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Prime and Maximal Spectra of Almost Distributive Lattices and 0 - Distributive Lattices

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Abstract

The present research work attempts to bring together the two different algebraic structures namely almost distributive lattice (ADL) and 0-distributive lattice with the help of a special mapping which leads us to significant results. A surjection from an ADL with the least element and maximal element to a bounded 0-distributive lattice satisfying certain conditions is defined and studied. It is proved that this mapping induces a homeomorphism between their prime and maximal spectra.

Keywords: Almost distributive lattice (ADL); 0-Distributive lattice; Prime filter; Maximal filter; Prime spectrum; Maximal spectrum.

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1. Introduction

An algebraic structure Almost Distributive Lattice (ADL) is introduced by Swamy and Rao [1]. This notion considers the class of distributive lattices in a broad sense. The concepts of ideal and filter introduced in an ADL are similar to those in a distributive lattice. This led many researchers to extend many existing notions from the theory of distributive lattices to ADLs. Ideals and filters play an important role in lattices and ADLs. Due to duality principle in lattice theory, the filters (dual ideals) did not gain much importance in lattice theory. However, in ADLs, the duality principle does not hold good. So, Rao and Ravikumar [2] have specifically studied properties of prime filters in a normal ADL and gave characterizations of normal ADLs in terms of their prime filters. Further, the hull kernel topology on the set of prime (maximal) filters of an ADL (with 0 and maximal elements) has been introduced by Swamy *et al*. [3] and used to characterize normal ADLs in terms of their prime (maximal) filters. Recently, Babu *et al*. [4] and Rafi *et al.* [5] have made contribution to the theory of ideals and filters in ADL. 0 - distributive lattices are

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introduced by Varlet [6], as an extension of distributive lattices with 0 on one side and the pseudo- complemented lattices on the other. They have been studied in detail by Varlet [6], Jayaram [7], Pawar *et al.* [8-11], Balsubramani and Venkatanarasimhan [12,13] and Razia *et al.* [14]. Balsubramani [13] has discussed in detail the hull kernel topology of the collection of prime (maximal) filters of a bounded 0- distributive lattice.

2. Preliminaries

Some necessary definitions for an almost distributive lattice and a 0-distributive lattice are collected in this section. For the basic concepts in almost distributive lattice theory, this work follows Swamy and Rao [1]. For the important details in lattice theory, reference is made to Grätzer [15].

Definition 2.1 [1] "An almost distributive lattice (ADL in short) with 0 is an algebra $A = (A, \wedge, \vee, 0)$ of the type (2,2,0) satisfying the following conditions for all $x, y, z \in A$

1. $x \vee 0 = x$
2. $0 \wedge x = 0$
3. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
4. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
5. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
6. $(x \vee y) \wedge y = y$ "

Definition 2.2 [1] "For any $a, b \in A$, define $a \leq b$ if $a \wedge b = a$ (or equivalently $a \vee b = b$). The relation $\leq$ is a partial ordering on A."

Definition 2.3 [1] "An element $m \in A$ is called maximal if it is a maximal element in the poset $(A, \leq)$."

Definition 2.4 [1] "A non-empty subset F of A is said to be a filter in A if i) $a, b \in F \Rightarrow a \wedge b \in F$ and ii) $a \in F, x \in A \Rightarrow x \vee a \in F$."

Let $\mathfrak{F}(A)$ denotes the set of all filters of A. A filter F of A is proper if $F \neq A$.

Definition 2.5 [1] "A proper filter P of A is prime if for any $x, y \in A$, $x \vee y \in P$ implies $x \in P$ or $y \in P$."

Let $\mathfrak{P}(A)$ denotes the set of all prime filters of A.

Definition 2.6 [1] "A proper filter M of A is maximal if it is not properly contained in any proper filter of A."

Let $\mathfrak{M}(A)$ denotes the set of all maximal filters of A.

