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Solving Mathematical Model by using Modified Fractional Differential Transform Method with Adomian Polynomials

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Abstract

In this article, the solution of nonlinear fractional differential equations (FDEs) for disease transmission is discussed in a population believed to maintain a stable size during an epidemic by using a new approach called the fractional differential transform technique (FDTM) along with Adomian polynomials. Also, this method is compared with those that the homotopy perturbation approach produces. Several charts are presented to demonstrate the consistency and simplicity of this method.

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1. Introduction

The study of disease transmission and its consequences is known as epidemiology. This spans a wide variety of disciplines, from biology to sociology and philosophy, all of which are used to get a better understanding and control of virus propagation. The SIR model for disease propagation, which consists of a set of three differential equations that represent changes in the number of susceptible, infected, and recovered persons in a given community, is a widely used epidemiological model. Various writers have examined and explored mathematical SIR models [1-7].

The differential equations with fractional order have recently proved to be valuable tools to the modeling of many real problems in different areas [8-11]. This is because of the fact that the realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the history of the previous time which can also be successfully achieved by using fractional calculus. For example, half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical

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models [12,13]. Lately, a large amount of studies developed concerning the application of fractional differential equations in various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering. An excellent account in the study of fractional differential equations can be found in [14-17]. Now, in this work, we'll look at a SIR model of fractional order that goes like this:

$$\begin{aligned} T_\alpha^\xi y_1 &= -\rho y_1(\xi) y_2(\xi) \\ T_\alpha^\xi y_2 &= \rho y_1(\xi) y_2(\xi) - \eta y_2(\xi) \\ T_\alpha^\xi y_3 &= \eta y_2(\xi) \end{aligned} \tag{1.1}$$

with given initial conditions, $y_1(0) = 20, y_2(0) = 15$ and $y_3(0) = 10$, where $0 < \alpha \leqslant 1$. Further the involve functions in the model obey $N(\xi) = y_1(\xi) + y_2(\xi) + y_3(\xi)$.

Many researchers solved the linear and non-linear mathematical-biological fractional model on various disease by different methods like VIM, HAM, ADM, LADM and HPM [1,2,4,5,7]. The differential transform method (DTM) [12,16] has been successfully applied to a wide class of differential equations arising in many areas of science and engineering. Since many physical phenomena are more faithfully modeled by fractional differential equations (FDEs), Arikoglu and Ozkol developed the fractional differential transform method (FDTM) [19] for their efficient solution. Also, Odibat and Shawagfeh suggested the same technique as a generalized Taylor's formula for solving FDEs [20]. The FDTM provides an iterative procedure for obtaining the series solution of both linear and nonlinear FDEs. Unlike the traditional series method, which requires symbolic computation, the FDTM transforms the FDEs into algebraic equations, which can be solved by an iterative procedure. Fractional differential equations (FDEs) are used to simulate a wide range of physical events, and they may be solved using a variety of transform methods [27-29] and recently efficient approach for solving nonlinear FDEs by using the FDTM with Adomian polynomials [18]. In this work is to find approximate solution of nonlinear SIR model of fractional order differetial equations by using FDTM with Adomian polynomial. Teppawar *et al.* [28,29], developed CFDTM with Adomian polynomials have been used to solved nonlinear and singular Lane-Emden FDEs. Momani and Kharrat [8,30] have used ADM in order to resolve fractional Riccati differential equations. Recently Pawar *et al.* [31] examined the fractional order mathematical model of drug resistant TB using a two-line therapy. They used the Caputo fractional derivative and the generalized Euler method (GEM) to analyze and compare the results with prior findings in integer order. Shatanawi [32] formulated novel fractional tuberculosis (TB) model with a generalized Atangana–Baleanu (GAB) fractional derivative, Sinan [33] discussed the Cutaneous Leishmaniasis disease model and numerically proposed model has used a nonstandard finite difference scheme. Siraj *et al.* [34] described a numerical scheme based on Laplace transform and numerical inverse Laplace transform for the approximate solution of fractal-fractional differential equations with order α, β and Kamal *et al.* [35] investigated dynamical system for the existence and uniqueness of at least one solution and used to Schauder and Banach fixed point theorems. Devi and Jakhar [36] introduced Sumudu-Adomian Decomposition Method (SADM) for finding the exact and approximate solutions of fractional order telegraph equations. Tyagi and Chandel [37] have obtained a method for solving inhomogeneous linear sequential fractional differential equations with constant coefficients (ILSFDE) involving Jumarie fractional derivatives in terms of Mittag-Leffler functions.

