Journal of Statistical Research *https://doi.org/10.3329/jsr.v57i12.72969*
2023, Vol. 57, No. 1-2, pp. 69-79 *ISSN 0256 - 422 X*

CONFIDENCE INTERVALS FOR MEAN RESPONSE TIMES OF A OPEN QUEUEING NETWORK: CALIBRATION APPROACH

SURESH B. PATHARE
School of Mathematics, Applied Statistics & Analytics,
SVKM's NMIMS (Deemed to be University), Navi Mumbai, Maharashtra, INDIA

ABSTRACT

Response time is a key factor in the analysis of the different queueing network model properties. This study computes a series of response times using a data-based recurrence relation. The true average response time is determined using the sample averages from those response times. Several confidence intervals are created for the open queueing network model's average response times using the calibration approach. Using a numerical simulation analysis, the accuracy of the various confidence intervals is evaluated.

Keywords and phrases: Calibration method, Relative coverage, Response time, Relative average length, Coverage percentage.

AMS Classification: 60K20, 60K25, 60K30, 68M20, 90B22

1 Introduction

The amount of time a user takes from the time they arrive until they depart is known as their response time. Examine the two-phase open queueing network architecture. Two nodes make up this system, and they have different service rates μ_1 and μ_2 respectively. The rate of external arrivals is λ. There is very few published research on statistical inference in queueing networks, and those that have largely dealt with parametric statistical inference, that is, in situations where the population distribution takes a certain form were conducted in the past. According to Jackson's (1957) theorem, every node functions as a separate queue. So far very few researchers have studied the non-parametric statistical inferences. Efron and Tibshirani (1993) developed and proposed the bootstrap technique to estimate the sampling distribution of any statistic. In addition to the conventional bootstrap (SB) method, the Bayesian bootstrap (BB) resampling technique was introduced by Rubin (1981). For an M/G/1 FCFS queueing system, Chu and Ke (2006) created new confidence intervals for the mean response time. Additionally, they computed the coverage probability and the average length of confidence intervals to assess the accuracy of bootstrap confidence intervals. Through the computation of a series average response time and the average response time's confidence intervals, Chu and Ke (2007) used simulation to create a data-driven recurrence relation for the G/G/1 queuing system. The nonparametric statistical estimation techniques for average response times of several queueing network models have been examined by Gedam and Pathare (2015 & 2019). For the mean reaction times of various queueing network models, they constructed confidence intervals. Further they developed a calibration technique to increase the coverage accuracy of confidence intervals in queueing network models. There is very little study on the calibration technique used in queueing networks. This encourages us to create confidence intervals for the average response times using the calibration approach and to derive nonparametric statistical conclusions about the average response times for queueing network models. Any approximate confidence interval system can have its coverage accuracy increased by using the calibrating procedure. The purpose of the bootstrap calibration method is to calibrate the confidence intervals by comparing them to the intended nominal level after estimating the true coverage of the intervals using bootstrap. Section 2 discusses estimation of average response time by nonparametric method. Section 3 discusses calibration method. In section 4, calibrated confidence intervals for the mean response time are covered. In Section 5, a numerical simulation analysis is performed. The corresponding tables display all of the simulation findings and highlight the various estimating methodologies' performances. Section 6 presents a few conclusions.

2 Estimation of Mean Response Time by Nonparametric Method

Consider $(A_i, S_i, i = 1, 2)$ are the continuous random variables where A_i represents inter-arrival times and S_i represents service times of distinct nodes of a queueing network. Service times and inter-arrival times are independent of each other.

Let $A_{11}, A_{12}, \ldots, A_{1n}$ and $A_{21}, A_{22}, \ldots, A_{2n}$ be a random sample drawn from A_1 and A_2 respectively, where A_{ij} stands for the times between arrivals. Let $S_{11}, S_{12}, \ldots, S_{1n}$ and $S_{21}, S_{22}, \ldots, S_{2n}$ be a random sample drawn from S_1 and S_2 respectively, where S_{ij} stands for the duration of service for j^{th} person at i^{th} queueing network node. Let U_{ij} & V_{ij} denotes respectively response time & j^{th} person's waiting time at the i^{th} node. Then

$$U_{ij} = V_{ij} + S_{ij}, i = 1, 2, j = 1, 2, \ldots, n. \quad (1)$$

Additionally, we can assess V_{ij} using recurrence relation as

$$V_{ij} = (U_{i,j-1} - A_{ij}) I(U_{i,j-1} > A_{ij}), \quad (2)$$

for $i = 1, 2, j = 1, 2, \ldots, n$ and $V_{i1} = 0, i = 1, 2$ and $I(.)$ denote the indicator function. Using equation (1) we get

$$U_{ij} = (U_{i,j-1} - A_{ij}) I(U_{i,j-1} > A_{ij}) + S_{ij}, \quad (3)$$

for $i = 1, 2, j = 2, 3, \ldots, n$ and $U_{i1} = S_{i1}, i = 1, 2$. Equation (3) is the exact data based recurrence relation for calculating response times $U_{ij}, i = 1, 2, j = 1, 2, \ldots, n$ that are exactly as a sequence of customer's response times for queueing network. Hence

$$\hat{u}_i = \frac{1}{n} \sum_{j=1}^{n} U_{ij}, \;\; i = 1, 2. \quad (4)$$

These response times' arithmetic average serves as the average response duration for a queueing network's natural estimator. By the Strong Law of Large Numbers (Rousses, 1997) , $\hat{u}_i$ is strongly consistent estimator of u_i . Since the precise distributions of A_i and S_i are rarely known in practice, it is challenging to determine the distribution of $\hat{u}_i$

