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Higher Dimensional Bianchi Type *III* Dark Energy Cosmological Model with Cosmic String in $f(R)$ Theory of Gravitation

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Abstract

This study investigates higher-dimensional Bianchi type-III spacetime within the framework of $f(R)$ gravity to derive cosmological solutions for Einstein's field equations (EFE) incorporating dark energy and cosmic strings. To obtain a deterministic solution, we utilized specific conditions: a power-law relationship between F and the average scale factor, a proportionality between the shear and expansion scalars, and a functional link between the Hubble parameter and the average scale factor. We investigated both exponential and power-law expansion models. Furthermore, the temporal evolution of the relevant physical parameters is analysed and illustrated graphically.

Keywords: Dark energy; Cosmic string; Bianchi Type III; $f(R)$ theory.

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1. Introduction

The search for a deeper understanding of the universe has long been regarded as a key objective since the inception of scientific research. As time has progressed, significant advancements have been achieved in fields including astronomy, astrophysics, cosmology, data science, and space science, and consequently, our comprehension has been considerably broadened. Following the recent discovery of the universe's accelerated expansion [1], modified gravitational theories have increasingly been considered by cosmologists, as viable explanations for the observed acceleration may be provided by these theories.

Within the spectrum of modified theories of gravitation, the $f(R)$ theory is widely recognized as particularly suitable, owing to the cosmological relevance of its models. In these models, higher-order curvature invariants are expressed as functions of the Ricci scalar. Several viable $f(R)$ gravity models [2] have been introduced, through which the unification of early-time inflation and late-time acceleration has been illustrated. It is widely

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believed that the issue of dark matter may potentially be resolved by viable $f(R)$ gravity models. In addition, a natural gravitational alternative to dark energy is presented by this theory. Extensive research has been carried out in the field of $f(R)$ gravitation.

Exact solutions of cosmological models in the framework of $f(R)$ gravity were investigated by Capozziello *et al*. [3]. In a comprehensive review of $f(R)$ theories, various topics such as inflation, dark energy, cosmological perturbations, local gravity constraints, and spherically symmetric solutions in weak and strong gravitational backgrounds were discussed by Felice *et al*. [4]. The dynamics of anisotropic LRS Bianchi type-I models in $f(R)$ gravity were analysed by Tripathy *et al*. [5], and solutions to the field equations were derived. The Bianchi type-I cosmological model in $f(R)$ gravity with a perfect fluid was examined by Agrawal *et al*. [6], and the dynamics of gravitational baryogenesis were discussed using two distinct forms of the Ricci scalar. The correspondence between HDE and quintessence cosmological models in various scenarios was studied by Santhi *et al*. [7]. An analytical solution to the general relativistic field equation in four and five dimensions was derived by Mete *et al*. [8] using a two-fluid Kasner-type Bianchi type-I cosmological model. The field equations of $f(R,T)$ gravity for Bianchi type-I spacetime were addressed by Adhav [9]. The evolution of the Bianchi type-I Kasner metric within the framework of $f(R)$ gravity was examined by Saaidi *et al*. [10], and it was noted that power-law $f(R)$ models exhibited behaviour similar to the quintessence model. The tilted Bianchi type-I cosmological model of Kasner form in Brans-Dicke theory was analysed by Pawar *et al.* [11]. The existence and stability of the Bianchi type-I Kasner solution in $f(T)$ gravity were demonstrated by Paliathanasis *et al.* [12]. The Bianchi type-III cosmological model with cosmic strings and domain walls was explored by Adhav et al. [13]. The Bianchi type-I perfect fluid string cosmological model in $f(T)$ gravity was investigated by Chirde *et al.* [14], and quintessence dark energy was found to be dominant. A plane symmetric model of the universe in $f(R, L_m)$ gravity was investigated by Bayaskar *et al*. [15]. A hybrid expansion law for bulk viscous $f(T)$ gravity was studied by Darunde *et al*. [16]. Cosmographical analysis toward viscous $f(T)$ gravity was carried out by Shekh *et al*. [17]. A logarithmic $f(Q)$ gravity model was investigated by Bayaskar *et al*. [18], and the corresponding energy conditions were examined. The reconstruction of the ΛCDM universe in $f(Q)$ gravity was studied by Gadbail *et al.* [19]. Models of holographic dark energy in $f(Q)$ gravity were investigated by Shekh *et al.* [20]. The $f(R,T)$ gravity theory was formulated by Harko *et al.* [21], its field equations were derived, and non-geodesic motion of test particles was shown due to an extra force. FRW cosmological models in $f(R,T)$ gravity were investigated by Myrzakulov *et al*. [22], and their dynamical behaviour in an accelerating universe was analysed. The Bianchi type-III cosmic string cosmological model in $f(R)$ gravity was studied by Adhav [23]. Locally rotationally symmetric Bianchi type-I string cosmological models in $f(R)$gravity were examined by Aditya *et al*. [24].