2. Basic Ideas of the Fractional Differential Transform Method (FDTM)

In this part, fractional calculus and fractional differential transform method (FDTM) are reviewed.

Definition 2.1. The Riemann-Liouville fractional integral of order $\gamma \geq 0$, is defined by [23-26]

$$\mathscr{I}^{\gamma}\varphi(\vartheta) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^{\vartheta} (\vartheta - \tau)^{\gamma-1}\varphi(\tau)d\tau & \gamma > 0, \quad \vartheta > 0. \\ \varphi(\vartheta) & \gamma = 0. \end{cases} \tag{2.1}$$

Definition 2.2. The definition of the Caputo fractional derivative of φ is defined as [23,26]

$$\mathcal{D}^{\mu}\varphi(\vartheta) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^{\vartheta} (\vartheta - \tau)^{n-\alpha-1}\varphi^{(n)}(\tau)d\tau & n-1 < \gamma < n \\ \dfrac{\mathrm{d}^n\varphi(\vartheta)}{\mathrm{d}\vartheta^n} & \gamma = n \end{cases} \tag{2.2}$$

where n is an integer. Caputo's integral operator has useful properties such as:

$$\mathcal{D}^{\gamma}\mathcal{J}^{\gamma}\varphi = \varphi(\vartheta)$$

$$\mathcal{J}^{\gamma}\mathcal{D}^{\gamma}\varphi(\vartheta) = \varphi(\vartheta) - \sum_{k=0}^{n-1} \varphi^{(k)}(0^{+})\frac{\vartheta^k}{k!},\ t \geq 0\ n-1 < \gamma \leqslant n$$

The fractional differentiation in Riemann-Liouville sense is defined by

$$\mathcal{P}^{\gamma}_{\vartheta_0}\varphi(\vartheta) = \frac{1}{\Gamma(m-\gamma)}\frac{\mathrm{d}^m}{\mathrm{d}\vartheta^m}\left[\int_{\vartheta_0}^{\vartheta} \frac{\varphi(t)}{(\vartheta - t)^{1+\gamma-m}}\ \mathrm{d}t\right] \tag{2.3}$$

for $m-1 \leqslant \gamma < m, m \in Z^{+}, \vartheta > \vartheta_0$. Let us expand the analytical and continuous function $\varphi(\vartheta)$ in terms of a fractional power series as follows:

$$\varphi(\vartheta) = \sum_{k=0}^{\infty} \Phi(k)(\vartheta - \vartheta_0)^{k/\alpha} \tag{2.4}$$

where α is the order of fraction and $\Phi(k)$ is the fractional differential transform of $\varphi(\vartheta)$. In order to avoid fractional initial and boundary conditions, we define the fractional derivative in the Caputo sense. The relation between the Riemann-Liouville operator and Caputo operator is given by

$$\mathcal{D}^{\gamma}_{*\vartheta_0}\varphi(\vartheta) = \mathcal{D}^{\gamma}_{\vartheta_0}\left[\varphi(\vartheta) - \sum_{k=0}^{m-1} \frac{1}{k!}(\vartheta - \vartheta_0)^k\varphi^{(k)}(\vartheta_0)\right] \tag{2.5}$$

Setting $\varphi(\vartheta) = \varphi(\vartheta) - \sum_{k=0}^{m-1} \frac{1}{k!}(\vartheta - \vartheta_0)^k\varphi^{(k)}(\vartheta_0)$ in Eq. (2.1) and using Eq. (2.3), fractional derivative is obtained in the Caputo sense [13] as follows:

$$\mathcal{D}^{\gamma}_{*\vartheta_0}\varphi(\vartheta) = \frac{1}{\Gamma(m-\gamma)}\frac{\mathrm{d}^m}{\mathrm{d}\vartheta^m}\left\{\int_{\vartheta_0}^{\vartheta}\left[\frac{\varphi(t) - \sum_{k=0}^{m-1}\ (1/k!)(t-\vartheta_0)^k\varphi^{(k)}(\vartheta_0)}{(\vartheta - t)^{1+\gamma-m}}\right]\mathrm{d}t\right\}$$

Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$\Phi(k)=\begin{cases}\text{If } k/\alpha\in Z^{+}, \dfrac{1}{(k/\alpha)!}\left[\dfrac{\mathrm{d}^{k/\alpha}\varphi(\vartheta)}{\mathrm{d}\vartheta^{k/\alpha}}\right]_{\vartheta=\vartheta_0} & \text{for } k=0,1,2,\dots,(\gamma\alpha-1)\\ \text{If } k/\alpha\notin Z^{+} & 0\end{cases} \tag{2.6}$$

where, γ is the order of fractional differential equation considered. The following theorems that can be deduced from Eqs. (2.3) and (2.4) are given below, proofs and details are reported [19]:

Theorem 2.3. If $\varphi(\zeta)=\psi(\zeta)\pm w(\zeta)$, then $\Phi(k)=\Psi(k)\pm\omega(k)$.
Theorem 2.4. If $\varphi(\zeta)=\psi(\zeta)w(\zeta)$, then $\Phi(k)=\sum_{l=0}^{k}\Psi(l)\omega(k-l)$.
Theorem 2.5. If $\varphi(\zeta)=\psi_1(\zeta)\psi_2(\zeta)\dots\psi_{n-1}(\zeta)\psi_n(\zeta)$, then

$$\Phi(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\Psi_1(k_1)\Psi_2(k_2-k_1)\dots\Psi_{n-1}(k_{n-1}-k_{n-2})\Psi_n(k-k_{n-1})$$

Theorem 2.6. If $\varphi(\zeta)=(\zeta-\zeta_0)^r$, then $\Phi(k)=\delta(k-\alpha r)$ where,

$$\delta(k)=\begin{cases}1 & \text{if } k=0\\ 0 & \text{if } k\neq 0\end{cases}$$

Theorem 2.7. If $\varphi(\zeta)=D_{\zeta_0}^{q}[\psi(\zeta)]$, then $\Phi(k)=\frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)}\Psi(k+\alpha q)$.

2.1. *Basic idea of FDTM with Adomian polynomials*

Consider the nonlinear FDE of the form

$$\mathcal{D}^{\alpha}y=\varphi\left(y,y^{(\alpha)}\right) \tag{2.1.1}$$

where $\varphi\left(y,y^{(\alpha)}\right)$ denotes a nonlinear function. Then, for $\varphi\left(y,y^{(\alpha)}\right)$ that is analytic in the dependent variables and its Adomian polynomials are analytic with respect to the given [8] conditions differential transform, recurrence scheme takes the form [18]

$$\frac{\Gamma\left(\alpha+1+\frac{k}{\theta}\right)}{\Gamma\left(1+\frac{k}{\theta}\right)}Y(k+\alpha\theta)=\tilde{A}_k \tag{2.1.2}$$

where $\tilde{A}_k$ is obtained from the Adomian polynomials of $\varphi\left(y,y^{(\alpha)}\right)$ by replacing each y_k and $\mathcal{D}^{\beta}y_k$ in the Adomian polynomial component A_k by $Y(k)$ and $\frac{\Gamma\left(\beta+1+\frac{k}{\theta}\right)}{\Gamma\left(1+\frac{k}{\theta}\right)}Y(k+\beta\theta)$, respectively.