Let $L = (L, \sqcup, \sqcap)$ be a lattice with 0. For the definitions of ideal, filter, prime filter, maximal filter of L the reader is referred to Grätzer [15]. Let us denote the set of all filters of L by $\mathfrak{F}(L)$, the set of all prime filters of L by $\mathfrak{P}(L)$ and the set of all maximal filters of L by $\mathfrak{M}(L)$.

Definition 2.7 [6] "A lattice L with 0 in which for $a, b, c \in L$, $a \sqcap b = 0$ and $a \sqcap c = 0$ imply $a \sqcap (b \sqcup c) = 0$ is called a 0 – distributive lattice."

3. Prime and Maximal Spectra

Unless otherwise specified, now onwards an ADL ($A, \vee, \wedge, 0$) with the least element 0 and maximal elements is denoted simply by A and a bounded lattice ($L, \sqcup, \sqcap, 0, 1$), with bounds 0 and 1 is denoted by L. Let $\mu: A \rightarrow L$, be a function satisfying the following conditions:

(C1). $\mu(a \vee b) = \mu(a) \sqcup \mu(b)$ for all $a, b \in A$.

(C2). $\mu(a \wedge b) = \mu(a) \sqcap \mu(b)$ for all $a, b \in A$.

(C3). $\mu(0) = 0$ and $\mu(m) = 1$ for all maximal elements m in A.

(C4). μ is surjective.

(C5). $\mu(a) \leq \mu(b) \Rightarrow a \leq b$.

Example 3.1. Let $A = \{0, u, v, w\}$. Define $\vee$ and $\wedge$ on A as follows:

$\vee$	0	u	v	w
0	0	u	v	w
u	u	u	u	u
v	v	u	v	u
w	w	u	u	w

$\wedge$	0	u	v	w
0	0	0	0	0
u	0	u	u	u
v	0	v	v	0
w	0	w	0	w

Then ($A, \vee, \wedge$) is an ADL with the least element 0 and maximal element u.
Consider $L = (L, \sqcup, \sqcap, 0, 1)$, a bounded lattice with bounds 0 and 1 where $L = \{0, a, b, 1\}$ and $\sqcup, \sqcap$ are defined as follows:

$\sqcup$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$\sqcap$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Define $\mu: A \rightarrow L$ by $\mu(0) = 0$, $\mu(u) = 1$, $\mu(v) = a$ and $\mu(w) = b$. Then the function μ satisfies the conditions (C1) to (C5)
The work begins with the following theorem.

Theorem 3.2. The following properties hold, for all a, b in A which are not maximal in A.

(i). If $\mu(a) = \mu(b)$, then $a \in F$ if and only if $b \in F$.

(ii). $\mu(a) = \mu(b)$ if and only *if* $[a) = [b)$.

(iii). For any $F \in \mathfrak{F}(A)$, $\mu(a) \in \mu(F)$ *if and only if* $a \in F$.

(iv). For any $G \in \mathfrak{F}(A)$, $\mu(G) \in \mathfrak{F}(L)$.

(v). For any $F \in \mathfrak{F}(L)$, $\mu^{-1}(F) \in \mathfrak{F}(A)$.

(vi). For any $Q \in \mathfrak{P}(A)$, $\mu(Q) \in \mathfrak{P}(L)$.

(vii). For any $P \in \mathfrak{P}(L)$, $\mu^{-1}(P) \in \mathfrak{P}(A)$.

Proof. There are straight forward proofs for the statements (i)-(v).

(vi) Let $Q \in \mathfrak{P}(A)$. Then by (iv), $\mu(Q) \in \mathfrak{F}(L)$. As Q is a proper filter of A, there exists $a \in A$ such that $a \notin Q$. By (iii), $\mu(a) \notin \mu(Q)$. This shows that $\mu(Q)$ is a proper filter of L. Let $x, y \in \mu(Q)$ such that $x \sqcup y \in \mu(Q)$. μ being surjective, there exist $a, b \in Q$ such that $x = \mu(a)$ and $y = \mu(b)$. Thus $\mu(a \vee b) = \mu(a) \sqcup \mu(b) \in \mu(Q)$. Therefore $a \vee b \in Q$ (by (iii)). Q is a prime filter of A implies $a \in Q$ or $b \in Q$ which in turn imply that $x = \mu(a) \in \mu(Q)$ or $y = \mu(b) \in \mu(Q)$. Hence $\mu(Q) \in \mathfrak{P}(L)$.