Hence assuming independence of A_i and S_i , the asymptotic distributions of $\hat{u}_i$ can be developed. By Slutsky's theorem (Hogg and Craig, 1995) we have

$$\sqrt{n}(\hat{u}_i - \; u_i) \xrightarrow{D} N(0, \sigma_i^2); i = 1, 2,$$

where $\hat{\sigma}_i^2$ is the variance of U_{ij} and $\xrightarrow{D}$ denotes convergence in distribution. Then $\hat{\sigma}_i^2, \ \ i = 1, 2$ is a strongly consistent estimator of σ_i^2. Again applying the Slutsky's theorem we have

$$\frac{\sqrt{n}(\hat{u}_i - \ u_i)}{\hat{\sigma}_i} \xrightarrow{D} N(0,1), \ \ i = 1, 2.$$

Thus $\hat{u}_i, i = 1, 2$ is a strongly consistent and asymptotically normal (CAN) estimator with approximate variances $\hat{\sigma}_i^2/n, \ \ i = 1, 2$.

3 Calibration Method

Loh (1987) introduced the bootstrap calibration technique. In most cases, the actual coverage and the desired coverage of a confidence procedure differ significantly. Most people are unaware of the calibration function $\beta_i(\alpha)$ where $\beta_i(\alpha) = P_{F_i}[u_i \leq \hat{u}_i[\alpha]]$ and where F_i is unknown continuous probability distribution. Accurate coverage might be obtained by calibrating an approximate confidence interval if the function $\beta_i(\alpha)$ was known. To estimate $\beta_i(\alpha)$ we use the bootstrap method. The bootstrap estimate of $\beta_i(\alpha)$ is $\hat{\beta}_i(\alpha) = P_{\hat{F}_i}[\hat{u}_i \leq \hat{u}_i[\alpha]^*]$ where $\hat{u}_i[\alpha]^*$ is the α^{th} confidence limit based on bootstrap dataset from $\hat{F}_i$. Also $\hat{F}_i$ & $\hat{u}_i$ are fixed. By taking B bootstrap data sets the estimate of $\hat{\beta}_i(\alpha)$ is obtained and seeing what proportion of them have $\hat{u}_i \leq \hat{u}_i[\alpha]^*$.

4 Different Calibrated Confidence Intervals for Mean Response Time

4.1 Calibrated consistent and asymptotically normal (CAN) confidence interval

Using CAN estimators, we construct confidence intervals for $u_i, i = 1, 2$. Assume z_α is the standard normal distribution's upper α^{th} quantile. Evaluate $\hat{\beta}(\alpha_1) = P[u_i \leq \hat{u}_i - z_{\alpha/2}\hat{\sigma}_i/\sqrt{n}]$ and $\hat{\beta}(\alpha_2) = P[u_i \leq \hat{u}_i + z_{\alpha/2}\hat{\sigma}_i/\sqrt{n}]$ where $\alpha_2 = 1 - \alpha_1$ and $0 \leq \alpha_1 \leq 1$. Then approximate $100(1 - \alpha)\%$ calibrated CAN confidence intervals for mean response times $u_i, i = 1, 2$ are given as

$$\left(\hat{u}'_i - z_{(\hat{\beta}(\alpha_1)/2)}\hat{\sigma}_i/\sqrt{n} \ , \ \hat{u}'_i + z_{(\hat{\beta}(\alpha_2)/2)}\hat{\sigma}_i/\sqrt{n}\right), \ \ i = 1, 2. \tag{5}$$

The calibrated CAN confidence intervals approach the calibrated normal confidence intervals for large enough values of n.

4.2 Calibrated Confidence Intervals using Students t distribution (Exact-t)

Let t_α be the upper α^{th} quantile of the student's t-distribution. Evaluate $\hat{\beta}(\alpha_3) = P[u_i \leq \hat{u}_i - t_{\alpha/2,(n-1)}\hat{\sigma}_i/\sqrt{n}]$ and $\hat{\beta}(\alpha_4) = P[u_i \leq \hat{u}_i + t_{\alpha/2,(n-1)}\hat{\sigma}_i/\sqrt{n}]$ where $\alpha_4 = 1 - \alpha_3$ and $0 \leq \alpha_3 \leq 1$ Then approximate $100(1 - \alpha)\%$ calibrated t-confidence intervals for mean response times $u_i, i = 1, 2$ are as follows:

$$\left(\hat{u}'_i - t_{(\hat{\beta}(\alpha_3)/2,(n-1)}\hat{\sigma}_i/\sqrt{n} \ , \ \hat{u}'_i + t_{(\hat{\beta}(\alpha_4)/2,(n-1)}\hat{\sigma}_i/\sqrt{n}\right), \ \ i = 1, 2. \tag{6}$$

