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Boundedness and Continuous Dependence of Solutions for Fractional Difference Equations with Nonlocal Condition

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Abstract

This study investigates the properties of solutions to nonlinear nonlocal fractional difference equations. By applying the Leray-Schauder alternative in combination with Bihari's integral inequality, we establish rigorous results con cerning boundedness and continuous dependence of solutions on initial data. The theoretical findings are further illustrated with examples, demonstrating the practical relevance and applicability of the proposed approach. These results provide a comprehensive framework for analyzing nonlinear fractional discrete systems with nonlocal conditions and offer a foundation for future research in this area.

Keywords: Difference equation; Initial value problem; Local existence and uniqueness solutions; Boundedness; Continuous dependence.

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1. Introduction

Atici *et al.* [1] explored a key problem in discrete fractional calculus of variations, derive the Euler-Lagrange relation, and solve a Gompertz fractional difference equation modeling tumor growth. Existence and uniqueness results for fractional difference equations, including two-point boundary value problems, are presented in literature [2-4]. Chen *et al.* Chen *et al.* [5] established existence results for nonlinear fractional difference equations using fixed-point techniques. Ibrahim and Jalab [6] investigated fractional difference equations subject to two-point boundary value conditions and established existence results using fixed-point methods. Deshpande *et al.* [7] analyzed chaotic features, showing that tent and Gauss fractional maps possess greater stability than their integer-order counterparts.

The connection between uncertain fractional forward difference solutions and their trajectories is examined [8]. Positive solutions under nonlocal conditions were obtained in [9] using Krasnosel'skii's and Schauder's fixed-point theorems. Discrete fractional Dirichlet-type problems are studied [10]. Anh *et al.* [11] employed the Z-transform to derive variation-of-constants formulas for fractional difference equations of Caputo and Riemann-Liouville type. Jonnalagadda *et al.* [12] discussed existence, uniqueness, boundedness, and

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stability using resolvent kernels. Stability criteria for delayed systems and systems with feedback are provided [13].

He *et al.* [14] investigated nonlinear equations with a Caputo-like operator, establishing existence via topological degree theory. Nonlinear nonlocal boundary value problems at resonance are addressed [15] through coincidence degree theory, coupled with a new property of the Gamma function. Anastassiou [16] developed a Caputo-like fractional difference operator, compares it with the Riemann-Liouville form, and studies discrete inequalities. Abdeljawad [17] introduced left and right Caputo fractional sums and differences, establishing their relationship with Riemann-Liouville operators and proposing discrete Mittag-Leffler functions.

Applications of discrete fractional calculus to physical models such as mass-spring-damper systems are found [18]. Mozyrska [19] analyzed several fractional difference types using L-transform methods. Mohan and his coworkers [20] defined the nabla discrete Sumudu transform and prove properties of Mittag-Leffler eigenfunctions. A unified fractional difference-sum framework is provided in literature [21]. Terminologies are defined elsewhere [4].

Sugiyama [22] investigated the problem of existence and uniqueness of solutions by employing Tychonov's fixed point theorem in conjunction with the successive approximation technique and the comparison approach.

$$\Delta^{\beta}\omega(t) = \Phi\big(\zeta, \omega(\zeta), \omega(\zeta-1)\big)\ (0 \le \zeta \le \zeta_1) \tag{1.1}$$

Governed by the condition

$$\omega(\zeta-1) = \phi(\zeta)\ (0 \le \zeta < \mathrm{b}) \tag{1.2}$$

$$\omega(0) = \omega_0 \tag{1.3}$$

where ω & $\Phi \in X^n$. Stokes [23] studied above problem for nonlinear differential equations. Sugiyama [24] investigated the difference-differential system (1.1)-(1.3) and established results on the existence, stability, and boundedness of its solutions. The approach relied on Tychonov's fixed point theorem, applied together with an extra restriction imposed on Φ. Subsequent contributions from other researchers, employing diverse mathematical frameworks, have examined the problems of existence, uniqueness, and various additional characteristics of solutions to Eq. (1.1) and its specific forms [25-31]. Pachpatte [32] investigated the existence and uniqueness of solutions for the problem Eqs. (1.1)-(1.3) by employing the Leray-Schauder alternative together with Bihari's integral inequality. It is worth noting that, while various authors have examined integrodifferential problems, the specific case involving nonlocal conditions combined with an infinitesimal generator of operators has received comparatively little attention.

Note: It is assumed throughout this work that the set $J = \{\zeta_{0,}\zeta_0 + 1, \zeta_0 + 2, \ldots\ldots, b\}$.

In this work, we address this gap by establishing results on the existence and uniqueness of solutions for a fractional difference equation of the following type:

$$\Delta^{\beta}\omega(\zeta) + A\omega(\zeta) = \Phi\left(\zeta, \omega(\zeta), \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)} k\big(\zeta, \eta, \omega(\eta)\big), \phi(\zeta-1)\right)\ \zeta \in J \tag{1.4}$$

subject to the conditions

$$\omega(\zeta-1)=\phi(\zeta) \tag{1.5}$$
$$\omega(0)+\rho(\omega)=\omega_0 \tag{1.6}$$

where A is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators $P(t)$ in $\chi, \Phi \in C(J\times\chi\times\chi,\chi), \rho \in C(C(J,\chi),\chi)$ and $\phi(t)$ is a function continuous for $\zeta \in J$. We denote $\phi(1-0)=c_0$,whrere c_0 is constant.

