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Generalized Derivations Acting as Homomorphisms and Anti-Homomorphisms of Gamma Rings

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Abstract

Let M be a prime Γ-ring and let I be a nonzero ideal of M. Suppose that D: $M \rightarrow M$ is a nonzero generalized derivation with associated derivation $d : M \rightarrow M$. Then we prove the following:

(i) If D acts as a homomorphism on I, then either $d = 0$ on M or M is commutative.
(ii) If M satisfies the assumption (*) (see below), and if D acts as an anti-homomorphism on I, then either $d = 0$ on M or M is commutative.

Keywords: Prime Γ-rings; Generalized derivations; Torsion free Γ-rings; Homomorphisms; Anti-homomorphisms.

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1. Introduction

In classical ring theory, Bell and Kappe [1] proved that if d is a derivation of a semiprime ring R which is either an endomorphism or an anti-endomorphism on R, then $d = 0$: whereas, the behavior of d is somewhat restricted in case of prime rings in the way that if d is a derivation of a prime ring R acting as a homomorphism or anti-homomorphism on a nonzero right ideal of R, then $d = 0$ on R.

Afterwards Yenigul and Argac [2] generalized these results with α-derivations and M. Ashraf, Rehman and Quadri [3] obtained the similar results with (σ,τ)-derivations. Analogously Rehman [4] extended the results for generalized derivation acting on nonzero ideals in case of prime rings. Recently Ali and Kumar [5] established the above mentioned result for generalized (θ,φ)-derivations in prime rings. By the same motivation, we extend the results in [4] of classical ring theory to the Γ-ring theory in the case of generalized derivation acts as a homomorphism and an anti-homomorphism of prime Γ-rings.

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Definition 1.1 [6]

Let M and Γ be additive abelian groups. M is called a Γ-ring if for all $x,y,z\in M$, $\alpha,\beta\in\Gamma$ the following conditions are satisfied:

(i) $x\beta y\in M$,

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$,

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

Definition 1.2

A Γ-ring M is called prime if for any $a,b\in M$, $a\Gamma M\Gamma b = 0$ implies that either $a = 0$ or $b = 0$.

Definition 1.3

An additive mapping $d : M \to M$ is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ holds for all $x,y\in M$, $\alpha\in\Gamma$.

For a fixed $a\in M$, $\alpha\in\Gamma$, the mapping $I^{\alpha}_{a} : M \to M$ given by $I^{\alpha}_{a}(x) = [x, a]_{\alpha}$ is a derivation which is said to be inner derivation. An additive function $D^{\alpha}_{a,b} : M \to M$ is called generalized inner derivation if $D^{\alpha}_{a,b}(x) = a\alpha x + x\alpha b$ for some fixed $a,b\in M$, $\alpha\in\Gamma$. It is straight forward to note that if $D^{\alpha}_{a,b}$ is a generalized inner derivation, then for any $x,y\in M$, $\alpha\in\Gamma$, $D^{\alpha}_{a,b}(x\alpha y) = D^{\alpha}_{a,b}(x)\alpha y + x\alpha I^{\alpha}_{b}(y)$ where I^{α}_{b} is an inner derivation. In view of the above observation, the concept of generalized derivation is introduced as follows:

Definition 1.4

An additive mapping $D : M \to M$ is called a generalized derivation associated with a derivation d if $D(x\alpha y) = D(x)\alpha y + x\alpha d(y)$ for all $x,y\in M$, $\alpha\in\Gamma$.

Definition 1.5

The commutator $x\alpha y - y\alpha x$ will be denoted by $[x, y]_{\alpha}$. We know that $[x\beta y, z]_{\alpha} = [x, z]_{\alpha}\beta y + x\beta[y, z]_{\alpha} + x[\beta,\alpha]_{z}y$ and $[x, y\beta z]_{\alpha} = y\beta[x, z]_{\alpha} + [x, y]_{\alpha}\beta z + y[\beta,\alpha]_{x}z$.

We take an assumption (*) $x\beta z\alpha y = x\alpha z\beta y$ for all $x,y,z \in M$ and $\alpha,\beta\in\Gamma$. Using the assumption the basic commutator identities reduces to $[x\beta y, z]_{\alpha} = [x, z]_{\alpha}\beta y + x\beta[y, z]_{\alpha}$ and $[x, y\beta z]_{\alpha} = y\beta[x, z]_{\alpha} + [x, y]_{\alpha}\beta z$.

Definition 1.6

Let M be a Γ-ring. An additive mapping ϕ on M is called a homomorphism if $\phi(x, y) = \phi(x)\alpha\phi(y)$, for every $x,y\in M$ and $\alpha\in\Gamma$.

Definition 1.7

A Γ-ring M is commutative if $x\alpha y = y\alpha x$ for all $x,y\in M$ and $\alpha\in\Gamma$.