Bianchi type models are generally regarded as the simplest and most effective anisotropic models, by which anisotropic effects are adequately described. Although Bianchi universes are anisotropic in nature, it has been suggested from a cosmological perspective that the early universe may have been anisotropic, and that such anisotropies might have been eliminated through evolutionary processes, thereby resulting in an isotropic and homogeneous universe. In recent years, various anisotropic Bianchi type cosmological models in $f(R)$ gravity have been extensively investigated. Exact vacuum solutions for Bianchi type-I, III, and Kantowski–Sachs spacetimes in the metric formulation of $f(R)$ gravity were studied by Shamir [25]. Non-vacuum Bianchi type-I and V models in $f(R)$ gravity were explored by Sharif *et al.* [26]. Plane symmetric vacuum solutions of Bianchi type-III cosmology in $f(R)$ gravity were examined by Shamir [27]. LRS Bianchi type-I string cosmological models with magnetic strings in $f(R)$ gravity were analysed by Sheykhi [28]. Bianchi type-II, VIII, and IX string cosmological models in $f(R)$ gravity were discussed by Katore [29]. Bianchi type-III bulk viscous string cosmological models in $f(R)$ gravity were recently investigated by Santhi *et al.* [30]. Bianchi-type dark energy cosmological models in various gravitational theories have been widely discussed in the literature [31–35]. Quark and strange quark matter in $f(R)$ gravity for Bianchi type-I and V spacetimes were studied by Yilmaz *et al.* [36]. A non-vacuum static cylindrically symmetric solution in $f(R)$ gravity and its energy distribution were explored by Sharif *et al.* [37]. Bianchi type string cosmological models in $f(R,T)$ gravity were discussed by Sahoo *et al.* [38]. Higher-dimensional Bianchi type- III, string cosmological models with dark energy in Saez–Ballester theory were studied by Dabgar *et al*. [39], while similar models in Brans–Dicke theory were investigated by Trivedi *et al.* [40]. A five-dimensional cosmological model with a cosmic string and zero-mass scalar field in Lyra manifold was investigated by Mete *et al.* [41]. Holographic dark energy and domain wall models in logarithmic $f(Q)$ gravity were investigated by Bayaskar *et al.* [42]. Cosmological models with cosmic strings and minimally interacting dark energy were examined by Mete *et al.* [43]. Anisotropic Bianchi type VI_0 cosmological models in modified $f(R,T)$ gravity were investigated by Ugale *et al*. [44]. Cosmic acceleration through analysis of physical and kinematical parameters in $f(R,\Sigma,\mathrm{T})$ gravity was investigated by Shekh *et al.* [45]. Interacting ghost dark energy with sign-changeable coupling in Brans–Dicke cosmology was investigated by Mehta *et al.* [46].

Motivated by the above studies, the anisotropic Bianchi type III spacetime in the presence of cosmic strings within the framework of $f(R)$ gravity is investigated in the present work. A solution of Einstein's field equations is obtained. The organization of the paper is structured as follows: in Sect. 1, background information is presented; in Sect. 2, $f(R)$ gravity formalism and metric tensor are discussed; in Sect. 3, solutions of the field equations with dark energy and string cloud are analysed; in Sect. 4, physical and geometrical properties of the model are described along with a brief discussion of the results; and finally, in Sect. 5, concluding remarks are provided.

2. $f(R)$ Gravity Formalism and Metric

We consider a class of modified gravity in which modifies Einstein-Hilbert action by replacing Ricci curvature scalar R by an arbitrary function of curvature $f(R)$ as follows,

$$S = \int \sqrt{-g} \left(\frac{f(R)}{2} + kL_m\right) d^4x \tag{1}$$

where g is the metric determinant, L_m is the matter Lagrangian.

The field equation of f(R)is given by

$$F(R)R_{ij} - \frac{1}{2} f(R)\, g_{ij} - \nabla_i \nabla_j F(R) + g_{ij} \Box F(R) = k^2 T_{ij} \quad (2)$$

where $F = \frac{df}{dR}$, ∇ is a covariant derivative, $\Box = \nabla^{\alpha}\nabla_{\alpha}$ is the d'Alembertian, T_{ij} is the matter energy-momentum tensor

The standard FRW model is in good agreement with the present-day universe. However, it does not give a clear description of the early stage of evolution of the universe. Bianchi type models provide a physically realistic description of the initial universe. It is well known that, near the Big Bang, the universe is neither isotropic nor spherically symmetric. Thus, anisotropic models play an important role in cosmology.