3. Solving System by FDTM with Adomian Polynomials Sheme

In this section, the system of fractional order differential equations will be solved using the Fractional Differential Transform Method (FDTM) with Adomian polynomials

$$\mathcal{D}^{\alpha_i}y_i(\zeta)=\mathcal{N}_i(y_1,\dots,y_n)\ i=1,2,\dots,m,\ n_{i-1}\le\alpha_i\le n_i \tag{3.1}$$

$$y_i^{(k)}(0)=c_k\ k=0,1,2,\dots,m \tag{3.2}$$

where $\mathcal{N}_i$ represent nonlinear operators, respectively. By employing FDTM with Adomian polynomials on system of equation 3.1 we get.

$$\frac{\Gamma\left(\alpha_i+1+\frac{k}{\theta_i}\right)}{\Gamma\left(1+\frac{k}{\theta_i}\right)} Y_{i,k+\alpha_i\theta_i}(k+\alpha_i\theta_i) = Y_{i,k}(k) + \tilde{A}_{i,k} \; k = 0,1,2,\dots,m, \; i = 1,2,\dots,m \qquad (3.3)$$

where $\tilde{A}_{i,k}$ is obtained from the Adomian polynomials of $\mathcal{N}_i(y_1,\dots,y_n)$ by replacing each y_k and $\mathcal{D}^{\beta_i} y_{i,k}$ in the Adomian polynomial component $A_{i,k}$ by $Y_{i,k}(k)$ and $\frac{\Gamma\left(\beta_i+1+\frac{k}{\theta_i}\right)}{\Gamma\left(1+\frac{k}{\theta_i}\right)} Y_{i,k+\beta_i\theta_i}(k+\beta_i\theta_i)$, respectively.

The initial conditions in equation 3.2 can be transformed by using equation 2.6 as follows

$$Y_{ik}(k) = C_k, \; k = 0,1,2,\dots,(\alpha_i\theta_i - 1), \; i = 1,2,\dots,m$$

Let

$$y_i = \sum_{m=0}^{\infty} Y_{im}, \; i = 1,2,\dots,n \qquad (3.4)$$

and

$$\mathcal{N}_i(y_1,\dots,y_n) = \sum_{m=0}^{\infty} A_{im}$$

with A_{im} are defined as Adomian polynomials and they are determined by the following relations [10, 12] ,

$$A_{im} = \left[\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\mathcal{N}_i\left(\sum_{m=0}^{\infty} y_{1m}\lambda^m, \dots, \sum_{m=0}^{\infty} y_{nm}\lambda^m\right)\right]_{\lambda=0}$$

Adomian polynomials can be written as, $\tilde{A}_{im}$

$$\tilde{A}_{im} = \left[\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\mathcal{N}_i\left(\sum_{m=0}^{\infty} Y_{1m}\lambda^m, \dots, \sum_{m=0}^{\infty} Y_{nm}\lambda^m\right)\right]_{\lambda=0}$$

The following recursive formula gives:

$$Y_{ik}(k) = C_k, \; k = 0,1,2,\dots,(\alpha_i\theta_i - 1), \; i = 1,2,\dots,n$$

Using inverse transformation rule in equation 2.4 becomes

$$y_i(\zeta) = \sum_{m=0}^{\infty} Y_{im}\zeta^{k/\alpha_i}, \; i = 1,2,\dots,m \qquad (3.6)$$

3.1. *Analysis of convergence and error estimate*

Theorem 3.1. If $\mathcal{B}$ be a Banach space, then the series solution of the system (3.1) converges to $S_j \in \mathcal{B}$ for $j \in N_n$, if $\exists \sigma_j \in [0,1)$ such that, $\|Y_{jn}\| \le \sigma_j \|Y_{j(n-1)}\| \forall n \in \mathbb{N}$.

Proof. Let the sequences $S_{jn}, j \in N_n$ be a partial sums of the series given by the system (3.5) as

$$\begin{cases} S_{j0} = Y_{j0}(\xi) \\ S_{j1} = Y_{j0}(\xi) + Y_{j1}(\xi) \\ S_{j2} = Y_{j0}(\xi) + Y_{j1}(\xi) + Y_{j2}(\xi) \\ \vdots \\ S_{jn} = Y_{j0}(\xi) + Y_{j1}(\xi) + Y_{j2}(\xi) + \cdots + Y_{jn}(\xi), j = 1,2,\dots,n \end{cases} \qquad (3.7)$$