It is clear that a Γ-ring M is commutative if and only if $[x, y]_{\alpha} = 0$ for every $x,y\in M$ and $\alpha\in\Gamma$.

Definition 1.8

Let S be a nonempty subset of M and D be a generalized derivation on M with associated derivation d. A generalized derivation D of M is said to act as a homomorphism on S if $D(x\alpha y) = D(x)\alpha y + x\alpha d(y) = D(x)\alpha D(y)$ for all $x,y\in S$ and $\alpha\in\Gamma$.

Definition 1.9

Let S be a nonempty subset of M and D be a generalized derivation on M with associated derivation d. A generalized derivation D of M is said to act as an anti-homomorphism on S if $D(x\alpha y) = D(x)\alpha y + x\alpha d(y) = D(y)\alpha D(x)$ for all $x,y\in S$ and $\alpha\in\Gamma$.

2. Results

Lemma 2.1

If d is a nonzero derivation of a prime Γ-ring M, then the left and right annihilators of $d(x) = 0$, $x\in M$. In particular $a\beta[b, x]_\alpha = 0$ or $[b, x]_\alpha\beta a = 0$ implies that $I_b(x) = 0$, ($b\in Z$, $\alpha,\beta\in\Gamma$) or $a = 0$.

Proof

In $a\alpha d(x) = 0$ for all $x\in M$, $\alpha\in\Gamma$, replace x by $x\beta y$. Then we get $a\alpha d(x\beta y) = 0 = a\alpha d(x)\beta y + a\alpha x\beta d(y) = a\alpha x\beta d(y)$, for all $x,y\in M$, $\alpha,\beta\in\Gamma$. If $d \neq 0$, that is $d(y) \neq 0$ for some y$\in M$, then, by the primeness of M we get $a = 0$.

Lemma 2.2

Let I be a nonzero right ideal in a prime Γ-ring M.

(a) If M has a derivation d which is zero on I, then d is zero on M.

(b) If M has homomorphism T which is the identity on I, then T is the identity on M.

Proof

(a) If $d(x) = 0$, $x\in I$, then $0 = d(x\alpha r) = d(x)\alpha r + x\alpha d(r) = x\alpha d(r)$, for all $x\in I$, $r\in M$, $\alpha\in\Gamma$. By lemma 2.1, d must be zero since I is nonzero.

(b) Let $x\in I$ and $a,b\in M$. Then $x\alpha a\beta b = T(x\alpha a\beta b) = T(x\alpha a)\beta T(b) = x\alpha a\beta T(b)$. Thus $x\alpha a\beta(b - T(b)) = 0$ and either $x = 0$ or $b - T(b) = 0$. But I is nonzero and so contains an $x \neq 0$. This forces $T(b) = b$ for all $b\in M$.

Lemma 2.3 If a prime Γ-ring M contains a nonzero commutative right ideal I, then M is commutative.

Proof

If $x\in I$, then $I_x(y) = [x, y]_\alpha = 0$, for all $y\in I$, $\alpha\in\Gamma$, since I is commutative. By lemma 2.2(a), $I_x(y) = 0$ on M and x is in the center. Thus $[x, r]_\alpha = 0$ for every $x\in I$, $r\in M$, $\alpha\in\Gamma$. Hence $I_a(x) = 0$ for $a\in M$ and again by lemma 2.2(a), $I_a(r) = 0$ and a is in the center for all $a\in M$. Therefore M is commutative.

Theorem 2.4. Let M be a prime Γ-ring and I be a nonzero ideal of M. Suppose $D : M \to M$ is a nonzero generalized derivation with associated derivation d. If D acts as a homomorphism on I and if $d \neq 0$ on I, then M is commutative.

Proof

If D acts as a homomorphism on I, then we have

$D(x\alpha y) = D(x)\alpha y + x\alpha d(y) = D(x)\alpha D(y)$ for all $x,y\in I$, $\alpha\in\Gamma$. (1)

For any $x,y,z\in I$, we find that

$D(x\alpha y\beta z) = D(x\alpha y)\beta z + x\alpha y\beta d(z) = D(x)\alpha D(y)\beta z + x\alpha y\beta d(z)$ for all $x,y,z\in I$, $\alpha,\beta\in\Gamma$ (2)

On the other hand,

$D(x\alpha y\beta z) = D(x)\alpha D(y\beta z) = D(x)\alpha D(y)\beta z + D(x)\alpha y\beta d(z)$ for all $x,y,z\in I$, $\alpha,\beta\in\Gamma$. (3)

On comparing (2) and (3), we get $(D(x) - x)\alpha y\beta d(z) = 0$ for all $x,y,z\in I$, $\alpha,\beta\in\Gamma$. Thus, primeness of M forces that either $(D(x) - x) = 0$ or $d(z) = 0$. If $d(z) = 0$ for all $z\in I$, then $d = 0$, a contradiction. On the other hand if $D(x) = x$ for all $x\in I$, then

$x\alpha y = D(x\alpha y) = D(x)\alpha y + x\alpha d(y)$ for all $x,y\in I$, $\alpha\in\Gamma$

and hence we find that $x\alpha d(y) = 0$.