The spatially homogeneous and anisotropic five-dimensional Bianchi Type-*III* metric is given by,

$$ds^2 = dt^2 - A^2 dx^2 - B^2 e^{-2sx} dy^2 - C^2 dz^2 - D^2 d\emptyset^2 \quad (3)$$

where A, B, C and D are the metric potentials and functions of cosmic time t only.

3. Solution of Field Equations for Dark Energy with String Cloud

The energy-momentum tensor for the derived model is expressed as:

$$T_{ij} = T_{ij}{}^{CS} + T_{ij}{}^{DE} \quad (4)$$

where the energy-momentum tensor is composed of the energy-momentum tensors of cosmic string ($T_{ij}{}^{CS}$) and dark energy ($T_{ij}{}^{DE}$).

The energy-momentum tensor of cosmic string is defined as:

$$T_{ij}{}^{CS} = (\rho + p)u_i u_j - p g_{ij} + \lambda x_i x_j \quad (5)$$

here, $u^i u_i = -x^i x_i = 1$ and $u^i x_i = 0$, where x^i represents the direction of cosmic strings along x-direction and u^idenotes the four-velocity vector. The fluid's pressure is denoted by p, the tension density of the string is represented by λ and the rest energy density of strings is given by ρ. Now,

$$T_{ij}{}^{CS} = \mathrm{diag}[-(p + \lambda), -p, -p, -p, \rho] \quad (6)$$

The energy-momentum tensor of dark energy is defined as

$$T_{ij}{}^{DE} = (\rho + p)u_i u_j - p g_{ij} \quad (7)$$

which finally takes the form of

$$T_{ij}{}^{DE} = \mathrm{diag}\left[-\omega_x, -\omega_y, -\omega_z, -\omega_\phi, 1\right]\rho_{de} \quad (8)$$

$$T_{ij}{}^{DE} = \mathrm{diag}[-\omega_{de}, -(\omega_{de} + \beta), -(\omega_{de} + \gamma), -(\omega_{de} + \gamma), 1]\rho_{de} \quad (9)$$

In the above equation, the parameterω_{de}denotes the equation of state (EoS) for dark energy and ρ_{de} represents the density of dark energy. Additionally, there are two skewness parameters β and γ, which describe the deviations from ω_{de} along the y, z and ϕ axes. The inclusion of deviations from the equation of state parameter of dark energy along specific axes in cosmological models serves to incorporate the possibility of anisotropy in the distribution of dark energy [24].

For simplification of the solution, we assume

$$\omega_{de_x} = \omega_{de} \quad \text{and} \quad \omega_{de_z} = \omega_{de_\phi} = \omega_{de} + \gamma$$

field Eq. (2) for the Bianchi Type III metric (3) with the help of Eqs. (4)-(6) becomes,

$$F\left\{\frac{A_{44}}{A}+\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}+\frac{A_4D_4}{AD}-\frac{s^2}{A^2}\right\}-\frac{1}{2}f+\frac{B_4}{B}F_4+\frac{C_4}{C}F_4+\frac{D_4}{D}F_4+F_{44} = -((p+\lambda)+\omega_{de}\rho_{de}) \tag{10}$$

$$F\left\{\frac{B_{44}}{B}+\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{B_4D_4}{BD}-\frac{s^2}{A^2}\right\}-\frac{1}{2}f+\frac{A_4}{A}F_4+\frac{C_4}{C}F_4+\frac{D_4}{D}F_4+F_{44} = -(p+(\omega_{de}+\beta)\rho_{de}) \tag{11}$$

$$F\left\{\frac{C_{44}}{C}+\frac{C_4B_4}{CB}+\frac{A_4C_4}{AC}+\frac{C_4D_4}{CD}\right\}-\frac{1}{2}f+\frac{B_4}{B}F_4+\frac{A_4}{A}F_4+\frac{D_4}{D}F_4+F_{44} = -(p+(\omega_{de}+\gamma)\rho_{de}) \tag{12}$$

$$F\left\{\frac{D_{44}}{D}+\frac{D_4B_4}{DB}+\frac{D_4C_4}{DC}+\frac{A_4D_4}{AD}\right\}-\frac{1}{2}f+\frac{B_4}{B}F_4+\frac{C_4}{C}F_4+\frac{A_4}{A}F_4+F_{44} = -(p+(\omega_{de}+\gamma)\rho_{de}) \tag{13}$$

$$F\left\{\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{D_{44}}{D}\right\}-\frac{1}{2}f+\frac{B_4}{B}F_4+\frac{C_4}{C}F_4+\frac{A_4}{A}F_4+\frac{D_4}{D}F_4 = \rho+\rho_{de} \tag{14}$$