We observed that the solutions of Eq. (1.4), for $\zeta \in J$, is a function $\omega(\zeta-1)$ which is unable to define as solution for $\zeta \in J$. Hence, we have to impose some condition, such as the condition Eq. (1.5). We note that, if $\zeta \in J$. , the problem reduces to the fractional difference equation

$$\Delta^{\beta}(\zeta)+A\omega(\zeta)=\Phi\left(\zeta,\omega(\zeta),\frac{1}{\Gamma(\beta)}\sum_{s=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}k\big(\zeta,\eta,\omega(\eta)\big),\phi(\zeta)\right) \tag{1.7}$$

with the initial condition

$$\omega(0)+\rho(\omega)=\omega_0 \tag{1.8}$$

Therefore to obtain the solutions of Eqs. (1.4)-(1.6) for $0\le\zeta\le b$, For the purposes of the forthcoming discussion, let us take $1\le b$. This study primarily focuses on establishing the global existence of solutions for Eqs. (1.4)-(1.6) through the use of Granas' topological transversality theorem [33], more widely recognized as the Leray-Schauder alternative.The uniqueness of solutions to Eqs. (1.4)–(1.6) is derived using Bihari's integral inequality. The obtained results extend and generalize those of Pachpatte [32] to a broader semigroup and nonlocal fractional framework.

Definition [34]: let $\beta>0$. The β^{th} fractional sum of f is defined by

$$\Delta^{-\beta}f(\zeta)=\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}f(\eta).$$

Here f is defined for $\eta=a\ mod\ (1)$ and $\Delta^{-\beta}f$ is defined for $\zeta=(\zeta_0+\beta)mod(1))$; in particular $\Delta^{-\beta}$ map function defined on N_{ζ_0} to function defined on $N_{a+\beta}$ where $N_\zeta=\{\zeta,\zeta+1,\zeta+2,\dots.\}$

2. Preliminaries and Hypotheses

At first we shall discuss some preliminaries and set forth hypotheses that will be used in our subsequent discussion.

Let χ be the Banach space with norm $\|\cdot\|$. Let $B=C(J,\chi)$ be the space of all continuous functions from J into χ endowed with the supremum norm

$$\|\omega\|_B=\{\sup\|\omega(\zeta)\|:\zeta\in J\}$$

Lemma 2.1 Let A be the infinitesimal generator of a C_0- semigroup $P(\zeta),\zeta\ge0$, on a Banach space χ. For $0\le\zeta<$ b, define $\omega\in B$ given by

$$\omega(\zeta)=P(\zeta)[\omega_0-\rho(\omega)]+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)$$
$$\times\ \Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right). \tag{2.1}$$

and For $1 \le \zeta \le b$ define $\omega \in B$ given by

$$\omega(\zeta) = P(\zeta)[\omega_0 - \rho(\omega)] + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)} P(\zeta-\eta)\Phi\left(\eta, \omega(\eta), \frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)} k(\eta,\sigma,\omega(\sigma)), \phi(\eta)\right) + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta-1)\right). \tag{2.2}$$

then ω is the mild solution of the problem (1.4)-(1.6).

Proof: $$\omega(\zeta) = P(\zeta)[\omega_0 - \rho(\omega)] + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right). \tag{I}$$

$$\Delta^\beta\omega(\zeta) = \Delta^\beta\{P(\zeta)[\omega_0 - \rho(\omega)\}] + \Delta^\beta\{\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right)\}$$

$$\Delta^\beta\omega(\zeta) = -AP(\zeta)[\omega_0 - \rho(\omega)\}] + \{\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)\Delta^\beta\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right)\}$$

$$\Delta^\beta\omega(\zeta) = -AP(\zeta)[\omega_0 - \rho(\omega)\}] - \frac{A}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right) + \Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right). \tag{II}$$

Multiplying (I) by A

$$A\omega(\zeta) = AP(\zeta)[\omega_0 - \rho(\omega)] + \frac{A}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right).$$

(III)

Adding (II) & (III)

$$\Delta^\beta\omega(\zeta) + A\omega(\zeta) = -AP(\zeta)[\omega_0 - \rho(\omega)\}] - \frac{A}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right) + \Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right) + AP(\zeta)[\omega_0 - \rho(\omega)] + \frac{A}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right).$$

$$\Delta^{\beta}\omega(\zeta)+A\omega(\zeta)=\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k\big(\eta,\sigma,\omega(\sigma)\big),\phi(\eta)\right),\zeta\in J$$

Put $\zeta=0$ in (I)
$\omega(0)=P(0)[\omega_0-\rho(\omega)]+0$
$\omega(0)=[\omega_0-\rho(\omega)]$.