4.3 Calibrated confidence intervals using standard bootstrap (SB) method

A simple random sample $(A^*_{ij}, S^*_{ij}, i = 1,2, j = 1,2,\ldots,n)$ is obtained using the empirical distribution function of $(A_{ij}, S_{ij}, i = 1,2, j = 1,2,\ldots,n)$ as per the bootstrap procedure. Using equation (3) we get sequence of person's response time as $u_{ij}, i = 1,2, j = 1,2,\ldots,n$. On similar way we can obtain $u^*_{ij}, i = 1,2, j = 1,2,\ldots,n$. It follows that $\hat{u}_i = \frac{1}{n}\sum_{j=1}^{n} u_{ij},\ \ i = 1,2$ is estimate of the queuing network's average response time $u_i, i = 1,2$ and its bootstrap estimate is $\hat{u}^*_i = \frac{1}{n}\sum_{j=1}^{n} u^*_{ij},\ \ i = 1,2$. The above re-sampling process can be repeated N times. The N bootstrap estimates $\hat{u}^*_{i1}, \hat{u}^*_{i2}, \ldots, \hat{u}^*_{iN}, i = 1,2$ can be computed from the bootstrap resamples. Averaging the N bootstrap estimates we get

$$\hat{u}_N(i) = \frac{1}{N}\sum_{j=1}^{N} \hat{u}^*_{ij},\ \ i = 1,2$$

is the bootstrap estimate of $u_i, i = 1,2$ and standard deviation as

$$sd(\hat{u}_N(i)) = \left[\frac{1}{N-1}\sum_{j=1}^{N}\left[u^*_{ij} - \hat{u}_N(i)\right]^2\right]^{\frac{1}{2}},\ \ i = 1,2.$$

The distribution of $\hat{u}_i, i = 1,2$ is approximately normal by central limit theorem. After computing $\hat{\beta}(\alpha_5) = P[u_i \leq \hat{u}_i - z_{\alpha/2}\hat{\sigma}_i/\sqrt{n}]$ and $\hat{\beta}(\alpha_6) = P[u_i \leq \hat{u}_i + z_{\alpha/2}\hat{\sigma}_i/\sqrt{n}]$ where $\alpha_6 = 1 - \alpha_5$ and $0 \leq \alpha_5 \leq 1$ we get approximate $100(1-\alpha)\%$ calibrated SB confidence intervals for mean response times $u_i, i = 1,2$ as

$$\left(\hat{u}'_i - z_{(\hat{\beta}(\alpha_5)/2)} sd(\hat{u}_N(i))\ ,\ \hat{u}'_i + z_{(\hat{\beta}(\alpha_6)/2)} sd(\hat{u}_N(i))\right),\ \ i = 1,2. \tag{7}$$

4.4 Calibrated confidence intervals using bootstrap - t Method

The N bootstrap estimates $\hat{u}^*_{i1}, \hat{u}^*_{i2}, \ldots, \hat{u}^*_{iN}, i = 1,2$ are obtained from the bootstrap resamples. Compute $Z^*_{ij} = \frac{(\hat{u}^*_{ij} - \hat{u}_N(i))}{sd(\hat{u}_N(i))}\ \ i = 1,2, j = 1,2,\ldots,N$ and sample $Z^*_{i1}, Z^*_{i2}, \ldots, Z^*_{iN}, i = 1,2$ follow roughly t-distribution. Now compute $\hat{\beta}(\alpha_7) = P[u_i \leq \hat{u}_i - z_{\alpha/2}\hat{\sigma}_i/\sqrt{n}]$ and $\hat{\beta}(\alpha_8) = P[u_i \leq \hat{u}_i + z_{\alpha/2}\hat{\sigma}_i/\sqrt{n}]$ where $\alpha_8 = 1 - \alpha_7$ and $0 \leq \alpha_7 \leq 1$. Then approximate $100(1-\alpha)\%$ calibrated SB confidence intervals for mean response times $u_i, i = 1,2$ as

$$\left(\hat{u}'_i - z_{(\hat{\beta}(\alpha_7)/2)} sd(\hat{u}_N(i))\ ,\ \hat{u}'_i + z_{(\hat{\beta}(\alpha_8)/2)} sd(\hat{u}_N(i))\right),\ \ i = 1,2. \tag{8}$$

4.5 Calibrated confidence intervals using variance-stabilized bootstrap-t (VST) method

Let $\hat{u}_i, i = 1,2$ is a strongly consistent and asymptotically normal estimator with approximate variances $\hat{\sigma}^2_i/n, i = 1,2$ and consider $\hat{\sigma}_i = \phi(\hat{u}_i)$.