$$\frac{A_4}{A}-\frac{B_4}{B} = 0, \tag{15}$$

where the derivative with respect to cosmic time t is denoted by the subscript ‘4’. Solving Eq. (15) we get,

$$A = B \tag{16}$$

Using Eq. (13), Eqs. (7)-(11) are given by

$$F\left\{\frac{A_{44}}{A}+\frac{{A_4}^2}{A^2}+\frac{A_4C_4}{AC}+\frac{A_4D_4}{AD}-\frac{s^2}{A^2}\right\}-\frac{1}{2}f+\frac{A_4}{A}F_4+\frac{C_4}{C}F_4+\frac{D_4}{D}F_4+F_{44} = -((p+\lambda)+\rho_{de}) \tag{17}$$

$$F\left\{\frac{A_{44}}{A}+\frac{{A_4}^2}{A^2}+\frac{A_4C_4}{AC}+\frac{A_4D_4}{AD}-\frac{s^2}{A^2}\right\}-\frac{1}{2}f+\frac{A_4}{A}F_4+\frac{C_4}{C}F_4+\frac{D_4}{D}F_4+F_{44} = -(p+(\omega_{de}+\beta)\rho_{de}) \tag{18}$$

$$F\left\{\frac{C_{44}}{C}+\frac{C_4B_4}{CB}+\frac{A_4C_4}{AC}+\frac{C_4D_4}{CD}\right\}-\frac{1}{2}f+2\frac{A_4}{A}F_4+\frac{D_4}{D}F_4+F_{44} = -(p+(\omega_{de}+\gamma)\rho_{de}) \tag{19}$$

$$F\left\{\frac{D_{44}}{D}+\frac{D_4C_4}{DC}+2\frac{A_4D_4}{AD}\right\}-\frac{1}{2}f+\frac{C_4}{C}F_4+2\frac{A_4}{A}F_4+F_{44} = -(p+(\omega_{de}+\gamma)\rho_{de}) \tag{20}$$

$$F\left\{2\frac{A_{44}}{A}+\frac{C_{44}}{C}+\frac{D_{44}}{D}\right\}-\frac{1}{2}f+\frac{C_4}{C}F_4+2\frac{A_4}{A}F_4+\frac{D_4}{D}F_4 = \rho_{de} \tag{21}$$

Subtracting Eq. (20) from Eq. (19), we get

$$\frac{C_4}{C}-\frac{D_4}{D} = \frac{KF}{V}, \text{where} \quad V = ABCD \tag{22}$$

The system of field Eqs. (17)-(21) consists of five independent equations with eleven unknowns. The above five equation are highly non-linear. Because of this, we take the help of following physically plausible condition to obtain a determinate solution.

(I). We have assumed a scaling relation between the shear scalar $'\rho'$ and expansion scalar $'\theta'$ [47] which led to relationship between the metric potential. As a result, we can take

$$D = C^u, \text{ where } u \neq 0 \text{ is a constant} \tag{23}$$

(II). Using power law relation between F and average scale factor 'a', established by Uddin *et al*. [48] in $f(R)$ theory of gravity i.e.

$F(R) \propto a^m$

where 'm' is arbitrary constant.

Above equation lead to

$$F(R) = d\, a(t)^m \tag{24}$$

where 'd' is proportionality constant.

(III). The value of deceleration parameter depends on the scale factor. The relation between average Hubble parameter H and average scale factor proposed by Berman [49] is given by, $H = \alpha[a^{-r}]$ (25)

where $\alpha > 0, r \geq 0$

And it provides constant value of the deceleration parameter.

Solving Eq. (25), we get

$$a(t) = (r\alpha t + k_1)^{\frac{1}{r}},\ r \neq 0 \tag{26}$$

$$a(t) = k_2 \exp(\alpha t),\ r = 0 \tag{27}$$

where k_1 and k_2 are constant of integration.

Thus, the preceding two values of average scale factor correspond to two different models of the universe.

The energy conservation equation involving cosmic string, cosmic energy density, dark energy and dark energy density is given by;

$$\dot{\rho} + 4H(p + \rho) + \lambda H_1 + \dot{\rho_{de}} + 4H(\omega_{de} + 1)\rho_{de} + (\beta H_2 + \gamma(H_3 + H_4))\rho_{de} = 0 \tag{28}$$

Assuming non-interacting behaviour between cosmic string and dark energy, above equation (25) can be separated into three equations.

$$\dot{\rho} + 4H(p + \rho) + \lambda H_1 = 0 \tag{29}$$

$$\dot{\rho_{de}} + 4H(\omega_{de} + 1)\rho_{de} = 0 \tag{30}$$

$$\beta \dot{H_2} + \gamma(H_3 + H_4))\rho_{de} = 0 \tag{31}$$

Eqs. (29) and (30) represents the energy conservation for cosmic string and dark energy. Eq. (31) represents a constraint equation that arises when assuming non-interacting behaviour between cosmic string and dark energy. The choice of these expansion laws is physically motivated by their alignment with current cosmological observations, such as SNIa and CMB data. The power-law model (r ≠ 0) effectively describes the universe's evolution during matter-dominated epochs, whereas the exponential model (r = 0) corresponds to the de Sitter phase, providing a vital description of the late-time accelerated expansion driven by dark energy.