Then must prove that in Banach space $\mathcal{B}$, $\{S_{jn}\}$ are Cauchy sequences. The following factors are examined in this regard:

$$\|S_{j(n+1)} - S_{jn}\| = \|Y_{j(n+1)}(\xi)\| \le \sigma_j \|Y_{jn}(\xi)\| \le \sigma_j^2 \|Y_{j(n-1)}(\xi)\| \le \cdots$$

For $n \ge m$ & $\forall n, m \in \mathbb{N}$, by using the system (3.7) and triangle inequality successively, we have,

$$\begin{aligned}\|S_{jn} - S_{jm}\| &= \|S_{j(m+1)} - S_{jm} + S_{j(m+2)} - S_{j(m+1)} + \cdots + S_{in} - S_{j(n-1)}\| \\ &\le \|S_{j(m+1)} - S_{jm}\| + \|S_{j(m+2)} - S_{j(m+1)}\| + \cdots + \|S_{jn} - S_{j(n-1)}\| \\ &\le \sigma_j^{m+1}\|Y_{j0}(\xi)\| + \sigma_j^{m+2}\|Y_{j0}(\xi)\| + \cdots + \sigma_j^{n}\|Y_{j0}(\xi)\| \\ &= \sigma_j^{m+1}\left(1 + \sigma_j + \cdots + \sigma_j^{n-m-1}\right)\|Y_{j0}(\xi)\| \\ &\le \sigma_j^{m+1}\left(\frac{1-\sigma^{n-m}}{1-\sigma_j}\right)\|Y_{j0}(\xi)\|.\end{aligned}$$

As $0 < \sigma_j < 1$, so $1 - \sigma_j^{n-m} \le 1$ then

$$\|S_{jn} - S_{jm}\| \le \frac{\sigma_j^{m+1}}{1-\sigma_j}\|Y_{j0}(\xi)\|$$

Since $Y_{j0}(\xi)$ is bounded, then

$$\lim_{n,m\to\infty}\|S_{jn} - S_{jm}\| = 0,\ j \in N_n$$

As a result, the sequences $\{S_{jn}\}$ in the Banach space $\mathcal{B}$ are Cauchy sequences, and the series solution specified in system (3.6) converges.

Theorem 3.2. The series solution (3.4) of the system (3.1) is determined to have a maximum absolute truncation error of (3.4).

$$\sup_{\xi\in\Theta}\left|Y_j(\xi) - \sum_{k=0}^{m} Y_{jk}(\xi)\right| \le \frac{\sigma_j^{m+1}}{1-\sigma_j}\sup_{\xi\in\Theta}\left|Y_{j0}(\xi)\right|, j \in N_n$$

where the region $\Theta \subset \mathbb{R}^{n+1}$.

Proof. The following is deduced from Theorem 3.1:

$$\|S_{jn} - S_{jm}\| \le \frac{\sigma_j^{m+1}}{1-\sigma_j}\sup_{(\xi)\in\Theta}\left|Y_{j0}(\xi)\right|, j \in N_n \tag{3.9}$$

However, suppose that $S_{jn} = \sum_{k=0}^{n} Y_{jk}(\xi)$ for $j = 1,2,\dots,n$, and since $n \to \infty$, then obtain $S_{jn} \to Y_j(\xi)$, so the system (3.9) can be rephrased as

$$\begin{aligned}\|Y_j(\xi) - S_{jm}\| &= \left\|Y_j(\xi) - \sum_{k=0}^{m} Y_{jk}(\xi)\right\| \\ &\le \frac{\sigma_j^{m+1}}{1-\sigma_j}\sup_{\xi\in\Theta}\left|Y_{j0}(\xi)\right|, j \in N_n.\end{aligned}$$

As a result, in the Θ region, the maximum absolute truncation error is

$$\sup_{\xi\in\Theta}\left|Y_j(\xi) - \sum_{k=0}^{m} Y_{jk}(\xi)\right| \le \frac{\sigma_j^{m+1}}{1-\sigma_j}\sup_{(\xi)\in\Theta}\left|Y_{j0}(\xi)\right|, j \in N_n$$

and this completes the proof.