By taking into account the expectations on both sides and expanding the Taylor series to the first order, we find a transformation $f(\hat{r}_i)$ that is $Var(f(\hat{u}_i))$ roughly constant.

$$f(\hat{u}_i) \approx f(u_i) + (\hat{u}_i - u_i)f'(u_i) \Rightarrow [f(\hat{u}_i) - f(u_i)]^2 \approx (\hat{u}_i - u_i)^2 (f'(u_i))^2,\ \ i = 1,2.$$

Taking expectations on both sides, we get

$$Var[f(\hat{u}_i)] \approx Var(\hat{u}_i)(f'(u_i))^2 = (\phi(u_i))^2(f'(u_i))^2, \;\; i = 1, 2.$$

Now consider $f(\hat{u}_i) = \sqrt{n}\log(\phi(\hat{u}_i)), i = 1, 2$ is the variance-stabilizing transformation. Then we have,

$$V[f(\hat{u}_i)] \approx \left(\frac{\sqrt{n}}{\phi(\hat{u}_i)}\right)^2 \;\; Var[\hat{u}_i] = \left(\frac{\sqrt{n}}{\hat{\sigma}_i}\right)^2 \;\; Var[\hat{u}_i] = \frac{n}{\hat{\sigma}_i^2}\frac{\hat{\sigma}_i^2}{n} = 1, i = 1, 2.$$

Consider N bootstrap estimates $\hat{u}_{i1}^*, \hat{u}_{i2}^*, \ldots, \hat{u}_{iN}^*, i = 1, 2$ computed from the bootstrap resamples and evaluate

$$\theta_{ij}^* = \sqrt{n}\log(\hat{u}_{ij}^*) - \sqrt{n}\log(\hat{u}_i), i = 1, 2, j = 1, 2, \ldots, N.$$

Now compute $\hat{\beta}(\alpha_9) = P[u_i \leq e^{\log(\hat{u}_i) - \frac{1}{\sqrt{n}}\hat{v}_i t_{1-\alpha/2}}]$ and $\hat{\beta}(\alpha_{10}) = P[u_i \leq e^{\log(\hat{u}_i) - \frac{1}{\sqrt{n}}\hat{v}_i t_{\alpha/2}}]$ where $\alpha_{10} = 1 - \alpha_9$ and $0 \leq \alpha_9 \leq 1$. Then approximate $100(1 - \alpha)\%$ calibrated confidence intervals for mean response times $u_i, i = 1, 2$ using VST as

$$\left(e^{\log(\hat{u}_i) - \frac{1}{\sqrt{n}}\hat{v}_i t_{\hat{\beta}(\alpha_9)}}, \;\; e^{\log(\hat{u}_i) - \frac{1}{\sqrt{n}}\hat{v}_i t_{\hat{\beta}(\alpha_{10})}}\right), \tag{9}$$

where $\hat{v}_i t_{\hat{\beta}(\alpha_9)}$ and $\hat{v}_i t_{\hat{\beta}(\alpha_{10})}$ are the percentiles of the random samples.

4.6 Calibrated confidence intervals using percentile bootstrap (PB) method

The bootstrap distribution of $\hat{u}_i, i = 1, 2$ is $\hat{u}_{i1}^*, \hat{u}_{i2}^*, \ldots, \hat{u}_{iN}^*, i = 1, 2$. Let order statistics be $\hat{u}_i^*(1), \hat{u}_i^*(2), \ldots, \hat{u}_i^*(N), i = 1, 2$ of $\hat{u}_{i1}^*, \hat{u}_{i2}^*, \ldots, \hat{u}_{iN}^*, i = 1, 2$. Now compute $\hat{\beta}(\alpha_{11}) = P[u_i \leq \hat{u}_i^*([N(\alpha/2)])$ and $\hat{\beta}(\alpha_{12}) = P[u_i \leq \hat{u}_i^*([N(1-\alpha/2)])$. The $100(1-\alpha)\%$ calibrated confidence interval for $u_i, i = 1, 2$ is then obtained by using the $100(\alpha/2)^{th}$ and $100(1 - \alpha/2)^{th}$ percentage points of the bootstrap distribution as

$$\left(\hat{u}_i^*\left(\left[N\left(\frac{\hat{\beta}(\alpha_{11})}{2}\right)\right]\right), \;\; \hat{u}_i^*\left(\left[N\left(\frac{\hat{\beta}(\alpha_{12})}{2}\right)\right]\right)\right), \;\; i = 1, 2, \tag{10}$$

where $[x]$ denotes the greatest integer less than or equal to x.