Thus the function ω defined by Eqs. (2.1) and (2.2) satisfies the fractional equation Eq. (1.4), the nonlocal condition Eq. (1.6), and the backward functional condition Eq. (1.5). Hence it is the mild solution of the problem Eqs. (1.4)-(1.6)

For investigating the completeness, we shall state the fixed point result given by Granas [33], often referred to as the Leray-Schauder alternative.

Lemma 2.2 (Leray-Schauder Alternative). Let Q be a convex subset of a normed linear space E and assume $0\in Q$. Let $\Psi\in \mathrm{CC(Q,Q)}$ be a completely continuous operator and let $U(\Psi)=\{\omega\in Q:\omega=\lambda\Psi\omega\}$ for some $0<\lambda<1$. Then either $U(\Psi)$ is unbounded or Ψ has a fixed point.

We shall also use the well known Bihari's inequality [13] that states as below.

Lemma 2.3 Let $\varrho(\zeta),\mu(\zeta)\in C(\mathbb{R}_+,\mathbb{R}_+),\mathbb{R}_+=[0,\infty)$. Let $\varpi(\varrho)$ be a continuous, nondecreasing function defined on $\mathbb{R}_+,\varpi(\varrho)>0$ for $\varrho>0$ and $\varpi(0)=0$. If

$$\varrho(\zeta)\le\frac{1}{\Gamma(\beta)}\sum\nolimits_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}p(s)\varpi(\varrho(\eta))$$

for $\zeta\in\mathbb{R}_+$, where $c\ge 0$ is a constant, then for $0\le\zeta\le\zeta_1$,

$$\varrho(\zeta)\le W^{-1}\left[W(c)+\frac{1}{\Gamma(\beta)}\sum\nolimits_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}\mu(\eta)\right],\mu(\zeta)\in C(\mathbb{R}_+,\mathbb{R}_+)$$

where,

$$W(r)=\frac{1}{\Gamma(\beta)}\sum\nolimits_{\eta=\zeta_0+r_0}^{r-\beta}(\zeta-\eta-1)^{(\beta-1)}\frac{1}{\varpi(\eta)},\ r>0,r_0>0$$

W^{-1} is the inverse function of W and $\zeta_1\in\mathbb{R}_+$be chosen so that

$$W(c)+\frac{1}{\Gamma(\beta)}\sum\nolimits_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}\mu(\eta)\in \mathrm{Dom}\,(W^{-1})$$

for all $\zeta\in\mathbb{R}_+$lying in the interval $0\le\zeta\le\zeta_1$.
Now we set forth following hypotheses:

(H_1) A is the infinitesimal generator of a semigroup of bounded linear operators $P(\zeta)$ in χ, which is compact for $\zeta>0$, and there exist constant $\mathcal{M}\ge 1$ such that $\| P(\zeta)\|\le\mathcal{M},\ \zeta\ge 0$.

(H_2) The function Φ in (1.4) satisfies the condition

$$\|\Phi(\zeta,\omega,y,z)\|\le\mu(\zeta)[\Omega(\|\omega\|)+\Omega(\|y\|)+\Omega(\|z\|)]$$

for every $\omega,y,z\in X$, where $\mu\in C(J,\mathbb{R}_+)$and $\Omega:\mathbb{R}_+\to(0,\infty)$ is continuous and increasing function satisfying $\Omega(\alpha(\zeta)\|\omega\|)\le\alpha(\zeta)\Omega(\|\omega\|)$, where α is defined as the function μ, $\alpha(\zeta)$ is scalar Quantity.

(H_3) There exists a continuous function $\tau:J\to\mathbb{R}_+$such that

$$\left\|\frac{1}{\Gamma(\beta)}\sum\nolimits_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\zeta,\eta,\omega(\eta))\right\|\le\tau(\zeta)\|\omega\|$$

for every $\zeta\ge\eta\ge 0$ and $\omega\in\chi$.

(H_4) There exist constant $G > 0$ such that $\| \rho(\omega) \| \leq G$ for every $\omega \in B$.

(H_5) For each $\zeta \in J$, the function $\Phi(\zeta,\cdot,\cdot,\cdot): J \times \chi \times \chi \times \chi \to \chi$ is continuous and for each $(\omega, y, z) \in \chi \times \chi \times \chi$, the function $f(\cdot, \omega, y, z): J \times \chi \times \chi \times \chi \to \chi$ is strongly measurable.

(H_6) For each $\zeta, \eta \in J$, the function $k(\zeta, \eta, \cdot): J \times J \times \chi \to \chi$ is continuous and for each $\omega \in \chi$, the function $k(\cdot,\cdot,\omega): J \times J \times \chi \to \chi$ is strongly measurable.

(H_7) For every positive integer m there exists $\alpha_m \in L^1(J)$ such that

$\sup \| \Phi(\zeta, \omega, y, z) \| \leq \alpha_m(\zeta)$ for $\zeta \in J$ a.e.