4. Numerical Approximation Solution of the Fractional Order SIR Model by FDTAM

Now, in this section we'll try to solve mathematical SIR model of fractional order by using this new technique:

$$\begin{aligned} T_\alpha^\xi y_1 &= -\rho y_1(\xi) y_2(\xi) \\ T_\alpha^\xi y_2 &= \rho y_1(\xi) y_2(\xi) - \eta y_2(\xi) \\ T_\alpha^\xi y_3 &= \eta y_2(\xi) \end{aligned} \tag{4.1}$$

In terms of an infinite power series, the FDTM with Adomian polynomials gives an analytical approximation solution. However, evaluating this solution and obtaining numerical numbers from the infinite power series is necessary in practise. To complete this work, the series is truncated as a result, and the practical approach is used. Now apply the FDTM with Adomian polynomials of equation 4.1 can be expressed as follows:

$$\begin{aligned} Y_{1,k+\alpha\theta_1}(k+\alpha\theta_1) &= \frac{\Gamma(1+k/\theta_1)}{\Gamma(\alpha+1+k/\theta_1)} \tilde{A}_{1k} \\ Y_{2,k+\beta\theta_1}(k+\beta\theta_1) &= \frac{\Gamma(1+k/\theta_2)}{\Gamma(\alpha+1+k/\theta_2)} (\rho \tilde{A}_{2k} - (\eta+\kappa) Y_2(k)) \\ Y_{3,k+\gamma\theta_3}(k+\gamma\theta_3) &= \frac{\Gamma(1+k/\theta_3)}{\Gamma(\alpha+1+k/\theta_3)} \eta Y_{2k}(k) \end{aligned} \tag{4.2}$$

where $\theta_1, \theta_2, \theta_3$ are the fractions of order α, β, γ and $Y_{1,k+\alpha\theta_1}(k+\alpha\theta_1), Y_{2,k+\beta\theta_2}(k+\alpha\theta_2)$ and $Y_{3,k+\gamma\theta_3}(k+\gamma\theta_3)$ are FDT of $y_1(\xi), y_2(\xi)$ and $y_3(\xi)$ respectively. The corresponding Adomian polynomials $A_{ij}, i = 1,2,3$ and $j = 0,1, ...$

$$\begin{aligned} \tilde{A}_{10} &= Y_{10} Y_{20}, \\ \tilde{A}_{11} &= Y_{11} Y_{20} + Y_{10} Y_{21}, \\ \tilde{A}_{12} &= Y_{12} Y_{20} + Y_{11} Y_{21} + Y_{10} Y_{22}, \\ \tilde{A}_{13} &= Y_{13} Y_{20} + Y_{12} Y_{21} + Y_{11} Y_{22} + Y_{10} Y_{23}, \\ \tilde{A}_{14} &= Y_{14} Y_{20} + Y_{13} Y_{21} + Y_{12} Y_{22} + Y_{11} Y_{23} + Y_{10} Y_{24}, \\ &\vdots \\ \tilde{A}_{20} &= Y_{10} Y_{20} \\ \tilde{A}_{21} &= Y_{11} Y_{20} + Y_{10} Y_{21} \\ \tilde{A}_{22} &= Y_{12} Y_{20} + Y_{11} Y_{21} + Y_{10} Y_{22} \\ \tilde{A}_{23} &= Y_{13} Y_{20} + Y_{12} Y_{21} + Y_{11} Y_{22} + Y_{10} Y_{23} \\ \tilde{A}_{24} &= Y_{14} Y_{20} + Y_{13} Y_{21} + Y_{12} Y_{22} + Y_{11} Y_{23} + Y_{10} Y_{24} \end{aligned}$$

For sake of convenience use following parameter in given model

Parameter	Representation
y_1	Represents those who are susceptible to infection.
y_2	Represents people who have been infected .
y_3	Represents people who have recovered from the Dengue virus.
η	Rate of change of infectives to immune population.
ρ	Rate of change of susceptibles to infective population.

We utilise the following numerical values for parameters from [1] to derive the approximate series solution of the above model.