(vii) Let $P \in \mathfrak{P}(L)$. By (v), $\mu^{-1}(P)$ is a filter of A. Since P is a proper filter of L, there exists an element $x \in L$ such that $x \notin P$. As μ is onto, for $x \in L$ there exists $a \in A$ such that $\mu(a) = x$. But then $x = \mu(a) \notin P$ implies $a \notin \mu^{-1}(P)$. Therefore $\mu^{-1}(P)$ is a proper filter of A. Let $a, b \in A$ so that $a \vee b \in \mu^{-1}(P)$. But then $\mu(a \vee b) \in P$ which implies $\mu(a) \sqcup \mu(b) \in P$. Since P is a prime filter of L it follows that $\mu(a) \in P$ or $\mu(b) \in P$. This is equivalent to $a \in \mu^{-1}(P)$ or $b \in \mu^{-1}(P)$. Hence $\mu^{-1}(P)$ is a prime filter of A.

The following consequence is derived from Theorem 3.2.

Corollary 3.3. Following statements hold in $\mathfrak{F}(A)$ and $\mathfrak{F}(L)$.

(a) For any $F \in \mathfrak{F}(A)$, $\mu^{-1}(\mu(F)) = F$.

(b) For any $H \in \mathfrak{F}(L)$, $\mu(\mu^{-1}(H)) = H$.

Consider the subset B of A which contains all the non-maximal elements of A and only one maximal element of A say m. Then $B = (B, \vee, \wedge, 0, m)$ is a sub ADL of the ADL A, where m is any fixed maximal element of A. Define a function $\gamma : B \to L$ as a restriction of the function $\mu : A \to L$ to the subset B of A.

Equip the set $\mathfrak{P}(L)$ of all prime filters of a bounded 0 - distributive lattice L with the topology τ for which $\{X(x) : x \in L\}$ is a base, where $X(x) = \{P \in \mathfrak{P}(L) : x \notin P\}$. The topological space ($\mathfrak{P}(L)$, τ) is called the prime spectrum of L (see [13]). The set $\mathfrak{P}(B)$ of all prime filters of the sub ADL $B = (B, \vee, \wedge, 0, m)$ together with the topology τ' for which $\{H(x) : x \in B\}$ is a base, where $H(x) = \{P \in \mathfrak{P}(B) : x \notin P\}$, is called the prime spectrum of B (see [3]). Now it is proved that the two spaces $\mathfrak{P}(L)$ and $\mathfrak{P}(B)$ are homeomorphic.

Theorem 3.4. Let L be a bounded 0 - distributive lattice and $B = (B, \vee, \wedge, 0, m)$ be a sub ADL of A where m is a fixed maximal element of A. Then the function $\gamma : B \to L$ which is a restriction of the function $\mu : A \to L$ (to the subset B) induces a homeomorphism between the prime spectrum of L and the prime spectrum of B.

Proof. Define the mapping γ^*: $\mathfrak{P}(L) \to \mathfrak{P}(B)$ by $\gamma^*(P) = \gamma^{-1}(P)$, for every $P \in \mathfrak{P}(L)$. Claim: γ^* is a homeomorphism.

(i) Let $Q \in \mathfrak{P}(B)$. Then part (vi) of Theorem 3.2 gives $\gamma(Q) \in \mathfrak{P}(L)$, and hence $\gamma^*(\gamma(Q)) = \gamma^{-1}(\gamma(Q)) = Q$. This proves that γ^* is surjective.

(ii) Let $P_1 \neq P_2$ in $\mathfrak{P}(L)$. Select $a \in P_1$ such that $a \notin P_2$. As γ: $B \to L$ is surjective, there exists $b \in B$ such that $\gamma(b) = a$. Thus $\gamma(b) \in P_1$ but $\gamma(b) \notin P_2$ which means $b \in \gamma^{-1}(P_1)$ but $b \notin \gamma^{-1}(P_2)$. Hence $\gamma^*(P_1) \neq \gamma^*(P_2)$. This proves that γ^*: $\mathfrak{P}(L) \to \mathfrak{P}(B)$ is injective.