(H_8) The function Φ in (1.4) satisfies the condition

$$\| \Phi(\zeta, \omega, y, z) - \Phi(\zeta, \bar{\omega}, \bar{y}, \bar{z}) \| \leq \bar{\mu}(\zeta)[\bar{\Omega}(\| \omega - \bar{\omega} \|) + \bar{\Omega}(\| y - \bar{y} \|) + \bar{\Omega}(\| z - \bar{z} \|)]$$

for every $\omega, y, z, \bar{\omega}, \bar{y}, \bar{z} \in \chi$, where $\bar{\mu} \in C(\mathbb{R}_+, \mathbb{R}_+)$and $\bar{\Omega}(\varrho)$ is a continuous and increasing function for $\varrho \geq 0, \bar{\Omega}(0) = 0$.

(H_9) The function k in (1.4) satisfies the condition

$$\| \tfrac{1}{\Gamma(\beta)} \textstyle\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta - \eta - 1)^{(\beta-1)} [\mathrm{k}(\zeta, \eta, \omega) - k(\zeta, \eta, \bar{\omega})\big] \| \leq \bar{\tau}(\zeta) \| \omega - \bar{\omega} \|$$

for every $\omega, \bar{\omega} \in \chi$, where $\bar{\tau} \in C(\mathbb{R}_+, \mathbb{R}_+)$.

(H_{10}) There exist constant $\bar{G} > 0$ such that

$\| \rho(\omega) - \rho(\bar{\omega}) \| \leq \bar{G} \| \omega - \bar{\omega} \|$

for every $\omega, \bar{\omega} \in B$ and $\mathcal{M}\bar{G} < 1$.

3. Existence and Uniqueness Results

Result for global existence of the solution of Eqs. (1.4)-(1.6) is proved in the following theorem.

Theorem 3.1 If (H1)-(H7) hold, then the problem Eqs. (1.4)-(1.6) has a solution $\omega(\zeta)$ defined on *J* provided *b* satisfies

$$\frac{1}{\Gamma(\beta)} \sum_{\eta=\zeta_0}^{b-\beta} (\zeta - \eta - 1)^{(\beta-1)} \mathcal{M}\big[\mu(\eta)\big(1 + \tau(\eta)\big) + \mu(\eta + 1)\big]$$

$$< \tfrac{1}{\Gamma(\beta)} \textstyle\sum_{\eta=\mathrm{c}}^{\infty} (\zeta - \eta - 1)^{(\beta-1)} \tfrac{1}{\Omega(\eta)} \tag{3.1}$$

Where

$$c = \mathcal{M}[\|\omega_0 + G\|] + \mathcal{M} \tfrac{1}{\Gamma(\beta)} \textstyle\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta - \eta - 1)^{(\beta-1)} \mu(\eta) \Omega(\| \phi(\eta) \|) \tag{3.2}$$

4. Boundedness of Solutions

Now we shall obtain estimates on the solutions of equations Eqs. (1.4)-(1.6) under suitable assumptions on the involved functions. Following theorem deals with the boundedness of the solution of the equation Eq. (1.4).

Theorem 4.1 Let hypotheses $(H1), (H3), (H6)$ hold and

$$d_1 = \sup_{\zeta \in \mathbb{R}_+} \left\| \tfrac{1}{\Gamma(\beta)} \textstyle\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta - \eta - 1)^{(\beta-1)} \Phi(\eta, 0,0,0) \right\| < \infty$$

If $\omega(\zeta), \zeta \in \mathbb{R}_+$, is a solution of problem (1.4) - (1.6), then

$$\| \omega(\zeta) \| \leq \Upsilon^{-1}\left[\Upsilon(d_2) + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}\mathcal{M}\bar{\mu}(\eta)\big(1+\bar{\tau}(\eta)\big)\right] \tag{4.1}$$

for $0 \leq \zeta < \mathrm{b}$ and

$$\| \omega(\zeta) \| \leq \Upsilon^{-1}\left[\Upsilon(d_2) + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}\mathcal{M}\left(\bar{\mu}(\eta)\big(1+\bar{\tau}(\eta)\big) + \bar{\mu}(\eta+1)\right)\right] \tag{4.2}$$

for $1 \leq \zeta < \infty$, where

$$d_2 = \mathcal{M}[\|\omega_0\| + G] + \mathcal{M}d_1 + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}\mathcal{M}\bar{\mu}(\eta)\bar{\Omega}(\| \phi(\eta) \|)$$

Moreover, if $sup_{\zeta\in R^+}\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}[\bar{\mu}(\eta)+\bar{\mu}(\eta+1)]$ is bounded on $\mathbb{R}_+$, then every solution of the problem Eqs. (1.4) - (1.6) is bounded on $\mathbb{R}_+$.

Proof. Let us suppose that $\omega(\zeta)$ is a solution of the problem Eqs. (1.4)-(1.6). We shall consider following two cases.