Table 1: Description of various queuing network models were simulated with $a_1 \geq 0$, $a_2 \geq 0$, $s_1 \geq 0$, and $s_2 \geq 0$

Queueing Network	Model for Inter-arrival time	Model for service time
$E_4/H_4^{Pe}/1$ to	$f(a_1) = \frac{128}{3}a_1^3e^{-4a_1}$	$f(s_1) = 0.02e^{-0.2s_1} + 0.16e^{-0.8s_1} + 0.48e^{-1.6s_1/3} + 2.56e^{-6.4s_1}$
$H_4^{Pe}/E_4/1$	$f(a_2) = 0.02e^{-0.2a_2} + 0.16e^{-0.8a_2} + 0.48e^{-1.6a_2/3} + 2.56e^{-6.4a_2}$	$f(s_2) = \frac{1}{96}s_2^3e^{-s_2/2}$
$H_4^{Pe}/H_4^{Po}/1$ to	$f(a_1) = \frac{3}{40}e^{-0.2a_1} + \frac{1}{10}e^{-0.8a_1} + \frac{2}{5}e^{-1.6a_1} + \frac{8}{5}e^{-6.4a_1}$	$f(s_1) = 2e^{-2s_1} + 4e^{-4s_1} + \frac{16}{3}e^{-16s_1/3} + 16e^{-16s_1}$
$H_4^{Po}/H_4^{Pe}/1$	$f(a_2) = 2e^{-2a_2} + 4e^{-4a_2} + \frac{16}{3}e^{-16a_2/3} + 16e^{-16a_2}$	$f(s_2) = 0.1e^{-s_2} + 0.4e^{-2s_2} + 0.8e^{-8s_2/3} + 3.2e^{-8s_2}$
$E_4/H_4^{Po}/1$ to	$f(a_1) = \frac{128}{3}a_1^3e^{-4a_1}$	$f(s_1) = e^{-s_1} + 2e^{-2s_1} + \frac{8}{3}e^{-8s_1/3} + 8e^{-8s_1}$
$H_4^{Po}/E_4/1$	$f(a_2) = e^{-a_2} + 2e^{-2a_2} + \frac{8}{3}e^{-8a_2/3} + 8e^{-8a_2}$	$f(s_2) = \frac{1}{96}s_2^3e^{-s_2/2}$

Table 2: Consistency of simulation analysis when $n = 100\ and\ 200$

		The mean of 1000 simulated response time	
Simulated Models	The true response time	$n = 100$	$n = 200$
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	r_1=1.02302	$\hat{r}_1$=1.02128	$\hat{r}_1$=1.02626
	r_2=0.58672	$\hat{r}_2$=0.58470	$\hat{r}_2$=0.58444
$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$	r_1=1.33233	$\hat{r}_1$=1.32091	$\hat{r}_1$=1.33013
	r_2=0.38887	$\hat{r}_2$=0.38964	$\hat{r}_2$=0.38930
$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	r_1=2.33751	$\hat{r}_1$=2.33527	$\hat{r}_1$=2.33276
	r_2=0.51244	$\hat{r}_2$=0.51167	$\hat{r}_2$=0.51245

Table 3: Consistency of simulation analysis when $n = 15$ and 25

		The average of 1000 simulated response times	
Simulated Models	The true response time	$n = 15$	$n = 25$
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	r_1=1.02302	$\hat{r}_1$=1.02434	$\hat{r}_1$=1.01430
	r_2=0.58672	$\hat{r}_2$=0.57245	$\hat{r}_2$=0.58277
$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$	r_1=1.33233	$\hat{r}_1$=1.29252	$\hat{r}_1$=1.30199
	r_2=0.38887	$\hat{r}_2$=0.38499	$\hat{r}_2$=0.38993
$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	r_1=2.33751	$\hat{r}_1$=2.28803	$\hat{r}_1$=2.31360
	r_2=0.51244	$\hat{r}_2$=0.50942	$\hat{r}_2$=0.51085