(iii) Let $a \in A$. Then $(\gamma^*)^{-1}(H(a)) = \{P \in \mathfrak{P}(L) : \gamma^*(P) \in H(a)\} = \{P \in \mathfrak{P}(L) : \gamma^{-1}(P) \in H(a)\} = \{P \in \mathfrak{P}(L) : a \notin \gamma^{-1}(P)\} = \{P \in \mathfrak{P}(L) : \gamma(a) \notin P\} = X(\gamma(a))$. This proves that

γ^* is continuous.

(iv) Let $t \in L$. Since γ is surjective, there exists $a \in B$ so that $\gamma(a) = t$. Now $\gamma^* (X(t)) = \gamma^* (X(\gamma(a))) = \{\gamma^* (P) : P \in X(\gamma(a))\} = \{\gamma^* (P) : P \in \mathfrak{P}(L), \gamma(a) \notin P\} = \{\gamma^*(P) : P \in \mathfrak{P}(L), a \notin \gamma^{-1}(P)\} = \{\gamma^*(P) : P \in \mathfrak{P}(L), a \notin \gamma^* (P)\} = \{Q \in \mathfrak{P}(B) : a \notin Q\} = H(a)$ (this is possible because γ^* is surjective). This shows that γ^* is open.

From parts (i), (ii), (iii) and (iv) it follows that γ^* is a homeomorphism.

Following lemmas are crucial for next result.

Lemma 3.5. Let L be a bounded 0 - distributive lattice. A proper filter M of L is a maximal filter of L if and only if $\gamma^{-1}(M)$ is a maximal filter of B.

Proof. Let M be a maximal filter of L. Then M is a proper filter of L. Part (v) of Theorem 3.2 implies $\gamma^{-1}(M) \in \mathfrak{F}(B)$. Select $t \in L$ such that $t \notin M$. Then γ being a surjection, there exists $b \in B$ such that $\gamma(b) = t$. Then $\gamma(b) = t \notin M$ implies $b \notin \gamma^{-1}(M)$, showing that $\gamma^{-1}(M)$ is a proper filter of B. By part (a) of the corollary 3.3, any proper filter of B is of the form $\gamma^{-1}(F)$, where F is a proper filter of L. Let $\gamma^{-1}(F)$ be a proper filter of B such that $\gamma^{-1}(M) \subseteq \gamma^{-1}(F)$. Then $\gamma (\gamma^{-1}(M)) \subseteq \gamma (\gamma^{-1}(F))$ i.e. $M \subseteq F$. Maximality of M yields $M = F$. But then $\gamma^{-1}(M) = \gamma^{-1}(F)$. This shows that $\gamma^{-1}(M)$ is maximal filter of B. Conversely suppose $\gamma^{-1}(M)$ is maximal filter of B. Then $\gamma^{-1}(M) \in \mathfrak{P}(B)$ which implies $M = \gamma (\gamma^{-1}(M)) \in \mathfrak{P}(L)$ and hence M is a proper filter of L. Now, let F be a proper filter of L such that $M \subseteq F$. But then $\gamma^{-1}(M) \subseteq \gamma^{-1}(F)$. As $\gamma^{-1}(M)$ is a maximal filter of B, $\gamma^{-1}(M) = \gamma^{-1}(F)$. Therefore $\gamma(\gamma^{-1}(M)) = \gamma (\gamma^{-1}(F))$ i.e. $M = F$. This proves that M is a maximal filter of L.

The following lemma can be proved in a similar way.

Lemma 3.6. Let L be a bounded 0 - distributive lattice. A proper filter K of B is a maximal filter of B if and only if $\gamma(K)$ is a maximal filter of L.