Case I: Let $0 \leq \zeta < \mathrm{b}$. Then as the solution of the problem Eqs. (1.4)-(1.6) is $\omega(\zeta)$ and by the hypotheses, we have

$$\begin{aligned}
\| \omega(\zeta) \| &= \| P(\zeta)[\omega_0 - \rho(\omega)] + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta) \\
&\quad \times \Phi\left(\eta, \omega(\eta), \frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta} (\zeta-\eta-1)^{(\beta-1)}\, k\big(\eta,\sigma,\omega(\sigma)\big), \phi(\eta)\right) \| \\
&\leq \mathcal{M}[\|\omega_0\| + G] + \mathcal{M}\left\|\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}\Phi(s,0,0,0)\right\| \\
&\quad + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}\mathcal{M}\bar{\mu}(\eta)\bar{\Omega}(\| \phi(\eta) \|) \\
&\quad + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}(\mathcal{M}\bar{\mu}(\eta)(1+\bar{\tau}(\eta))\bar{\Omega}(\| \omega(\eta) \|) \\
&\leq \mathcal{M}[\|\omega_0\| + G] + \mathcal{M}d_1 + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}\mathcal{M}\bar{\mu}(\eta)\bar{\Omega}(\| \phi(\eta) \| \\
&\quad + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}\mathcal{M}\bar{\mu}(\eta)\big(1+\bar{\tau}(\eta)\big)\bar{\Omega}(\| \omega(\eta) \|) \\
&= d_2 + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}\mathcal{M}\bar{\mu}(\eta)\big(1+\bar{\tau}(\eta)\big)\bar{\Omega}(\| \omega(\eta) \|)
\end{aligned} \tag{4.3}$$

Applying Lemma 2.3 to the equation Eq. (4.3) we obtain required Eq. (4.1).

Case II: Let $1 \leq \zeta < \infty$. Then as the solution of the problem Eqs. (1.4)-(1.6) is $\omega(\zeta)$ and by the hypotheses, we have

$$\begin{aligned}
\| \omega(\zeta) \| &= \| P(\zeta)[\omega_0 - \rho(\omega)] \\
&\quad + \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\delta} (\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta) \\
&\quad \times \Phi\left(\eta, \omega(\eta), \frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}\, k\big(\eta,\sigma,\omega(\sigma)\big), \phi(\eta)\right) \\
&\quad + \frac{1}{\Gamma(\beta)}\sum_{\eta=\delta+1}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta) \| \\
&\quad \times \Phi\left(\eta, \omega(\eta), \frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta} (\zeta-\eta-1)^{(\beta-1)}\, k\big(\eta,\sigma,\omega(\sigma)\big), \phi(\eta-1)\right) \\
&\leq \mathcal{M}[\|\omega_0\| + G] + \| \frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\delta} (\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)
\end{aligned}$$

$$\times\left[\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}\,k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right)-\Phi(\eta,0,0,0)\right.$$

$$\left.+\Phi(\eta,0,0,0)\right]\|$$

$$+\left\|\frac{1}{\Gamma(\beta)}\sum_{\eta=\delta+1}^{\zeta-1}(\zeta-\eta-1)^{(\beta-1)}P(\zeta-\eta)[\Phi(\eta,\omega(\eta),\omega(\eta-1))-\Phi(\eta,0,0,0)+\right.$$

$$\left.\Phi(\eta,0,0,0)]\right\|$$

$$\leq M[\|\omega_0\|+G]+M\left\|\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}\Phi(\eta,0,0,0)\right\|$$

$$+\sum_{\eta=\zeta_0}^{\delta}M\bar{\mu}(\eta)\bar{\Omega}(\|\phi(\eta)\|)$$

$$+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\delta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)(1+\bar{\tau}(\eta))\bar{\Omega}(\|\omega(\eta)\|)$$

$$+\frac{1}{\Gamma(\beta)}\sum_{\eta=\delta+1}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)(1+\bar{\tau}(\eta))\bar{\Omega}(\|\omega(\eta)\|)$$

$$+\frac{1}{\Gamma(\beta)}\sum_{\eta=\delta+1}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)\bar{\Omega}(\|\omega(\eta-1)\|)$$

$$\leq M[\|\omega_0\|+G]+Md_1+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\delta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)\bar{\Omega}(\|\phi(\eta)\|)$$

$$+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)(1+\bar{\tau}(\eta))\bar{\Omega}(\|\omega(\eta)\|)$$

$$+\frac{1}{\Gamma(\beta)}\sum_{\eta=\delta+1}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)\bar{\Omega}(\|\mu(\eta-1)\|)$$

$$=d_2+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)\big(1+\bar{\tau}(\eta)\big)\bar{\Omega}(\|\omega(\eta)\|)+I_1 \qquad (4.4)$$

where $I_1=\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)\bar{\Omega}(\|\omega(\eta-1)\|)$.

By the change variable .we observe that

$$I_1\leq\frac{1}{\Gamma(\beta)}\sum_{\eta=\delta+1}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta+1)\bar{\Omega}(\|\omega(\eta)\|) \qquad (4.5)$$

Now, by using (4.5) in Eq. (4.4), we get

$$\|\omega(\zeta)\|\leq d_2+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)\big(1+\bar{\tau}(\eta)+\mu(\eta+1)\big)\bar{\Omega}(\|\omega(\eta)\|) \quad (4.6)$$

Applying Lemma 2.3 to Eq. (4.6), we obtain required Eq. (4.2).

