): In example 3.2, It can be verified that (*X*, *t*) is $FR_1(5)$, But it is not R_0^1 [9].

Proof (e): Suppose (*X*, *t*) is an $FR_1(4)$-fts. Let $x, y \in X$, $x \ne y$ such that $\overline{1_y}(x) \notin \{0,1\}$. This implies that $\exists\ m \in t^c$ such that $m(y) = 1$ and $0 < m(x) < 1$. Put $w = 1 - m \in t$. Now, $w(x) > 0$ and $w(y) = 0$. Then, $w(x) > \alpha$ for some $\alpha \in I_{0,1}$. Since (*X*, *t*) is an $FR_1(4)$-fts, $\exists$ $u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \wedge v = 0$. Put $n = 1 - u \in t^c$. Now, $n(x) = 0$ and $n(y) = 1$. Therefore, $\overline{1_y} \le n$ and so $\overline{1_y}(x) = 0$, which is a contradiction. Again let $\overline{1_y}(x) \ne \overline{1_x}(y)$. Without any loss of generality, let $0 = \overline{1_x}(y) < \overline{1_y}(x) = 1$. This implies that $\exists\ \lambda_1, \lambda_2 \in t^c$ such that $\lambda_1(x) = \lambda_2(x) = \lambda_2(y) = 1$ and $\lambda_1(y) = 0$. Take $w = 1 - \lambda_1$. Now $w \in t$ such that $w(x) = 0$ and $w(y) = 1 > \alpha\ \ \forall\ \alpha \in I_{0,1}$. Since (*X*, *t*) is an $FR_1(4)$-fts, $\exists\ p, q \in t$ such that $p(x) = 1 = q(y)$ and $p \wedge q = 0$. Put $n_1 = 1 - p$ and $n_2 = 1 - q$. Now, $n_1, n_2 \in t^c$ such that $n_1(x) = 0 = n_2(y)$ and $n_1(y) = 1 = n_2(x)$. Clearly, $\overline{1_y} \le n_1$ and so $\overline{1_y}(x) = 0$, which is also a contradiction. Therefore, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0,1\}$. Therefore (*X*, *t*) is an R_0^7-fts.

Proof (f):

Example-4.2 [12]: Let *X* be an infinite set. For $x, y \in X$, we define $U_{xy} \in I^X$ as follows:

$$U_{xy}(z) = \begin{cases} 0 \text{ if } z \in \{x, y\} \\ 1 \text{ if } z \notin \{x, y\} \end{cases}$$

Let *t* be the fuzzy topology generated by $\{U_{xy} : x, y \in X\} \cup \{\text{constants}\}$. It can be checked that if $x \ne y$, then $\overline{1_x}(y) = 0$. Therefore, (*X*, *t*) is R_0^4, R_0^7 and R_0^9. But (*X*, *t*) is neither $FR_1(4)$ nor $FR_1(6)$ as there exists no u, $v \in t$ such that $u \leq 1 - v$. Therefore, $R_0^m \nRightarrow FR_1(k)$, where $k \in \{i - xviii\}$ and $m \in \{1-9\}$. □

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