Table 4: Simulation Results of different models

Estimation	n=100		n=200		Coverage Percentages		Relative Coverage		Average Lengths		Relative Average Length	
Approches	α	$1-\alpha$	α	$1-\alpha$	$n=100$	$n=200$	$n=100$	$n=200$	$n=100$	$n=200$	$n=100$	$n=200$
						$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$						
Normal1	0.059	0.924	0.046	0.906	0.808	0.801	**4.443**	**6.157**	0.182	0.130	**0.178**	**0.127**
Normal2	0.120	0.798	0.136	0.844	0.443	0.469	**7.021**	**9.800**	0.063	0.048	**0.108**	**0.082**
Boott1	0.035	0.928	0.024	0.918	0.879	0.889	4.124	5.663	0.213	0.157	0.209	0.153
Boott2	0.019	0.883	0.026	0.924	0.854	0.892	5.533	7.339	0.154	0.122	0.264	0.208
VST1	0.057	0.935	0.052	0.920	0.846	0.840	4.124	5.827	0.205	0.144	0.201	0.140
VST2	0.081	0.850	0.077	0.898	0.647	0.707	5.280	7.098	0.123	0.100	0.210	0.170
SB1	0.043	0.935	0.028	0.922	0.867	0.877	4.053	5.557	0.214	0.158	0.209	0.154
SB2	0.032	0.902	0.032	0.941	0.831	0.894	4.938	6.871	0.168	0.130	0.288	0.223
PB1	0.053	0.918	0.047	0.898	0.797	0.789	4.223	5.934	0.189	0.133	0.185	0.130
PB2	0.058	0.854	0.072	0.890	0.687	0.636	5.301	6.804	0.130	0.093	0.185	0.130
						$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$						
Normal1	0.149	0.743	0.156	0.766	0.320	0.319	**2.196**	**2.944**	0.146	0.108	**0.110**	**0.081**
Normal2	0.054	0.872	0.095	0.897	0.686	0.688	**12.011**	**18.163**	0.057	0.038	**0.147**	**0.097**
Boott1	0.013	0.871	0.030	0.912	0.850	0.876	1.731	2.334	0.491	0.375	0.372	0.282
Boott2	0.017	0.905	0.031	0.931	0.873	0.903	10.102	14.678	0.086	0.062	0.222	0.158
VST1	0.077	0.847	0.086	0.854	0.629	0.618	1.567	2.127	0.401	0.291	0.304	0.218
VST2	0.044	0.897	0.071	0.920	0.787	0.784	10.039	14.638	0.078	0.054	0.201	0.138
SB1	0.019	0.911	0.039	0.921	0.863	0.863	1.418	2.148	0.609	0.402	0.461	0.302
SB2	0.019	0.912	0.042	0.936	0.854	0.891	9.365	14.624	0.091	0.061	0.234	0.157
PB1	0.074	0.841	0.085	0.859	0.618	0.613	1.544	2.141	0.400	0.286	0.303	0.215
PB2	0.043	0.880	0.057	0.883	0.698	0.730	10.192	14.772	0.068	0.049	0.303	0.215
						$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$						
Normal1	0.114	0.772	0.130	0.818	0.393	0.463	**1.408**	**2.184**	0.279	0.212	**0.120**	**0.091**
Normal2	0.055	0.921	0.061	0.918	0.829	0.812	**10.653**	**15.039**	0.078	0.054	**0.152**	**0.105**
Boott1	0.009	0.870	0.023	0.916	0.858	0.881	1.172	1.596	0.732	0.552	0.313	0.237
Boott2	0.038	0.928	0.048	0.937	0.884	0.867	9.835	13.600	0.090	0.064	0.176	0.124
VST1	0.038	0.845	0.067	0.889	0.711	0.712	1.102	1.539	0.645	0.463	0.276	0.198
VST2	0.071	0.933	0.066	0.936	0.835	0.825	10.022	13.596	0.083	0.061	0.163	0.118
SB1	0.015	0.896	0.029	0.933	0.811	0.883	0.961	1.485	0.844	0.595	0.361	0.255
SB2	0.043	0.937	0.051	0.940	0.882	0.873	9.660	13.647	0.091	0.064	0.178	0.125
PB1	0.052	0.864	0.053	0.877	0.694	0.683	1.107	1.587	0.627	0.430	0.268	0.184
PB2	0.052	0.917	0.060	0.920	0.852	0.810	9.808	13.740	0.087	0.059	0.268	0.184

5 A Numerical Simulation Study

To assess the accuracy of calibrated confidence intervals numerical simulation is conducted. Based on simulation studies, we discover that higher confidence interval coverage percentages are frequently the result of higher interval estimate methods' standard deviations. Moreover, lower coverage percentages are frequently the outcome of more constrained confidence ranges. Therefore, neither the average length nor the coverage percentage are useful when evaluating interval estimate techniques. We take into account a metric called relative coverage to assess the effectiveness of interval estimate techniques. The coverage percentage divided by the calibrated confidence interval standard deviation yields the relative coverage. Furthermore, lower coverage percentages are often the result of more restricted confidence ranges. The related calibrated confidence intervals perform better the higher the relative coverage. The ratio of average length to response time is known as relative average length. It is more informative if the interval is modest for a given confidence level. Shorter relative average length, therefore, indicates better results from the associated calibrated confidence range. The coverage accuracy, relative average length and relative coverage of the various calibrated confidence intervals are assessed, but the consistency of $u_i, i = 1, 2$ is investigated by comparing the true value of $u_i, i = 1, 2$ with the average of simulated estimates $\hat{u}_i, i = 1, 2$.

In a simulated study, we have selected queueing network models, as indicated in Table 1, to achieve these goals. The true values of u_i and simulated sample values of $\hat{u}_i$ for large sample size $n \geq 10^7$ are shown in Table 1. Tables 2 and 3 display the estimated average response time for other queueing network model considered for the research. Here $H_4^{P_e}$ represents 4-stage hyper-exponential distribution, $H_4^{P_o}$ represents 4-stage hypo-exponential distribution, E_4 represents 4-stage Erlang distribution.

Therefore, for each queueing network listed in Table 1, sample of size(n) $15, 25, 100, 200$ are drawn from the original samples. From the original samples $N = 1000$ bootstrap resamples are drawn. Then from equations (5) to (10) we obtain calibrated Normal, CAN, Exact-t, Boot-t, SB, PB and VST confidence intervals for response time u_i with confidence level 90%. The aforementioned simulation process is repeated 1000 times and we calculate average lengths, relative average lengths, coverage percentages, and relative coverage. All results are shown in Tables 4 & 5.