3.1. ***Power law model ($r \neq 0$)***

Solving Eq. (22) by using Eqs. (23), (24) and (26), we have

$$C = k_2 e^{\left[\frac{kd(r\alpha t + k_1)^{\frac{r+m-4}{r}}}{(1-u)\alpha(r+m-4)}\right]} \tag{32}$$

Solving Eq. (24), we get

$$D = k_3 e^{\left[\frac{kdu(r\alpha t+k_1)^{\frac{r+m-4}{r}}}{(1-u)\alpha(r+m-4)}\right]} \tag{33}$$

Using Eqs. (16), (23), (26) and (33), we have

$$A = B = \frac{(r\alpha t+k_1)^{\frac{2}{r}}}{k_4} e^{\left[\frac{kd(1+u)(r\alpha t+k_1)^{\frac{r+m-4}{r}}}{2(u-1)\alpha(r+m-4)}\right]} \tag{34}$$

Using Eqs. (32), (33) and (34) the line element is given by,

$$ds^2 = dt^2 - \frac{(r\alpha t+k_1)^{\frac{2}{r}}}{k_4} e^{\left[\frac{kd(1+u)(r\alpha t+k_1)^{\frac{r+m-4}{r}}}{2(u-1)\alpha(r+m-4)}\right]^2} dx^2 - \frac{(r\alpha t+k_1)^{\frac{2}{r}}}{k_4} e^{\left[\frac{kd(1+u)(r\alpha t+k_1)^{\frac{r+m-4}{r}}}{2(u-1)\alpha(r+m-4)}\right]^2} e^{-2sx} dy^2 - k_2 e^{\left[\frac{kd(r\alpha t+k_1)^{\frac{r+m-4}{r}}}{(1-u)\alpha(r+m-4)}\right]^2} dz^2 - k_3 e^{\left[\frac{kdu(r\alpha t+k_1)^{\frac{r+m-4}{r}}}{(1-u)\alpha(r+m-4)}\right]^2} d\emptyset^2 \tag{35}$$

The physical quantities that are important in cosmology are, volume V, expansion scalar θ, shear scalar σ and Hubble's parameter H. They have derived the following expressions for the power-law model.

Volume, $V = ABCD = (r\alpha t+k_1)^{\frac{4}{r}}$ (36)

Hubble Parameter, $H = \frac{\alpha}{r\alpha t+k_1}$ (37)

Expansion scalar, $\theta = 4H = \frac{4\alpha}{r\alpha t+k_1}$ (38)

Deceleration parameter, $q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = r - 1$, where $r \neq 0$ (39)

Shear scalar, $\sigma^2 = \frac{3}{2}H^2\Delta = \frac{3}{2}\left[\frac{\alpha^2}{(r\alpha t+k_1)^2} + \frac{k\alpha d(1+u)}{u-1}(r\alpha t+k_1)^{\frac{m-r-4}{r}} + \left(\frac{kd}{u-1}\right)^2 \frac{3u^2+2u+3}{8}(r\alpha t+k_1)^{\frac{2m-8}{r}}\right]$ (40)

Anisotropy Parameter,

$$\Delta = \left[1 + \frac{kd(1+u)}{\alpha(u-1)}(r\alpha t+k_1)^{\frac{m+r-4}{r}} + \left(\frac{kd}{\alpha(u-1)}\right)^2 \frac{3u^2+2u+3}{8}(r\alpha t+k_1)^{\frac{2m+2r-8}{r}}\right] \tag{41}$$

To determine the string density, we use $\lambda = \delta\rho$ and $p = \eta\rho$ (42)
where δ and η are non-evolving state parameters.

$$\rho = \frac{\rho_0}{(r\alpha t+k_1)^{\frac{2\delta+4(\eta+1)}{r}}} e^{\left[\frac{\delta kd(1+u)(r\alpha t+k_1)^{\frac{r+m-4}{r}}}{2(u-1)\alpha(r+m-4)}\right]} \tag{43}$$

where ρ_0 is the rest energy density at the present epoch.