Further, if $\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}\bar{\mu}(\eta)\big(1+\bar{\tau}(\eta)+\mu(\eta+1)\big)$ is bounded on $\mathbb{R}_+$, then from Eqs. (4.1) and (4.2) we conclude that every solution of the problem Eqs. (1.4)-(1.6) is bounded on $\mathbb{R}_+$.

5. Continuous Dependence

In the following theorem we have investigated the continuous dependence of solutions of Eq. (1.4) satisfying the given initial data.

Theorem 5.1 Let hypotheses $(H1),(H8)-(H10)$ hold and $\omega_1(\zeta),\omega_2(\zeta)$ be the solutions of Eq. (1.4) with the initial conditions

$$\omega_1(\zeta-1)=\phi_1(\zeta) \quad (0 \le \zeta < \mathrm{b}), \quad \omega_1(0)+\rho(\omega_1)=\omega_0 \tag{5.1}$$

$$\omega_2(\zeta-1)=\phi_2(\zeta) \quad (0 \le \zeta < \mathrm{b}), \quad \omega_2(0)+\rho(\omega_2)=\bar{\omega}_0 \tag{5.2}$$

respectively, where $\omega_0, \bar{\omega}_0$ are elements of χ, then

$$\|\omega_1(\eta)-\omega_2(\eta)\| \le \Upsilon^{-1}\left[\Upsilon(d_3)+\frac{1}{\Gamma(\beta)}\sum_{\eta=\delta+1}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}\frac{M\bar{\mu}(\eta)\big(1+\bar{\tau}(\eta)\big)}{1-M\bar{G}}\right] \tag{5.3}$$

for $0 \le \zeta < \mathrm{b}$ and

$$\|\omega_1(\eta)-\omega_2(\eta)\| \le \Upsilon^{-1}\left[\Upsilon(d_3)+\frac{1}{\Gamma(\beta)}\sum_{\eta=\delta+1}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}\frac{M\big(\bar{\mu}(\eta)\big(1+\bar{\tau}(\eta)\big)+\bar{\mu}(\eta+1)\big)}{1-M\bar{G}}\right] \tag{5.4}$$

for $1 \le \zeta < \infty$, where

$$d_3=\frac{M[\|\omega_0-\bar{\omega}_0\|}{1-M\bar{G}}+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\delta}(\zeta-\eta-1)^{(\beta-1)}\frac{M\bar{\mu}(\eta)}{1-M\bar{G}}\bar{\Omega}(\|\phi_1(\eta)-\phi_2(\eta)\|). \tag{5.5}$$

Proof. Let us suppose that $\varrho(\zeta)=\|\omega_1(\eta)-\omega_2(\eta)\|$ for $\zeta \in \mathbb{R}_+$. We shall consider following two cases.

Case I: Let $0 \le \zeta < \mathrm{b}$, then from the hypotheses, we obtain

$$\begin{aligned}
\varrho(\zeta) &\le M[\|\omega_0-\bar{\omega}_0\|+\bar{G}\varrho(\zeta)] \\
&\quad +\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\ \Big\|\ \Phi\left(\eta,\omega_1(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-\right. \\
&\left. 1)^{(\beta-1)}\,k\big(\eta,\sigma,\omega_1(\sigma)\big),\phi(\eta)\right) \\
&\quad -\Phi\left(\eta,\omega_2(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}\,k\big(\eta,\sigma,\omega_2(\sigma)\big),\phi(\eta)\right)\Big\| \\
&\le M[\|\omega_0-\bar{\omega}_0\|+\bar{G}\varrho(\zeta)] \\
&\quad +\sum_{\eta=\zeta_0}^{\zeta-\beta}M\bar{\mu}(\eta)[\bar{\Omega}(\varrho(\eta))+\bar{\Omega}(\bar{\tau}(\eta)\varrho(\eta))+\bar{\Omega}(\|\phi_1(\eta)-\phi_2(\eta)\|)] \\
&\le M[\|\omega_0-\bar{\omega}_0\|+\bar{G}\varrho(\zeta)] \\
&\quad +\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)\bar{\Omega}(\|\phi_1(\eta)-\phi_2(\eta)\|) \\
&\quad +\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)(1+\bar{\tau}(\eta)\bar{\Omega}\big(\varrho(\eta)\big)
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\Rightarrow \varrho(\eta) &\le \frac{M[\|\omega_0-\bar{\omega}_0\|}{1-M\bar{G}}+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}\frac{M\bar{\mu}(\eta)}{1-M\bar{G}}\bar{\Omega}(\|\phi_1(\eta)-\phi_2(\eta)\|) \\
&\quad +\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}\frac{M\bar{\mu}(\eta)(1+\bar{\tau}(\eta))}{1-M\bar{G}}\bar{\Omega}(\varrho(\eta)) \\
&= d_3+\frac{1}{\Gamma(\beta)}\sum_{\eta=\delta+1}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}\frac{M\bar{\mu}(\eta)(1+\bar{\tau}(\eta))}{1-M\bar{G}}\bar{\Omega}(\varrho(\eta))
\end{aligned} \tag{5.7}$$

Applying Lemma 2.3 (with $c=d_3$) to the Eq. (5.7), we obtain Eq. (5.3).