Let $y_1(0) = 20, y_2(0) = 15$ and $y_3(0) = 10$. Now by using equation 2.6, the initial conditions can be transformed as $Y_{10}(0) = 20, Y_{20}(0) = 15$ and $Y_{30}(0) = 10$. If we taking the values of $\theta_1 = \frac{1}{\alpha}, \theta_2 = \frac{1}{\beta}, \theta_3 = \frac{1}{\gamma}$ and values of other parameters are $\eta = 0.02, \rho = 0.01$,. We have Using the equation 2.1.2 the foruth approximations are calculated for $y_1(\xi), y_2(\xi)$ and $y_3(\xi)$, respectively.
In view of these values, if

$$\theta_1 = \frac{1}{\alpha}, \theta_2 = \frac{1}{\beta}, \theta_3 = \frac{1}{\gamma} \text{ then}$$

$$y_1(\xi) = 20 - \frac{\rho ab}{\Gamma(\alpha+1)}\xi^{\alpha} - \frac{\rho\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(-\frac{\rho ab^2}{\Gamma(\alpha+1)} + \frac{a\rho(ab-\eta b)}{\Gamma(\beta+1)}\right)\xi^{2\alpha} - \frac{\rho\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left[\begin{array}{r}\left(\frac{-\rho\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(\frac{-\rho ab^2}{\Gamma(\alpha+1)} + \frac{a(ab-\eta b)}{\Gamma(\beta+1)}\right)\right)b + \frac{-\rho ab(ab-\eta b)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \\ + \frac{a\Gamma(\beta+1)}{\Gamma(2\beta+1)}\left(-\frac{\rho^2 ab^2}{\Gamma(\alpha+1)} + (ab-\eta b)\frac{(\rho a-\eta)}{\Gamma(\beta+1)}\right)\end{array}\right]\xi^{3\alpha} + \cdots$$

$$y_2(\xi) = 15 + \frac{\rho ab - \eta b}{\Gamma(\beta+1)}\xi^{\beta} + \left(\frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)}\left(-\frac{\rho^2 ab^2}{\Gamma(\alpha+1)} + \frac{(\rho ab-\eta b)(\rho a-\eta)}{\Gamma(\beta+1)}\right)\right)\xi^{2\beta} + \frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)}\left[\begin{array}{r}\frac{-\rho^2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(\frac{-\rho ab^2}{\Gamma(\alpha+1)} + \frac{a(\rho ab-\eta b)}{\Gamma(\beta+1)}\right)b + \rho\left(\frac{-\rho ab(ab-\eta b)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right) \\ + \left(\frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)}\left(-\frac{\rho ab^2}{\Gamma(\alpha+1)} + \rho(\rho ab-\eta b)\frac{(\rho a-\eta)}{\Gamma(\beta+1)}\right)\right)(a-\eta)\end{array}\right]\xi^{3\beta} + \cdots$$

$$y_3(\xi) = 10 + \frac{\eta b}{\Gamma(\gamma+1)}\xi^{\gamma} + \frac{\Gamma(1+\gamma)}{\Gamma(2\gamma+1)}\frac{\eta(\rho ab-\eta b)}{\Gamma(\beta+1)}\xi^{2\gamma} + \frac{\eta\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}\frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)}\left[\begin{array}{r}\frac{(\rho ab-\eta b)(\rho a-\eta)}{\Gamma(\beta+1)} \\ -\frac{\rho^2 ab^2}{\Gamma(\alpha+1)} \\ +\cdots\end{array}\right]\xi^{3\gamma} + \cdots$$

Take $\alpha = \beta = \gamma = 1$, then get

$$\begin{aligned} y_1(\xi) &= 20 - 3\xi - 0.045\xi^2 + 0.02805\xi^3 + \cdots \\ y_2(\xi) &= 15 + 2.7\xi + 0.018\xi^2 - 0.02817\xi^3 + \cdots \\ y_3(\xi) &= 10 + 0.3\xi + 0.027\xi^2 + 0.00012\xi^3 + \cdots \end{aligned} \tag{4.3}$$

These findings (4.3) are shown in Fig. 1, where an increase in the number of infected individuals followed by a decrease in the number of susceptible persons throughout the epidemic is demonstrated by the graphs. During this period, an increase in the number of immunized individuals compared to the findings of the immune population obtained using HPM [5] is observed. A comparison between the findings obtained via FDTAM and those obtained through HPM [5] demonstrates that the outcomes of the fourth-term approximations of FDTAM and HPM are the same. Therefore, it is concluded that this approach will work well in the epidemic scenario.