It is known that any maximal filter in an ADL with maximal elements is a prime filter and hence $\mathfrak{M}(B) \subseteq \mathfrak{P}(B)$. Similarly, any maximal filter in a bounded 0 - distributive lattice L is a prime filter which means $\mathfrak{M}(L) \subseteq \mathfrak{P}(L)$. Hence let us restrict the topology τ defined on $\mathfrak{P}(L)$ to $\mathfrak{M}(L)$ and the topology τ_1 defined on $\mathfrak{P}(B)$ to $\mathfrak{M}(B)$ and denote them by τ^* and τ_1^* respectively. Denote the basic open set for τ^* by $V(a)$, where for $a \in L$, $V(a) = \{M \in \mathfrak{M}(L) : a \notin M\}$ and the basic open set for τ_1^* by $Y(a)$, where for $a \in B$, $Y(a) = \{M \in \mathfrak{M}(L) : a \notin M\}$. For the topological spaces $(\mathfrak{M}(L), \tau^*)$ and $(\mathfrak{M}(B), \tau_1^*)$ the following interesting property is proved.

Theorem 3.7. Let L be a bounded 0 - distributive lattice. Define a mapping $\Gamma: \mathfrak{M}(L) \to \mathfrak{M}(B)$ by $\Gamma(M) = \gamma^{-1}(M)$, for all $M \in \mathfrak{M}(L)$. Then Γ is a homeomorphism.

Proof.

(i). Let $M \in \mathfrak{M}(B)$. Then by Lemma 3.5, $\gamma(M) \in \mathfrak{M}(L)$. According to part (a) of Corollary 3.2, $\Gamma(\gamma(M)) = \gamma^{-1}(\gamma(M)) = M$. This proves that Γ is a surjection.

(ii). Let $M_1 \neq M_2$ in $\mathfrak{M}(L)$. Hence there exists $a \in M_1$ such that $a \notin M_2$. As γ is an onto map, there exists $b \in B$ such that $\gamma(b) = a$. Thus $\gamma(b) \in M_1$ and $\gamma(b) \notin M_2$ i.e. $b \in \gamma^{-1}(M_1)$ and $b \notin \gamma^{-1}(M_2)$. Hence $\gamma^{-1}(M_1) \neq \gamma^{-1}(M_2)$ i. e. $\Gamma(M_1) \neq \Gamma(M_2)$. This proves that Γ is injective.

(iii). Let $a \in B$. Then

$\Gamma^{-1}(Y(a)) = \{M \in \mathfrak{M}(L) : \Gamma(M) \in Y(a)\}$

$= \{M \in \mathfrak{M}(L) : \gamma^{-1}(M) \in Y(a)\}$
$= \{M \in \mathfrak{M}(L) : a \notin \gamma^{-1}(M)\}$
$= \{M \in \mathfrak{M}(L) : \gamma(a) \notin M\}$
$= V(\gamma(a))$

This shows that Γ is continuous.

(iv). Let $t \in L$. Since γ is surjective, there exists $a \in B$ such that $\gamma(a) = t$.

$\Gamma(V(t)) = \Gamma(V(\gamma(a)))$
$= \{\Gamma(M) : M \in V(\gamma(a))\}$
$= \{\Gamma(M) : M \in \mathfrak{M}(L), \gamma(a) \notin M\}$
$= \{\Gamma(M) : M \in \mathfrak{M}(L), a \notin \gamma^{-1}(M)\}$
$= \{\Gamma(M) : M \in \mathfrak{M}(L), a \notin \Gamma(M)\}$
$= \{Q \in \mathfrak{M}(B) : a \notin Q\}$ … (since Γ is onto)
$= Y(a)$

This shows that Γ is open mapping.

From parts (i) to (iv), Γ is a homeomorphism.

4. Conclusion

The present work shows that it is possible to study two different algebraic structures, ADL and 0 – distributive lattice, together with the help of a suitable mapping. The result that, the prime (maximal) spectrum of an ADL is homeomorphic with the prime (maximal) spectrum of a bounded 0 - distributive lattice, is significant. In the light of this work, similar study can be carried out for spectrum of prime α - ideals in these two algebraic structures.

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