Replace x by $x\beta z$ in $x\alpha d(y) = 0$,

we get $x\beta z\alpha d(y) = 0$, for all $x,y,z\in I$, $\alpha,\beta\in\Gamma$, (4)

Similarly, replacing x by $z\beta x$, we get

$z\beta x\alpha d(y) = 0$, all $x,y,z\in I$, $\alpha,\beta\in\Gamma$, (5)

Subtracting (5) from (4), we get, $[x, z]_\beta\alpha d(y) = 0$, all $x,y,z\in I$, $\alpha,\beta\in\Gamma$.

Replacing y by $y\delta r$, $r\in I$, we get for all $x,y,z,r\in I$, $\alpha,\beta,\delta\in\Gamma$,

$[x, z]_\beta\alpha d(y\delta r) = [x, z]_\beta\alpha d(y)\delta r + [x, z]_\beta\alpha y\delta d(r) = [x, z]_\beta\alpha y\delta d(r) = 0$,

Since $d(r) \neq 0$ on I, we get $[x, z]_\beta = 0$, for all $x,z\in I$, $\beta\in\Gamma$ by the primeness of M. By lemma 2.3, M is commutative.

Theorem 2.5. Let M be a prime Γ-ring satisfying the condition (*) and I be a nonzero ideal of M. Suppose $D : M \to M$ is a nonzero generalized derivation with associated derivation d. If D acts as an anti-homomorphism on I and if $d \neq 0$ on I, then M is commutative.

Proof

If D acts as an anti-homomorphism

$D(x\alpha y) = D(x)\alpha y + x\alpha d(y) = D(y)\alpha D(x)$ for all $x,y\in I$, $\alpha\in\Gamma$. (6)

Replacing x by $x\beta y$ in (6) and (*), we get

$$x\alpha y\beta d(y) = D(y)\alpha x\beta d(y), \text{ for all } x,y\in I,\ \alpha,\beta\in\Gamma. \quad (7)$$

Now, replace x by $z\delta x$ in (7), to get

$$z\delta x\alpha y\beta d(y) = D(y)\alpha z\delta x\beta d(y), \text{ for all } x,y,z\in I,\ \alpha,\beta,\delta\in\Gamma \quad (8)$$

Left multiplying (7) by z, we obtain

$$z\delta x\alpha y\beta d(y) = z\delta D(y)\alpha x\beta d(y), \text{ for all } x,y,z\in I,\ \alpha,\beta,\delta\in\Gamma \quad (9)$$

Comparing (8) and (9), we find that $[D(y), z]_\alpha \alpha x\beta d(y) = 0$, for all $x,y,z\in I$, $\alpha,\beta\in\Gamma$. Replacing x by $x\lambda r$, we get $[D(y), z]_\alpha \delta x\lambda r\beta d(y) = 0$, $x,y,z\in I$, $r\in M$, $\alpha,\beta,\delta,\lambda\in\Gamma$. By the primeness of M either $[D(y), z]_\alpha \delta x = 0$ or $d(y) = 0$. By lemma 2.1,either $[D(y), z]_\alpha = 0$ or $d(y) = 0$ Now, let $A = \{y\in I \mid [D(y), z]_\alpha = 0, \text{ for all } z\in I\}$, $B = \{y\in I \mid d(y) = 0\}$. Thus A and B are additive subgroups of I and $I = A \cup B$. But a group can not be a union of two proper subgroups and hence $I = A$ or $I = B$. If $I = B$ then $d(y) = 0$ for all $y\in I$ and hence $d = 0$, a contradiction. On the other hand, if $I = A$, then $[D(y), z]_\alpha = 0$, for all $y,z\in I$, $\alpha\in\Gamma$. Now, replace y by $y\lambda z$ to get $[y, z]_\alpha \lambda d(z) + y\lambda[d(z), z]_\alpha = 0$. Again replacing y by $x\delta y$ we get $[x, z]_\alpha \delta y\lambda d(z) = 0$ for all $x,y,z\in I$, $\alpha,\delta,\lambda\in\Gamma$. Thus primeness of M implies that for each $z\in I$ either $[x, z]_\alpha = 0$ or $d(z) = 0$. If $d(z) = 0$ for all $z\in I$ then $d = 0$. Now, if $[x, z]_\alpha = 0$ for all $x,z\in I$, $\alpha\in\Gamma$, then by Lemma 2.3 we get the required result. This completes the proof of the theorem.

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