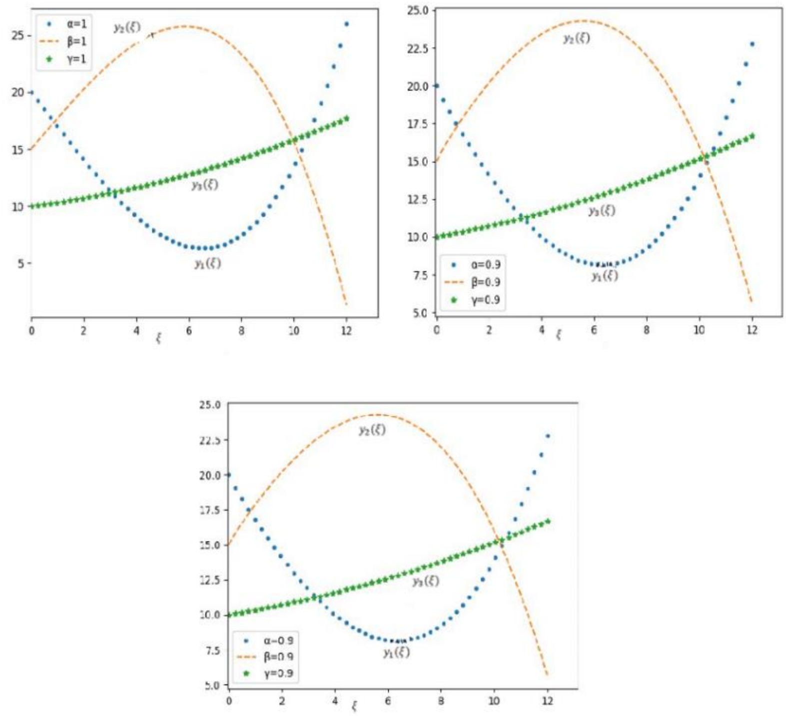

Fig. 1. Fourth approximate solution of 4.1 for different value of α, β, γ.

Table 1. Comparison of fourth-order HPM solution with fourth-order FDTAM solution $y_1(\xi)$ for $\alpha = 1$

ξ	Approximate by HPM	Approximate by FDTAM	Absolute Error
2	14.0444	14.0444	0
4	9.0752	9.0752	0
6	6.4388	6.4388	0
8	7.4816	7.4816	0
10	13.55	13.55	0
12	25.9904	25.9904	0

Table 2. Comparison of fourth-order HPM solution with fourth-order FDTAM solution $y_2(\xi)$ for $\alpha = 1$.

ξ	Approximate by HPM	Approximate by FDTAM	Absolute Error
2	20.24664	20.24664	0
4	24.28512	24.28512	0
6	25.76328	25.76328	0
8	23.32896	23.32896	0
10	15.63	15.63	0
12	1.31424	1.31424	0

Table 3. Comparison of fourth-order HPM solution with fourth-order FDTAM solution $y_3(\xi)$ for $\alpha = 1$.

ξ	Approximate by HPM	Approximate by FDTAM	Absolute Error
2	10.70896	10.70896	0
4	11.63968	11.63968	0
6	12.79792	12.79792	0
8	14.18944	14.18944	0
10	15.82	15.82	0
12	17.69536	17.69536	0

5. Conclusion

In this study, the nonlinear system of fractional differential equations governing the epidemic model was solved using the fractional differential transform method (FDTM) with Adomian polynomials. We examined the recommended strategy for convergence analysis and absolute error, utilizing this novel approach to approximate solutions to the model of nonlinear equations. It is important to note that the technique can reduce the amount of computing effort required while maintaining high accuracy, resulting in an improvement in the approach's performance. A comparison between the findings obtained via FDTAM and those obtained through a precise method reveals that the outcomes of the fourth-term approximations of FDTAM and the precise method are consistent, thereby supporting the theoretical conclusions and the efficacy of the numerical approximation. Moreover, our analysis was discussed graphically using Python software.

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