String Pressure, $p = \frac{\eta\rho_0}{(r\alpha t+k_1)^{\frac{2\delta+4(\eta+1)}{r}}} e^{\left[\frac{\delta kd(1+u)(r\alpha t+k_1)^{\frac{r+m-4}{r}}}{2(u-1)\alpha(r+m-4)}\right]}$ (44)

$$\text{String Tension density, } \lambda = \frac{\delta\rho_0}{(r\alpha t + k_1)^{\frac{2\delta+4(\eta+1)}{r}}} \, e^{\left[\frac{\delta k d(1+u)(r\alpha t + k_1)^{\frac{r+m-4}{r}}}{2(u-1)\alpha\,(r+m-4)}\right]} \tag{45}$$

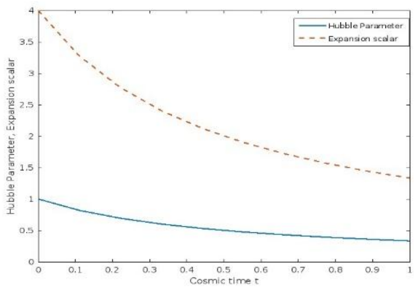

Fig. 1. Hubble parameter, expansion scalar vs cosmic time (t).

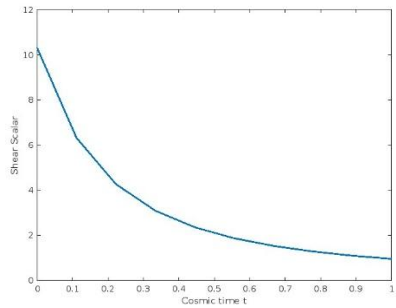

Fig. 2. Shear scalar vs cosmic time (t).

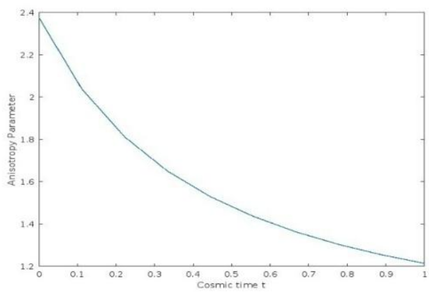

Fig. 3. Anisotropic parameter vs cosmic time (t).

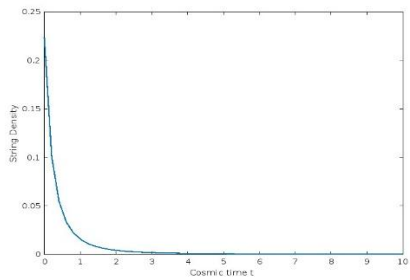

Fig. 4. String density vs cosmic time (t).

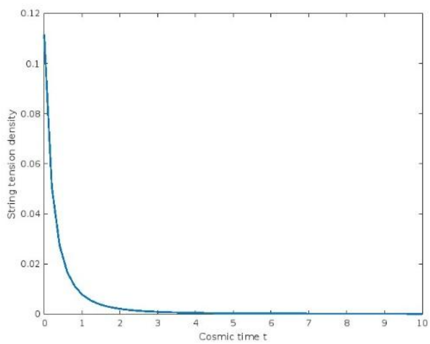

Fig. 5. String tension density vs cosmic time (t).

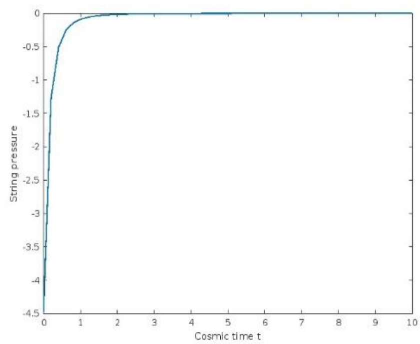

Fig. 6. String pressure vs cosmic time (t).

Fig. 1 illustrates the evolution of the Hubble parameter (H) and the expansion scalar (θ) with respect to cosmic time (t) for our derived model. It is evident that both parameters start at their maximum values in the early universe (t → 0) and decrease as time progresses. Importantly, at late times, they asymptote to a small positive value, successfully demonstrating the current accelerated expansion of the universe predicted by our model.

The geometrical evolution of the model is depicted in Figs. 2 and 3, which show the shear scalar (σ^2) and the mean anisotropy parameter, respectively. Both parameters are observed to be significantly high at the initial singularity ($t \to 0$) and decay rapidly as time progresses. This behavior signifies that the universe in our model begins with a highly anisotropic state, which is a characteristic of Bianchi type spacetimes. Crucially, the eventual decay of both shear and anisotropy towards zero at late times indicates a natural isotropization mechanism. This result is physically significant as it aligns with the observed large-scale homogeneity and isotropy of the present-day universe, making the model observationally viable. The dynamics of the matter content, specifically the cosmic strings, are illustrated in Figs. 4-6. Fig. 4 (String Density, ρ) and Fig. 5 (String Tension Density, λ) show that these quantities are extremely high at the initial moment of creation but rapidly decrease and approach zero as the universe expands. This implies that the cosmic strings have a substantial dynamical influence only during the primordial era, potentially playing a role in early structure formation, but their effect becomes negligible at later times. Furthermore, Fig. 6 shows that the string pressure (p) begins with a large negative value before quickly evolving towards zero. This initial negative pressure is a key feature that could contribute to the early expansion dynamics.