According to the simulation findings we see that coverage percentages and relative coverage are increasing but average lengths and relative average lengths are decreasing with sample size n. Out of nearly all confidence intervals, the Percentile Bootstrap approach has the highest coverage percentage. As n increases, the coverage percentage gets closer to 90. With increasing sample size n, the relative average lengths of all methods decrease, with the Normal approach having the lowest relative average length. The normal method has the smallest relative average lengths and the highest relative coverage among all estimation techniques.

Lastly, of all the estimation techniques for large samples, the best calibrated confidence intervals for the average response time are constructed using the normal method; for small samples, the best calibrated confidence intervals are constructed using the CAN method (Results are shown in Table 6).

Table 5: Simulation Results of different models

Estimation Approches	n=15 α	n=15 $1-\alpha$	n=25 α	n=25 $1-\alpha$	Coverage Percentages $n=15$	Coverage Percentages $n=25$	Relative Coverage $n=15$	Relative Coverage $n=25$	Average Lengths $n=15$	Average Lengths $n=25$	Relative Average Length $n=15$	Relative Average Length $n=25$
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$												
CAN1	0.041	0.878	0.033	0.911	0.737	0.819	**1.712**	**2.163**	0.430	0.379	0.420	0.373
CAN2	0.100	0.810	0.094	0.799	0.524	0.474	**3.315**	**3.710**	0.158	0.128	**0.276**	**0.219**
Exact-t1	0.031	0.893	0.026	0.918	0.794	0.845	1.609	2.042	0.494	0.414	0.482	0.408
Exact-t2	0.092	0.829	0.089	0.811	0.557	0.499	3.195	3.681	0.174	0.136	0.305	0.233
Boott1	0.021	0.861	0.017	0.911	0.785	0.867	1.708	2.057	0.460	0.421	0.449	0.415
Boott2	0.007	0.861	0.009	0.859	0.861	0.839	2.526	3.039	0.341	0.276	0.595	0.474
VST1	0.069	0.907	0.047	0.937	0.774	0.847	1.693	2.023	0.457	0.419	0.446	0.413
VST2	0.049	0.885	0.054	0.858	0.757	0.708	2.278	2.795	0.332	0.253	0.580	0.435
SB1	0.029	0.877	0.022	0.915	0.772	0.845	1.595	1.951	0.484	0.433	0.472	0.427
SB2	0.012	0.894	0.013	0.878	0.829	0.778	1.809	2.165	0.458	0.359	0.801	0.617
PB1	0.029	0.862	0.038	0.911	0.706	0.783	1.675	2.110	0.422	0.371	**0.412**	**0.366**
PB2	0.042	0.832	0.043	0.817	0.624	0.616	2.341	2.830	0.267	0.218	0.412	0.366
$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$												
CAN1	0.109	0.707	0.129	0.739	0.364	0.354	**1.097**	**1.274**	0.332	0.278	**0.257**	**0.213**
CAN2	0.067	0.839	0.065	0.846	0.652	0.658	**5.176**	**6.504**	0.126	0.101	**0.327**	**0.259**
Exact-t1	0.098	0.727	0.122	0.748	0.393	0.367	1.064	1.249	0.369	0.294	0.286	0.226
Exact-t2	0.051	0.853	0.060	0.852	0.699	0.685	4.862	6.400	0.144	0.107	0.373	0.274
Boott1	0.005	0.793	0.005	0.828	0.741	0.799	0.887	1.002	0.835	0.798	0.646	0.613
Boott2	0.013	0.841	0.011	0.864	0.789	0.832	4.447	5.489	0.177	0.152	0.461	0.389
VST1	0.052	0.811	0.059	0.832	0.639	0.657	0.729	0.861	0.876	0.763	0.678	0.586
VST2	0.062	0.881	0.054	0.892	0.749	0.763	4.285	5.362	0.175	0.142	0.454	0.365
SB1	0.008	0.840	0.008	0.869	0.717	0.782	0.539	0.641	1.330	1.220	1.029	0.937
SB2	0.019	0.856	0.021	0.883	0.790	0.812	3.864	4.930	0.204	0.165	0.531	0.422
PB1	0.038	0.797	0.049	0.809	0.558	0.573	0.789	0.887	0.707	0.646	0.547	0.496
PB2	0.044	0.829	0.042	0.838	0.692	0.702	4.479	5.776	0.155	0.122	0.547	0.496
$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$												
CAN1	0.079	0.762	0.088	0.773	0.471	0.436	**0.657**	**0.775**	0.717	0.563	**0.314**	**0.243**
CAN2	0.051	0.883	0.049	0.907	0.782	0.811	**4.252**	**5.328**	0.184	0.152	**0.361**	**0.298**
Exact-t1	0.070	0.773	0.080	0.790	0.507	0.464	0.643	0.763	0.788	0.608	0.345	0.263
Exact-t2	0.039	0.894	0.044	0.917	0.827	0.837	3.959	5.105	0.209	0.164	0.410	0.321
Boott1	0.002	0.793	0.003	0.839	0.712	0.808	0.505	0.638	1.411	1.267	0.617	0.548
Boott2	0.031	0.879	0.028	0.909	0.815	0.864	4.084	5.017	0.200	0.172	0.392	0.337
VST1	0.032	0.841	0.037	0.860	0.697	0.726	0.439	0.577	1.588	1.258	0.694	0.544
VST2	0.076	0.913	0.072	0.931	0.794	0.812	4.051	4.987	0.196	0.163	0.385	0.319
SB1	0.003	0.829	0.004	0.875	0.724	0.777	0.342	0.432	2.114	1.799	0.924	0.778
SB2	0.036	0.887	0.031	0.917	0.812	0.858	3.890	4.798	0.209	0.179	0.410	0.350
PB1	0.025	0.797	0.031	0.809	0.561	0.620	0.532	0.622	1.055	0.997	0.461	0.431
PB2	0.049	0.864	0.041	0.899	0.760	0.801	4.032	5.123	0.188	0.156	0.461	0.431