Case II: Let $1 \le \zeta < \infty$. Then from the hypotheses and following similar arguments as in Case II of the proof of Theorem 3.3, it follows that

$$\begin{aligned}
\varrho(\zeta) &\le M[\|\omega_0-\bar{\omega}_0\|+\bar{G}\varrho(\zeta)] \\
&\quad +\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta}(\zeta-\eta-1)^{(\beta-1)}M\bar{\mu}(\eta)\bar{\Omega}(\|\phi_1(\eta)-\phi_2(\eta)\|)
\end{aligned}$$

$$+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)} M\bar{\mu}(\eta)(1+\bar{\tau}(\eta)\bar{\Omega}\big(\varrho(\eta)\big)+I_2 \tag{5.8}$$

where $I_2=\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)} M\bar{\mu}(\eta)\bar{\Omega}(\varrho(\eta-1))$

by change of variable, we observe that

$$I_2\le\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)} M\bar{\mu}(\eta)\bar{\Omega}(\varrho(\eta)) \tag{5.9}$$

Now, using this inequality (5.9) in (5.8), we obtain

$$\begin{aligned}
\varrho(\zeta) &\le M[\|\omega_0-\bar{\omega}_0\|+\bar{G}\varrho(\zeta)]\\
&\quad+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\delta} (\zeta-\eta-1)^{(\beta-1)} M\bar{\mu}(\eta)\bar{\Omega}(\|\phi_1(\eta)-\phi_2(\eta)\|)\\
&\quad+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)} M[\bar{\mu}(\eta)(1+\bar{\tau}(\eta)+\bar{\mu}(\eta+1))]\bar{\Omega}\big(\varrho(\eta)\big)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \varrho(\zeta) &\le M[\|\omega_0-\bar{\omega}_0\|+\bar{G}\varrho(\zeta)]\\
&\quad+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\delta} (\zeta-\eta-1)^{(\beta-1)} \frac{M\bar{\mu}(\eta)}{1-M\bar{G}}\bar{\Omega}(\|\phi_1(\eta)-\phi_2(\eta)\|)\\
&\quad+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)} \frac{M[\bar{\mu}(\eta)(1+\bar{\tau}(\eta)+\bar{\mu}(\eta+1))]}{1-M\bar{G}}\bar{\Omega}(\|\varrho(\eta)\|)\\
&= d_3+\frac{1}{\Gamma(\beta)}\sum_{\eta=\zeta_0}^{\zeta-\beta} (\zeta-\eta-1)^{(\beta-1)} \frac{M[\bar{\mu}(\eta)(1+\bar{\tau}(\eta)+\bar{\mu}(\eta+1))]}{1-M\bar{G}}\bar{\Omega}(\|\varrho(\eta)\|)
\end{aligned} \tag{5.10}$$

Applying Lemma 2.3 (with $= d_3$) to the equation Eq. (5.10), we obtain Eq. (5.4). Hence from the inequalities Eqs. (5.3) and (5.4), it follows that the solutions of equation Eq. (1.4) depends on the given initial data.

Example: Consider the fractional difference equation of order $\beta=\frac{1}{2}$

$$\Delta^{0.5}\omega(\zeta)-2\omega(\zeta)=\zeta+\omega(\zeta)^2+\frac{1}{\sqrt{\pi}}\sum_{\eta=\zeta_0}^{\zeta-0.5}(\zeta-\eta-1)^{\frac{-1}{2}}\big(\zeta+\eta+\omega(\zeta)\big)+\omega(\zeta-1)\ \ for\ \zeta\in J$$

subject to the conditions $\omega(-1)=1\,,\omega(0)+\frac{1}{2}\omega(1)=3$.

Solution: Let $P(\zeta)=e^{-2\zeta}$ be the C_0-semigroup generated by $A=-2$.

$$\omega(\zeta)=e^{-2\zeta}[\omega_0-\rho(\omega)]+\frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{\eta=\zeta_0}^{\zeta-\frac{1}{2}}(\zeta-\eta-1)^{\left(\frac{1}{2}\right)}e^{-2(\zeta-\eta)}\Phi\left(\eta,\omega(\eta),\frac{1}{\Gamma(\beta)}\sum_{\sigma=\zeta_0}^{\eta-\beta}(\zeta-\eta-1)^{(\beta-1)}k(\eta,\sigma,\omega(\sigma)),\phi(\eta)\right)$$

Therefore, all the conditions of Theorems 4.1 & 4.2 are satisfied. the problem has a mild solution.

6. Conclusion

This study provides a detailed analysis of nonlinear nonlocal fractional difference equations, establishing existence, uniqueness, and key qualitative properties of their solutions. By utilizing the Leray-Schauder alternative alongside Bihari's integral inequality, we en sure that the solutions are bounded and depend continuously on the initial data. The illustrative examples confirm the practical applicability of the theoretical results, high lighting their relevance for discrete fractional systems with nonlocal conditions. Overall, the findings offer a solid foundation for further exploration of more complex nonlinear and nonlocal fractional difference models.