3.2. *Exponential model*

Solving Eq. (22) by using Eqs. (23), (24) and (27), we have,

$$C = k_3 e^{\left[\frac{kd_0 k_2^{(m-4)} e^{\alpha(m-4)t}}{(1-u)\alpha\,(m-4)}\right]}. \qquad (46)$$

From Eq. (23), we get $D = k_3^{\,u} e^{\left[\frac{kd_0 u k_2^{(m-4)} e^{\alpha(m-4)t}}{(1-u)\alpha\,(m-4)}\right]}$ (47)

Using Eqs. (16), (23), (27) and (47) we have

$$A = B = \frac{k_4}{k_3^{\frac{(1+u)}{2}}} e^{\left[2\alpha t + \frac{kd_0 u k_2^{(m-4)} e^{\alpha(m-4)t}}{2(u-1)\alpha\,(m-4)}\right]} \qquad (48)$$

The directional Hubble parameters are given by;

$$H_1 = H_2 = \left[2\alpha + \frac{kd_0(1+u)k_2^{(m-4)} e^{\alpha(m-4)t}}{2(u-1)}\right] \qquad (49)$$

$$H_3 = \frac{kd_0 k_2^{(m-4)} e^{\alpha(m-4)t}}{(1-u)\alpha} \qquad (50)$$

$$H_4 = \frac{kd_0 u k_2^{(m-4)} e^{\alpha(m-4)t}}{(1-u)\alpha} \qquad (51)$$

In cosmology, the key physical parameters include the volume (V), expansion scalar (θ), shear scalar (σ), and Hubble parameter (H). For the exponential model, the corresponding expressions for these quantities have been obtained as follows.

The average Hubble parameter, $H = \alpha$ (52)

Expansion scalar, $\theta = 4\alpha$ (53)

Deceleration parameter, $q = -1$ (54)

Spatial Volume, $V = a^4 = k_2\, e^{\alpha t}$ (55)

Anisotropy Parameter

$$\Delta = \left[1 + \frac{kd_0(1+u)}{\alpha(u-1)} k_2^{(m-4)} e^{\alpha(m-4)t} + \left(\frac{kd_0 k_2^{(m-4)}}{\alpha(u-1)}\right)^2 \left(\frac{3u^2+2u+3}{8}\right) e^{2\alpha(m-4)t}\right] \quad (56)$$

Shear scalar

$$\sigma^2 = \frac{3}{2} H^2 \Delta = \frac{3}{2}\left[\alpha^2 + \frac{k\alpha d_0(1+u)}{(u-1)} k_2^{(m-4)} e^{\alpha(m-4)t} + \left(\frac{kd_0 k_2^{(m-4)}}{(u-1)}\right)^2 \left(\frac{3u^2+2u+3}{8}\right) e^{2\alpha(m-4)t}\right] \quad (57)$$

To determine the string density, using

$$\lambda = \delta\rho \text{ and } p = \eta\rho \quad (58)$$

where δ and η are non-evolving state parameters.

$$\rho = \rho_0 exp.\left[(4\alpha(\eta+1) + 2\alpha\delta)t + \frac{kd_0(1+u)k_2^{(m-4)}\ \delta e^{\alpha(m-4)t}}{2\alpha(u-1)(m-4)}\right] \quad (59)$$

Where ρ_0 is the rest energy density at the present epoch.

String Pressure, $p = \eta\rho_0 e^{\left[(4\alpha(\eta+1)+2\alpha\delta)t + \frac{kd_0(1+u)k_2^{(m-4)}\ \delta e^{\alpha(m-4)t}}{2\alpha(u-1)(m-4)}\right]}$ (60)

String Tension density, $\lambda = \delta\rho_0 e^{\left[(4\alpha(\eta+1)+2\alpha\delta)t + \frac{kd_0(1+u)k_2^{(m-4)}\ \delta e^{\alpha(m-4)t}}{2\alpha(u-1)(m-4)}\right]}$ (61)

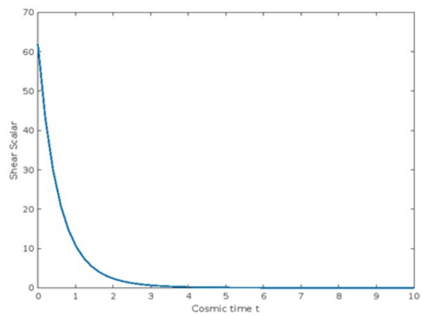

Fig. 7. Shear scalar vs cosmic time (t).