Note that among estimation approaches, boldface indicates the highest relative coverage and the shortest relative average lengths.

Table 6: Results based on various response time estimation techniques for different Queuing Network models

	Method of estimation with			
	Maximum Relative Coverage		Shortest Relative Average Length	
Large Sample size (n)	$n = 100$	$n = 200$	$n = 100$	$n = 200$
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	Normal	Normal	Normal	Normal
$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$	Normal	Normal	Normal	Normal
$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	Normal	Normal	Normal	Normal
Small Sample size (n)	$n = 15$	$n = 25$	$n = 15$	$n = 25$
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	CAN	CAN	CAN/PB	CAN/PB
$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$	CAN	CAN	CAN	CAN
$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	CAN	CAN	CAN	CAN

6 Conclusions

The calibrated confidence intervals for the average response time of the two-stage open queueing network are presented in this paper. For the two-stage open queueing network, we derive a response time sequence using a recurrence relation to inter-arrival and service times. Calibrated confidence intervals for average response time are produced by applying various estimation techniques, including Normal, CAN, Exact-t, Boot-t, SB, PB, and VST. To comprehend, compare, and evaluate the performance of the resulting calibrated confidence intervals, relative average lengths and the relative coverage are utilized. According to the simulation results, the CAN method performs best for small samples, while among nearly all estimation methods for large samples the Normal method performs best for simulated models. The approaches mentioned above can be implemented in real-world queueing networks.

References

Chu, Y. K. and Ke J. C. (2016), "Confidence intervals of mean response time for an M/G/1 queueing system: Bootstrap simulation," *Applied Mathematics and Computation*, 180, 255–263.

Chu, Y. K. and Ke, J. C. (2007), "Mean response time for a G/G/1 queueing system: Simulated computation," *Applied Mathematics and Computation*, 196, 772–779.

Efron, B. and Tibshirani, R. (1993), *An Introduction to the bootstrap*, Chapman and Hall, New York.

Gedam, V. K. and Pathare, S. B. (2015), "Estimation Approaches of Mean Response Time for a Two Stage Open Queueing Network Model," *Statistics, Optimization and Information Computing*, 3, 249–258.

Gedam, V. K. and Pathare, S. B. (2015), "Use of the calibration approach in Confidence intervals for mean response times of an open queueing network with feedback," *SIMULATION: Transactions of the Society for Modeling and Simulation International*, 91, 553–565.

Hogg, R. V. and Craig, A. T. (1995), *Introduction to Mathematical Statistics*, Prentice-Hall, Inc., Englewood Cliffs, N.J Prentice Hall.

Jackson, J. R. (1957), "Networks of Waiting Lines," *Operations Research*, 5, 518–521.

Loh, W. Y. (1987), "Calibrating confidence coefficient," *J. Amer. Statist. Assoc.*, 82, 155–162.

Pathare, S. B. and Gedam, V. K. (2019), "Mean Response Time of a Two Stage Open Queueing Network Model with feedback," *International Journal of Operational Research*, 35, 397–423.

Pathare, S. B. (2019), "Confidence Intervals for Mean Response Time of M/G/1 to G/G/1 Queueing Network Model with Feedback," *International Journal of Research and Analytical Reviews*, 6, 90–101.

Pathare, S. B. and Gedam, V. K. (2019), "Some Confidence Regions for Traffic Intensity Vector," *Statistics, Optimization & Information Computing*, 7, 360–369.

Rousses, G. G. (1997), *A Course in Mathematical Statistics*, 2nd ed., Academic Press, New York.

Rubin, D. B. (1981), "The Bayesian bootstrap," *The Annals of Statistics*, 9, 130-134.

Received: August 5, 2023

Accepted: January 4, 2024