References

1. F. M. Atici and S. Sengul, J. Math. Anal. Appl. **369**, 1 (2010). https://doi.org/10.1016/j.jmaa.2010.02.009
2. J. Mohan and G. V. S. R. Deekshitulu, Int. J. Difference Equ. **2012**, ID 780619 (2012). https://doi.org/10.1155/2012/780619
3. F. M. Atici and P. W. Eloe, J. Difference Equ. Appl. **17**, 445 (2011). https://doi.org/10.1080/10236190903029241
4. W. G. Kelley and A. C. Peterson, Difference Equations: An Introduction with Applications (Academic Press, USA, 2001).
5. F. Chen, X. Luo and Y. Zhou, Adv. Difference Equ. **2011**, ID 713201 (2011). https://doi.org/10.1155/2011/713201
6. R. W. Ibrahim and H. A. Jalab, Adv. Difference Equ. **2015**, ID 269 (2015). https://doi.org/10.1186/s13662-015-0609-1
7. A. Deshpande and V. Daftardar-Gejji, Pramana – J. Phys. **87**, 49 (2016). https://doi.org/10.1007/s12043-016-1231-9
8. H. M. Srivastava and P. O. Mohammed, Front. Phys. **8**, ID 120 (2020). https://doi.org/10.3389/fphy.2020.00280
9. A. Slavova and Z. Zafirova, AIP Conf. Proc. **2172**, ID 030013 (2019). https://doi.org/10.1063/1.5133502
10. C. S. Goodrich, Int. J. Dyn. Syst. Differ. Equ. **3**, 145 (2011).
11. P. Anh, A. Babiarz, A. Czornik, M. Niezabitowski, and S. Siegmund, Arch. Control Sci. **28**, 617 (2018). https://doi.org/10.24425/acs.2018.125486
12. J. M. Jonnalagadda, Kragujevac J. Math. **43**, 219 (2019).
13. D. D. Joshi, S. Bhalekar, and P. M. Gade, Qualitative Analysis of Fractional Difference Equations, SSRN Preprint (2023). https://doi.org/10.2139/ssrn.4499671
14. J. W. He, L. Zhang, Y. Zhou, and B. Ahmad, Adv. Difference Equ. **2018**, ID 153 (2018).
15. H. Chen, Y. Cui, S. Kang, Y. Lu, and W. Feng, Math. Methods Appl. Sci. **146**, 881 (2021). https://doi.org/10.1111/sapm.12368
16. G. A. Anastassiou, Intell. Math. Comput. Anal. **5**, 575 (2011).
17. T. Abdeljawad, Comput. Math. Appl. **62**, 1602 (2011). https://doi.org/10.1016/j.camwa.2011.03.036
18. O. Ozturk, New Trends Math. Sci. **4**, 137 (2016). https://doi.org/10.20852/ntmsci.2016422559
19. D. Mozyrska and M. Wyrwas, Discrete Dyn. Nat. Soc. **2015**, ID 852734 (2015). https://doi.org/10.1155/2015/852734
20. J. J. Mohan and G. V. S. R. Deekshitulu, Malaya J. Math. **3**, 7 (2013). https://doi.org/10.26637/mjm103/002
21. J.-F. Cheng and Y.-M. Chu, Abstract Appl. Anal. **2012**, ID 918529 (2012). https://doi.org/10.1155/2012/918529
22. S. Sugiyama, Kodai Math. Sem. Rep. **12**, 179 (1960). https://doi.org/10.2996/kmj/1138844324
23. A. Stokes, Ann. Math. Stud. **45**, 173 (1960). https://doi.org/10.1515/9781400882649-008
24. S. Sugiyama, Proc. Japan Acad. **38**, 145 (1962). https://doi.org/10.3792/pja/1195523438
25. R. Bellman and K. L. Cooke, Differential-Difference Equations (Academic Press, USA, 1963). https://doi.org/10.1063/1.3050672
26. L. Byszewski, J. Math. Anal. Appl. **162**, 494 (1991). https://doi.org/10.1016/0022-247X(91)90164-U
27. L. Byszewski, Nonlinear Anal. **34**, 65 (1998). https://doi.org/10.1016/S0362-546X(97)00693-7
28. J. K. Hale, Theory of Functional Differential Equations (Springer-Verlag, USA, 1977).
29. B. G. Pachpatte, Funkcial. Ekvac**.** **26**, 1 (1983).
30. B. G. Pachpatte, Inequalities for Differential and Integral Equations (Nova Science Publishers, USA, 2007).
31. B. G. Pachpatte, Electron. J. Differ. Equ. **2008**, 1 (2008). https://doi.org/10.1155/2008/718408
32. B. G. Pachpatte, Electron. J. Differ. Equ. **2009**, ID 49 (2009).

33. J. Dugundji and A. Granas, Fixed Point Theory (PWN, Poland, 1982).
34. G. A. Anastassiou, Math. Comp. Modell. **54**, 2966 (2011).