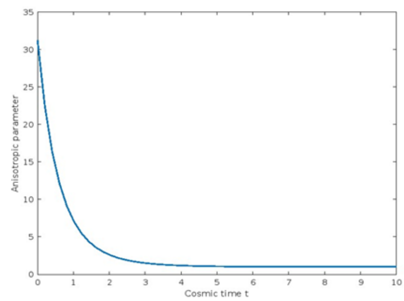

Fig. 8. Anisotropic parameter vs cosmic time (t).

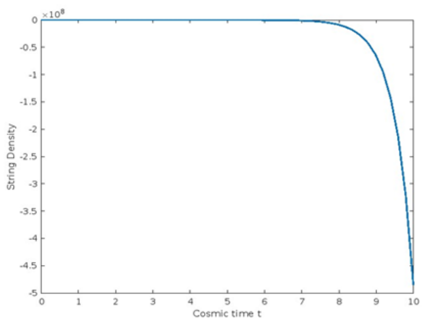

Fig. 9. String density vs cosmic time (t).

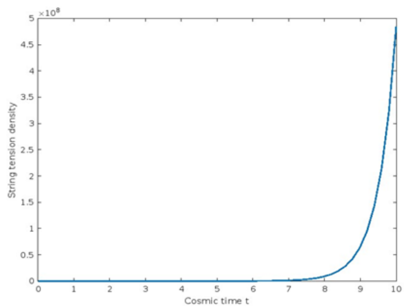

Fig. 10. String tension density vs cosmic time (t).

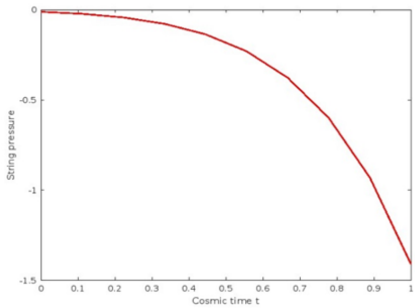

Fig. 11. String pressure vs cosmic time (t).

The dynamical evolution of the exponential model is presented in Figs. 7 through 11. The geometrical properties of the universe are depicted in Figs. 7 and 8, which illustrate the shear scalar (σ^2) and the anisotropy parameter, respectively. Both parameters are extremely high at the initial moment ($t = 0$) but exhibit a sharp exponential decay, effectively vanishing by t ≈ 3. This behavior demonstrates a rapid isotropization process, indicating that the model efficiently evolves from a highly anisotropic primordial state to the isotropic universe observed today, which is a physically desirable outcome. However, the matter dynamics reveal a more complex and problematic nature at late times. Fig. 9 shows that the string density (ρ) remains negligible for a significant period before abruptly becoming large and negative as t approaches 10. In stark contrast, Fig. 10 shows the string tension density (λ) growing to a very large positive value in the same time frame. This late-time behavior, where energy density becomes negative, is physically unconventional and suggests the possible onset of a future singularity, such as a "Big Rip." This is complemented by Fig. 11, which shows the string pressure (p) becoming increasingly negative over time, acting as the driving force for this exotic late-time evolution. In summary, while the model successfully achieves isotropization, its late-time dynamics are dominated by an unstable string fluid with negative energy density, pointing to potential physical limitations of the model at later cosmic epochs.

5. Conclusion

In this work, a higher-dimensional Bianchi type- III cosmological model was constructed within the framework of $f(R)$ gravity by incorporating cosmic strings and dark energy. By adopting physically motivated assumptions, including a power-law form of $F(R)$, a proportionality relation between shear and expansion scalars, and Berman's law for the Hubble parameter, exact solutions to the non-linear field equations were obtained. This approach resulted in two distinct cosmological models: a power-law model and an exponential model, both describing anisotropic cosmic evolution in modified gravity. The analysis reveals that the power-law model provides a physically viable and observationally consistent cosmological scenario. It successfully describes the transition from an initially anisotropic universe to the present isotropic state, with decaying shear and string energy density, and explains the shift from early deceleration to late-time accelerated expansion. In contrast, although the exponential model leads to rapid isotropization and a de Sitter–like expansion, it exhibits unphysical late-time behavior due to negative string energy density and unbounded tension. Therefore, the power-law solution emerges as the more realistic model, demonstrating the potential of $f(R)$ gravity to describe anisotropic cosmological evolution while emphasizing the sensitivity of solutions to underlying assumptions.

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